COT 3100 Section 201 Exam #1 (1/31/2024) Solutions

1) (10 pts) Use the laws of logic to show that the two following Boolean expressions are logically equivalent. **Please separate out Commutative and Associative steps.**

(a) $(p \wedge (q \vee (r \wedge q))) \vee (q \wedge \overline{(\bar{r} \vee p) \wedge p}$

(b) q

Please use the format shown in class to show the equivalence.

$(p \wedge (q \vee (r \wedge q))) \vee (q \wedge \overline{(\bar{r} \vee p) \wedge p}$	Given
$(p \wedge (q \vee (r \wedge q))) \vee (q \wedge \overline{p \wedge (\bar{r} \vee p)}$	Commutative Law
$(p \wedge (q \vee (r \wedge q))) \vee (q \wedge \overline{p \wedge (p \vee \bar{r})}$	Commutative Law
$(p \wedge (q \vee (r \wedge q))) \vee (q \wedge \bar{p})$	Absorption Law
$(p \wedge (q \vee (q \wedge r))) \vee (q \wedge \bar{p})$	Commutative Law
$(p \wedge q) \vee (q \wedge \bar{p})$	Absorption Law
$(q \wedge p) \vee (q \wedge \bar{p})$	Commutative Law
$q \wedge (p \vee \bar{p})$	Distributive Law
$q \wedge T$	Inverse Law
q	Identity Law

Grading: Holistic, roughly 7 pts for steps, 3 pts for reasons

2) (9 pts) The following truth table was correctly filled out by a student, but the column headers (in their place, A, B and C are written), which were supposed to be Boolean expressions were erased.

p	q	r	A	B	C
F	F	F	T	T	T
F	F	T	T	F	F
F	T	F	F	T	F
F	T	T	F	T	F
T	F	F	T	T	T
T	F	T	T	T	T
T	T	F	F	T	F
T	T	T	F	T	F

Determine possible expressions for A, B and C, limiting yourself to the variables, p, q and r. In addition to your answers, explain briefly how you came up with them.

For A, we can glance at the second column and see that all eight entries in this column (A) are the opposite of is shown in the second column, which is for q.

For B, notice that most items are true. When either p or q is true (bottom 6) our expression is true. Finally, the other option that isn't covered is when both p and q are false, r is also false.

Finally for C, we can see by looking at A and B that precisely the rows that are both true in A and B are true in C, which corresponds to and.

A: $\overline{\boldsymbol{q}}$ B: $\boldsymbol{p} \vee \boldsymbol{q} \vee \overline{\boldsymbol{r}}$ C : $\overline{\boldsymbol{q}} \wedge (\boldsymbol{p} \vee \boldsymbol{q} \vee \overline{\boldsymbol{r}})$

Grading: Holistic, but generally speaking, credit was given for only the answer listed above for A, several possible choices for B, and only for the intersection of whatever the student wrote for A and B for C.

3) (6 pts) Jesse races in the Daytona 500, a 500 mile race. When he is driving, he averages 160 miles per hour. He ends up having to stop for pit stops every 47 miles (first stop is at mile 47, next at mile 94, etc.) Each stop takes 1 minute, 15 seconds. What is his actual average speed over the course of the whole race, from start to finish., in miles per hour?

Time driving $= \frac{500}{160} = \frac{25}{8}$ hours. There are 10 pit stops (47, 94, …, 470) since the one after 470 would be after 500. This means the pitstops last $10 \times \frac{5}{4} \times \frac{1}{60} = \frac{5}{24}$ hours. Thus, the actual total time it took Jesse to complete the race was $\frac{25}{8} + \frac{5}{24} = \frac{80}{24} = \frac{10}{3}$ hours.

It follows that is overall average speed was $\frac{500\ miles}{\frac{10}{3} hours} = \boldsymbol{150\ mph}$

Grading: Holistic, but time for driving, time for stopping and overall average speed calculations need to be accurate and easily communicated.

4) (10 pts) Prove or disprove the following statement for finite sets A, B, C and D:

If $A \subseteq B \cup C$ and $B \subseteq D$, then $A \cap C \subseteq D$.

