COT 3100 Final Exam - Part B - 100 pts (4/25/2024) Solution

1) (4 pts) Fill out the following truth table.

p	q	r	$p \to q$	$r \to p$	$(p \to q) \to (r \to p)$
F	F	F	T	T	T
F	F	T	T	F	F
F	T	F	T	T	T
F	T	T	T	F	F
T	F	F	F	T	T
T	F	T	F	T	T
T	T	F	T	T	T
T	T	T	T	T	T

Grading: ½ per row, round down, row must be correct fully to get ½ pt.

2) (8 pts) For each of the two statements over the universe of real numbers, determine if they are true or false, and provide justification for reason. (Your answer is worth 1 pt for each part and the justification is worth 3 points.) Circle

(a) $\forall x \exists y[y^2 = 2xy - x^2]$ Is it true? **Yes** No

Justification

For all x, the value of y that exists is y = x, plugging that in the LHS = x^2, RHS = $2x^2 - x^2 = x^2$, proving that for the selection of y = x, the statement is satisfied.

(b) $\exists x \forall y[y^2 = 2xy - x^2]$ Is it true? Yes **No**

Justification

There is not a single value of x which makes the given statement always true. For any arbitrarily chosen value of x, let y = x + 1, then the LHS = $x^2 + 2x + 1$,

RHS = $2x(x+1) - x^2 = 2x^2 + 2x - x^2 = x^2 + 2x$.

Notice that it's impossible for these two to equal each other (in fact, they are guaranteed to be unequal). Thus, no matter what value is selected for x, it can be proven that there exists a value of y that makes the statement false (namely y = x+1), thus, the original statement is false.

Grading: 1 pt yes/no selection, 3 pts justification upto grader discretion.

3) (10 pts) Let A = {1, 2} and B = {3, 4}. Answer the following questions about the set $\wp(A \times B)$.

(a) How many elements are in the set? **16 (Grading: 2 pts)**

(b) What is the size of the largest element of $\wp(A \times B)$? **4 (Grading: 2 pts)**

(c) List each element of $\wp(A \times B)$ that contains exactly 2 elements. (Note: more blanks than necessary are provided to obfuscate how many elements satisfy the requirement.)

{(1,3), (1,4)}, {(1,3), (2,4)}, {(2,3), (1,4)}, {(2,3), (2,4)}, {(1,3), (2,3)},

{(1,4), (2,4)} **Grading: +1 for each correct answer, -1 for each incorrect answer, cap at 0.**

4) (6 pts) Let x and y be integers such that 13 | (2x + 5y). Prove that 13 | (73x – 19y).

Let 2x + 5y = 13c, for some integer c, based on the given information.

$$\begin{aligned} 73x - 19y &= 65x + 8x - 39y + 20y \\ &= (8x + 20y) + 65x - 39y \\ &= 4(2x + 5y) + 65x - 39y \\ &= 4(13c) + 65x - 39y \\ &= 13(4c + 5x - 3y) \end{aligned}$$

Since, c, x and y are all integers, it follows that 4c + 5x – 3y is also an integer. Thus, we can conclude that 13 | (73x – 19y) as desired.

Grading: 4 pts to rewrite given information, 2 pts to get to conclusion

Alternate solution: since 2x + 5y = 13c, we have 2x = 13c – 5y

$$\begin{aligned} 73x - 19y &= 13x + 60x - 19y \\ &= 13x + 30(2x) - 19y \\ &= 13x + 30(13c - 5y) - 19y \\ &= 13x + 390c - 150y - 19y \\ &= 13x + 390c - 169y \\ &= 13(x + 30c - 13y) \end{aligned}$$

Since x, c and y are all integers, so is x + 30c – 13y. It follows that 13 | (73x – 19y) as desired.

5) (10 pts) Find all ordered pairs of integers (x, y) that satisfy the equation 294x + 128y = 10.

Divide equation through by 2: 147x + 64y = 5. First, let's find a base solution, (x, y) to the equation 147x + 64y = 1, via the Extended Euclidean Algorithm:

147 = 2 x 64 + 19
64 = 3 x 19 + 7
19 = 2 x 7 + 5
7 = 1 x 5 + 2
5 = 2 x 2 + 1

5 – 2 x 2 = 1
5 – 2(7 – 5) = 1
3 x 5 – 2 x 7 = 1
3(19 - 2 x 7) – 2 x 7 = 1
3 x 19 – 8 x 7 = 1
3 x 19 – 8(64 – 3 x 19) = 1
27 x 19 – 8 x 64 = 1
27(147 – 2 x 64) – 8 x 64 = 1
27 x 147 – 62 x 64 = 1

Multiply this equation through by 5:

147(27 x 5) + 64(-62 x 5) = 5
147(135) + 64(-310) = 5

Thus, one solution to the original equation is x = 135, y = -310. It follows that all solutions can be expressed as:

{(x, y) | x = 135+64c, y =-310-147c, $c \in Z$}

Another way of expressing this solution set is:

{(x, y) | x = 7+64c, y =-16-147c, $c \in Z$}

Grading: 2 pts Euclidean, 4 pts Extended, 1 pt mult 5, 1 pt extract base solution, 2 pts complete solution

6) (4 pts) How many unique primes appear in the prime factorization of 50!?

