<u>COT 3100H Section 201 Exam #2 Solutions 3/5/2024</u>

1) (10 pts) Prove or disprove: if p and q are prime numbers, then at least one of the following five numbers is also prime: pq – 2, pq – 1, pq, pq + 1, pq + 2.

The claim is false. Consider p = 2, q = 13, both of which are prime. But, pq = 26 is NOT prime. Similarly, 24 = 4 x 6, 25 = 5 x 5, 27 = 3 x 9 and 28 = 4 x 7 are all also not prime, disproving the claim.

Grading: 2 pts or less for saying this is true. (Grader discretion)
3 pts for saying it's false.
3 pts for explicitly stating any valid values for p and q (1 pt for stating values that don't work)
4 pts for showing that each pq-2, pq-1, pq+1 and pq+2 are all composite.

2) (10 pts) What is the largest power of 12 that divides evenly into 1000! ? Please show your work for full credit and put a box around your final answer.

12 = $2^2 3$, thus we must determine both the maximal power of 2 that divides into 1000! and the maximal power of 3 that divides into 1000!

2	1000	500
2	500	250
2	250	125
2	125	62
2	62	31
2	31	15
2	15	7
2	7	3
2	3	+ 1
2	1	--------
		994

3	1000	333
3	333	111
3	111	37
3	37	12
3	12	4
3	4	+ 1
3	1	-------
		498

This means that 2^2 divides into 1000! a total of 994/2 = 497 times, while 3 divides into 1000! a total of 498 times. It follows that the largest power of 12 that divides evenly into 1000! is **<u>497.</u>**

Grading: 4 pts for determining how many times 2 divides into 1000!
4 pts for determining how many times 3 divides into 1000!
1 pt for determine how many times 2^2 divides into 1000!
1 pt for the correct final answer based on this evidence.

3) (15 pts) (a) Find the ordered pair of integers (x, y) that satisfies 248x + 111y = 3, with the minimum possible **positive** value of x.

(b) In addition, determine 111^{-1} mod 248. (Your answer must be a positive integer less than 248.)

Run the Extended Euclidean Algorithm with 248 and 111:

248 = 2 x 111 + 26
111 = 4 x 26 + 7
26 = 3 x 7 + 5
7 = 1 x 5 + 2
5 = 2 x 2 + 1

5 – 2 x 2 = 1
5 – 2(7 – 5) = 1
5 – 2 x 7 + 2 x 5 = 1
3 x 5 – 2 x 7 = 1
3(26 – 3 x 7) – 2 x 7 = 1
3 x 26 – 9 x 7 – 2 x 7 = 1
3 x 26 – 11 x 7 = 1
3 x 26 – 11(111 – 4 x 26) = 1
3 x 26 – 11 x 111 + 44 x 26 = 1
47 x 26 – 11 x 111 = 1
47(248 – 2 x 111) – 11 x 111 = 1
47 x 248 – 94 x 111 – 11 x 111 = 1
47 x 248 – 105 x 111 = 1

Multiply this through by three to get 141 x 248 – 315 x 111 = 1. Thus, one solution is (141, -315). All solutions are of the form { (141+111c, -315-248c) | c $\in$ Z }. We can see that 141 > 111 but 141 < 222. It follows that we can get the solution with minimal positive x by setting c = -1. The corresponding solution is **(30, -67).**

Take the equation (equal to 1) above mod 248 to get

0 – 105 x 111 $\equiv$ 1 (mod 248)

It follows that $111^{-1} \equiv -105 \equiv$ **143 (mod 248)**

(a) (30 , -67) **(b) 143**

Grading: 3 pts Euclidean, 7 pts Extended, 1 pt mult by 3,
2 pts to logic out correct answer to (a),
2 pts to give the correct answer to (b), partial possible for each part.

4) (12 pts) (a) Plug in values to make a guess for a closed-form solution to the following summation, in terms of n: $\sum_{k=1}^{n} k(k!)$. (Copy the summation, put an equal sign and put your formula in terns of n on the right hand side.)

The first few terms of the sum are 1 x 1! = 1, 2 x 2! = 4, 3 x 3! = 18 and 4 x 4! = 96. The corresponding sums are 1, 5, 23, and 119. Notice that the first few factorials are 1, 2, 6, 24 and 120. Thus, a reasonable guess is that the sum equals (n+1)! – 1. Thus, our conjecture is:

$\sum_{k=1}^{n} k(k!) = (n+1)! - 1$ **(3 pts)**

(b) Use mathematical induction to prove that is your guess is correct for all positive integers n.