Since 50! is the product of each positive integer upto 50, we just need to count the number of primes less than 50. These are:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, and 47.

There are 15 such primes, so the answer to the question is 15.

15

Grading: 2 pts for stating in English conceptually what the answer is 2 pts to count the primes.

7) (8 pts) A class has 15 boys and 18 girls. A team of 8 students, 4 boys and 4 girls must be selected. In how many ways can the team be selected? **Express your answer in prime factorized form.** (Note: 4 pts will be awarded for the symbolic answer and 4 pts will be awarded for the work on paper to simplify that to prime factorized form.)

$$\binom{15}{4}\binom{18}{4} = \frac{15 \times 14 \times 13 \times 12}{4!} \times \frac{18 \times 17 \times 16 \times 15}{4!}$$

$$= (15 \times 7 \times 13) \times (3 \times 17 \times 4 \times 15)$$

$$= 3 \times 5 \times 7 \times 13 \times 3 \times 17 \times 2^2 \times 3 \times 5$$

$$= \mathbf{2^2 \times 3^3 \times 5^2 \times 7 \times 13 \times 17}$$

Grading: 4 pts combination answer, 4 pts simplification to prime factorized form

8) (10 pts) Using induction on n, prove, for all positive integers n, that

$$\sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Base case: n = 1, LHS = $\sum_{i=1}^{1} \frac{1}{i(i+1)(i+2)} = \frac{1}{1\times2\times3} = \frac{1}{6}$, RHS = $\frac{1(1+3)}{4(1+1)(1+2)} = \frac{4}{24} = \frac{1}{6}$. It follows that the given formula is true for n = 1 and the base case holds.

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = k that

$$\sum_{i=1}^{k} \frac{1}{i(i+1)(i+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$$

Inductive step: Propve for n = k+1 that

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)(i+2)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)(i+2)} = \left[\sum_{i=1}^{k} \frac{1}{i(i+1)(i+2)}\right] + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}, \text{ using the IH}$$

$$= \frac{k(k+3)(k+3)}{4(k+1)(k+2)(k+3)} + \frac{4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+3)(k+3)+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k+1)(k+4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

This completes the proof of the inductive step. It follows that for all positive integers n,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Grading: 1 pt BC, 1 pt IH, 1 pt IS, 7 pts proof, grader decides partial

9) (10 pts) Let a continuous random variable be defined as follows:

$$f(x) = c(x-1)^2, for\ 0 \le x \le 4$$
$$f(x) = 0, otherwise$$

(a) (3 pts) Determine the value of c.

$$\int_0^4 c(x-1)^2 dx = 1$$

$$\frac{c(x-1)^3}{3}\Big|_0^4 = \frac{c}{3}\big(27-(-1)\big) = 1 \rightarrow c = \frac{3}{28}$$

Grading: 1 pt set up integral, 2 pts solve for c

(b) (4 pts) What is the expected value of the continuous random variable described above?

$$\int_0^4 \frac{3}{28}x(x-1)^2 dx = \frac{3}{28}\int_0^4 (x^3-2x^2+x)dx = \frac{3}{28}\left[\frac{x^4}{4}-\frac{2x^3}{3}+\frac{x^2}{2}\right]\Big|_0^4 = \frac{3}{28}\left(\frac{4^4}{4}-\frac{2(4)^3}{3}+\frac{4^2}{2}\right)$$

$$= \frac{3}{28}\left(64-\frac{128}{3}+8\right) = \frac{3}{28}\left(72-\frac{128}{3}\right) = \frac{3}{28}\times\frac{88}{3} = \frac{22}{7}$$

Grading: 1 pt integral setup, 3 pts solve

(c) (3 pts) What is the probability that x is in between 2 and 3 for this continuous random variable?

$$\int_2^3 \frac{3}{28}(x-1)^2 dx = \frac{(x-1)^3}{28}\Big|_2^3 = \frac{8}{28}-\frac{1}{28} = \frac{7}{28} = \mathbf{\frac{1}{4}}$$

Grading: 1 pt integral, 2 pts solve

10) (10 pts) The Griswold family is buying some vacation souvenir t-shirts. They've agreed to by exactly 18 t-shirts total out of four different types of t-shirts, each representing a different tour the family took. Let the t-shirt types be A, B, C and D. The tourist shop only has 3 copies of t-shirt B and 7 copies of t-shirt D left. The Griswold children (there are 2 of them)) each insist on having at least one copy of t-shirt type A, thus at least 2 copies of t-shirt A must be purchased. In how many ways can the family buy their shirts while adhering to these requirements? (Leave your answer in factorials, combinations, etc.)

First, buy 2 copies of shirt A, so that there are 16 t-shirts left to buy. Let a, b, c, and d be the number of shirts of each corresponding type they buy. Then we have

$a + b + c + d = 16$

We want the number of non-negative solutions to this equation with $b \le 3$ and $d \le 7$.

Using the combinations with repetition formula, we find that $\binom{16+4-1}{4-1} = \binom{19}{3}$ total ways to buy the shirts without restrictions on b and d.

Let's count the number of combinations with 4 or more shirts of type B. This necessitates us buying 4 of these shirts, leaving 12 shirts to buy, so we seek the number of solutions to

$a + b' + c + d = 12$

Thus, there are $\binom{12+4-1}{4-1} = \binom{15}{3}$ total ways to buy the shirts with at least 4 or type B. These should be subtracted from our total.

Let's do the same for the shirts of type D. Out of the 16, we buy 8 of these, leaving 8 more to buy. Thus, we want the number of solutions to

$a + b + c + d' = 8$.

Thus, there are $\binom{8+4-1}{4-1} = \binom{11}{3}$ total ways to buy the shirts with at least 8 or type D. These should be subtracted from our total.

But in doing the last two parts, we've double subtracted out the combinations with at least 4 shirts of type B and 8 shirts of type D. Thus, this needs to be added back in. Buy those 12 shirts out of 16, leaving 4 more to buy. We need the number of solutions to $a + b' + c + d' = 4$

Thus, there are $\binom{4+4-1}{4-1} = \binom{7}{3}$ total ways to buy the shirts with at least 4 or type B AND 8 of type D. These should be added back in since they were subtracted out twice.

It follows that our final answer is $\binom{19}{3} - \binom{15}{3} - \binom{11}{3} + \binom{7}{3}$. **Grading: First term 2 pts, 2 middle terms 3 pts, last term 2 pts.**

11) (10 pts) Let R be a relation defined over the universe of **non-empty sets of positive integers**, as follows:

$$R = \{(A, B) | A \cap B \neq \emptyset\}$$

With proof, determine if R is (a) reflexive, (b) irreflexive, (c) symmetric, (d) anti-symmetric, and (e) transitive.

(a) R is reflexive. Since the universe is non-empty subsets, for an arbitrarily chosen non-empty subset A, $A \cap A = A \neq \emptyset$, since A is non-empty. It follows that for all non-empty subsets A, (A, A) belongs to R.

(b) R is NOT irreflexive because for A = {1}, (A, A) is in R.

(c) R is symmetric. Let (A, B) be an element of R, it follows that $A \cap B \neq \emptyset$. But note that $A \cap B = B \cap A \neq \emptyset$, proving that (B, A) is in R as desired.

(d) R is not anti-symmetric. Let A = {1,2}, B = {2, 3}. We have (A, B) and (B, A) are both in R (because both sets share the element 2), but A ≠ B.

(e) R is not transitive. Consider the following counter-example. Let A = {1, 2}, B = {2, 3} and C = {3, 4}. In this example, $(A, B) \in R \wedge (B, C) \in R$, but $(A, C) \notin R$. The first is true because A and B share 2 in common, the second is true because B and C share 3 in common. But, A and C aren't related because they don't have any elements in common.

Grading: 1 pt answer, 1 pt reason, can only get the second point if the answer is correct.

12) (9 pts) Let $f(x) = \frac{2}{x+3} - 4$, with a domain of all real x except for x = -3.

(a) Determine $f^{-1}(x)$. Please express your answer in the form: $-(\frac{ax+b}{cx+d})$, where a, b, c and d are all positive integers.

Switch x and y and solve for y:

$$x = \frac{2}{y+3} - 4$$

$$x + 4 = \frac{2}{y+3}$$

$$y + 3 = \frac{2}{x+4}$$

$$f^{-1}(x) = \frac{2}{x+4} - 3$$

(b) What is the domain for $f^{-1}(x)$?

The domain of the inverse function is all real x except for x = -4. (We can see that this makes the function undefined.)

(c) What is the range for $f^{-1}(x)$?

This is the same as the domain for f(x), thus, this range is all real y except y = -3.

Grading: 5 pts part (a), 2 pts part (b), 2 pts part (c)

13) (1 pt) What geometric shape are the Pyramids of Giza? **<u>Pyramids</u> (Give to All)**