Base case: n = 1, LHS = $\sum_{k=1}^{1} k(k!) = 1 \times 1! = 1$ RHS = (1 + 1)! – 1 = 2 – 1 = 1

Thus, the base case holds. **(1 pt)**

Inductive hypothesis: Assume for an arbitrarily chosen positive integer n = m that: **(1 pt)**

$$\sum_{k=1}^{m} k(k!) = (m+1)! - 1$$

Inductive step: Prove for n = m + 1 that **(1 pt)**

$$\sum_{k=1}^{m+1} k(k!) = (m+2)! - 1$$

$$\sum_{k=1}^{m+1} k(k!) = \left(\sum_{k=1}^{m} k(k!)\right) + (k+1)(k+1)!$$

(1 pt)

$= (k+1)! - 1 + (k+1)(k+1)!$, plugging in the inductive hypothesis **(2 pts)**

$= (k+1)!\,[1 + k + 1] - 1$, factoring out (k+1)! **(1 pt)**

$= (k+1)!\,(k+2) - 1$ **(1 pt)**

$= (k+2)! - 1$ **(1 pt)**

This completes the proof of the inductive step. It follows that for all positive integers n,

$$\sum_{k=1}^{n} k(k!) = (n+1)! - 1$$

5) (12 pts) Define the recurrence relation t(n) for all non-negative integers as follows:

$$t(0) = 4,\ t(1) = 11,\ t(n) = 7t(n-1) - 10t(n-2), \text{ for all } n \geq 2.$$

Prove via strong induction on n with 2 base cases that for all non-negative integers n,

$$t(n) = 3 \times 2^n + 5^n$$

Base cases: n = 0, LHS = t(0) = 4, RHS = $3 \times 2^0 + 5^0 = 3 \times 1 + 1 = 4$ **(1 pt)**
n = 1, LHS = t(1) = 11, RHS = $3 \times 2^1 + 5^1 = 3 \times 2 + 5 = 11$ **(1 pt)**

It follows that the given assertion is true for n = 0 and n = 1, so theses two base cases hold.

Inductive hypothesis: Assume for all non-negative integers n ≤ k, where k is an arbitrarily chosen positive integer 1 or greater that $t(n) = 3 \times 2^n + 5^n$. **(2 pts)**

Inductive step: Prove for n = k + 1 that $t(k+1) = 3 \times 2^{k+1} + 5^{k+1}$. **(1 pt)**

$t(k+1) = 7t(k) - 10t(k-1)$, using the given recurrence **(1 pt)**
$= 7(3 \times 2^k + 5^k) - 10(3 \times 2^{k-1} + 5^{k-1})$, using IH twice. We can do this because k+1≥ 2, so k≥1 and k-1≥0, so IH applies in both cases. **(2 pts)**

$= 21 \times 2^k + 7 \times 5^k - 30 \times 2^{k-1} - 10 \times 5^{k-1}$ **(1 pt)**

$= 21 \times 2^k + 7 \times 5^k - 15 \times 2^k - 2 \times 5^k$, factoring out 2 and 5, respectively. **(1 pt)**

$= 6 \times 2^k + 5 \times 5^k$, combining like terms **(1 pt)**

$= 3 \times 2^{k+1} + 5^{k+1}$, which completes the proof of the inductive step. **(1 pt)**

6) (12 pts) Prove, via induction on n, that for all non-negative integers n, $7 \mid (3 + 4 \times 8^{2n})$.

Base case: n = 0, $3 + 4 \times 8^{2(0)} = 3 + 4 \times 1 = 7$. Since 7 | 7, the base case holds. **(1 pt)**

Inductive hypothesis: Assume for an arbitrarily chosen non-negative integer n = k that **(1 pt)**

$$7 \mid (3 + 4 \times 8^{2k})$$

Thus, there exists an integer c such that $7c = 3 + 4 \times 8^{2k}$.

Inductive step: Prove for n = k+1 that $7 \mid (3 + 4 \times 8^{2(k+1)})$. **(1 pt)**

$$\begin{aligned}
3 + 4 \times 8^{2(k+1)} &= 3 + 4 \times 8^{2k+2} \\
&= 3 + 4 \times 8^{2k} \times 8^2 \\
&= 3 + 4 \times 8^{2k} \times 64 \\
&= 3 + 63(3) + 4 \times 8^{2k} \times 64 - 3(63) \\
&= 64 \times 3 + 4 \times 8^{2k} \times 64 - 3(63) \\
&= 64 \times (4 \times 8^{2k} + 3) - 3(63) \\
&= 64 \times (7c) - 3(63) \\
&= 7(64c - 27)
\end{aligned}$$

Grading: 8 pts for the algebra to factor out 7, 1 pt for arguing that what's leftover is an integer.

Since c is integer, 64c – 27 is also in integer. It follows that $7 \mid (3 + 4 \times 8^{2(k+1)})$, proving the inductive step.

It follows that for all non-negative integers n, $7 \mid (3 + 4 \times 8^{2n})$.

7) (8 pts) 3 workers, each working at the same uniform rate, take 5 days to build 2 widgets. How many days would it take 2 of these workers, working at the same uniform rate, to build 7 of the same widgets? Please express your answer as a mixed fraction in lowest terms.

Thus, one worker would take 15 days to build 2 widgets, meaning that a single worker builds a single widget in 15/2 days.

If two workers are working, we want each to build 3.5 widgets so that together they build 7 widgets.

A single worker builds 1 widget in $\frac{15}{2}$ days. It follows that they'll build $\frac{7}{2}$ widgets each in $\frac{15}{2} \times \frac{7}{2} = \frac{105}{4} = \mathbf{26\frac{1}{4}}$ **days.**

Grading: 3 pts to logic out how long it takes one worker to make one widget, or some sort of equivalent unit piece of information.

3 pts to logic out the answer to the question as an improper fraction or decimal.

1 pt to convert to a mixed fraction in lowest terms.

8) (10 pts) Jenna does 5 push ups on day 1. She vows to complete exactly 2 more push ups on each subsequent day than the day before. On which day number does she complete 107 push ups? How many total push ups has she done starting on day 1 through the day she did 107 push ups?

The number of push ups Jenna does forms an arithmetic series with $a_1 = 5$ and $d = 2$, where a_k is the number of push ups she does on day k.

$a_k = 107 = a_1 + (k = 1)d = 5 + (k – 1)2$
$107 = 5 + 2k – 2$
$104 = 2k$
$k =$ **52**, so she did 107 push ups on day 52.

Total # of push ups = $\frac{(a_1+a_n)n}{2} = \frac{(5+107)52}{2} = 112 \times 26 = \mathbf{2912}$

Day with 107 push ups: **52 ,** Total Number of push ups: **2912**

Grading: 5 pts solving for # of days (grader discretion for partial)
5 pts to solve sum (grader discretion for partial)

9) (10 pts) Let x be the smallest integer with precisely 35 divisors. It turns out that there exists a positive integer y such that $x = y^2$. With proof, determine the value of y. (Put a box around your answer.)

Only perfect squares have an odd number of divisors. More specifically, the product of the exponents plus one of the primes of this number must equal 35. It follows that the number must be either of the form p^{34} or p^6q^4, where p and q are primes. (Since 7 and 5 are primes, we can't have more than 2 separate primes in the prime factorization of an integer with precisely 35 divisors.) The smallest number of the form p^{34} is very clearly 2^{34}, since 2 is the smallest prime number. For the other form, since 6 > 4, we want to minimize p, followed by minimizing q. (If $p < q$, then $p^6q^4 < p^4q^6$.) Thus, it makes sense to set p = 2, the smallest prime and q = 3, the second smallest prime.

By inspection, we should be able to see that $2^63^4 < 2^{34}$. (The former is the product of two numbers under 100 while the latter is over one billion.)

Thus, $x = 2^63^4 = y^2$. It follows that $y = 2^33^2 = 8 \times 9 =$ **72.**

Grading: 3 pts for stating formula for number of divisors.
1 pt for listing p^{34} as a possible form for the answer.
1 pt for removing this as a possible final answer.
2 pts for listing p^6q^4 as a possible form for the answer.
2 pts for using logic to determine p = 2, q = 3.
1 pt arriving at the final answer for y.

10) (1 pt) The famous Boston Massacre took place on March 5, 1770 (exactly 254 years ago). In which city did it occur?

Boston (1 pt - Give to All)