This is false. Consider the following counter-example:

A = {1} B = {2}

C = {1} D = {2}

In this counter-example, $B \cup C = \{1,2\}$, thus $A \subseteq B \cup C$. It can also be seen that $B \subseteq D$ because both sets are equal. But, $A \cap C = \{1\}$ and 1 doesn't belong to D, thus $A \cap C \subseteq D$ is false.

Grading: **Automatic 2/10 or less if they try to prove**
Holistic for disproofs. Looking for reasonable explanation of counter-example
Invalid counter-examples only get 1 or 2/10.

5) (10 pts) Prove of disprove the following statement for finite sets A, B and C:

If $A \subset B$ and $C \subseteq A$, then $A - C \subset B - C$

This was given as an exercise at the end of one of the class periods. This assertion is true.

Let's use direct proof to prove it. We must show two things:

(1) $A - C \subseteq B - C$, and

(2) $\exists x | (x \in B - C) \wedge (x \notin A - C)$

To prove (1), let x be an arbitrarily chosen element of $A - C$. We must show that $x \in B - C$.

By definition of set difference, $x \in A \wedge x \notin C$. Since we're given that $A \subset B$, it follows that $x \in B$, by definition of proper subset. Finally, by definition of set difference, we can conclude that $x \in B - C$, as desired. This completes the proof of (1).

To prove #2, we first use the given information that $A \subset B$. By definition of proper subset, there must exist some element, x such that $x \in B \wedge x \notin A$. Since $C \subseteq A$, by definition of subset, we can deduce that $x \notin C$. By definition of set difference, since $x \in B \wedge x \notin C$, we can conclude that $x \in B - C$. Finally, since $x \notin A$, it also follows by definition of set difference that $x \notin A - C$. This completes the proof of (2), since we've found an element that belongs to B – C, but not A – C.

Grading: Holistic, but looking for formal proof. Automatic 0/10 for all disproofs (since I said in class this was true…). Roughly 7 points for proof of subset, and 3 more for proper subset. Intuitive proofs can range from 1 – 6 pts depending on how convincing they are.

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6) (10 pts) Each of the 100 students at Hilldale Elementary speak English, French or Spanish. 85 students speak English or French, 70 students speak French or Spanish and 95 students speak English or Spanish. If we tallied 1 point for each student who spoke English, 1 point for each student who spoke French and 1 point for each student who spoke Spanish, 160 total points would be tallied. (Some students contribute 1, others 2 and a few 3, to this tally.) How many students speak all three languages out of the 100 total student? (Most of the credit will be given for the justification via the Inclusion-Exclusion Principle and not the answer.)

Let the sets be E, F and S respectively. The given info is as follows:

$$|E \cup F| = 85, |F \cup S| = 70, |E \cup S| = 95, |E| + |F| + |S| = 160$$

We can write down and add the first three statements via the 2 set Inclusion-Exclusion Principle as follows:

$$\begin{aligned}
|E \cup F| &= |E| + |F| - |E \cap F| = 85 \\
|F \cup S| &= |F| + |S| - |F \cap S| = 70 \\
|E \cup S| &= |E| + |S| - |E \cap S| = 95 \\
&\text{-------------------------------} \\
&2(|E| + |F| + |S|) - |E \cap F| - |F \cap S| - |E \cap S| = 250 \\
&2(160) - |E \cap F| - |F \cap S| - |E \cap S| = 250 \\
&|E \cap F| + |F \cap S| + |E \cap S| = 320 - 250 = 70
\end{aligned}$$

Now, we're ready to plug into the three set Inclusion-Exclusion Principle:

$$100 = |E \cup F \cup S| = |E| + |F| + |S| - (|E \cap F| + |F \cap S| + |E \cap S|) + |E \cap F \cap S|$$

$$100 = 160 - 70 + |E \cap F \cap S|$$

$$|E \cap F \cap S| = 10$$

Grading: Holistic, but roughly 6 points to discover sum of intersections by 2 with formalized proof, 4 points to finish from there, 3 out of 10 for correct intuitive answer without clear reference to formally plugging into I/E principle.

7) (7 pts) Solve the following equation for x:

$$log_4 x = log_x 16$$

Express your answer in the form a^b, where a is a positive integer and b is irrational.

Use a common base: $\frac{\ln x}{ln4} = \frac{\ln(4^2)}{\ln x} \rightarrow \frac{\ln x}{ln4} = \frac{2\ln 4}{\ln x} \rightarrow ln^2 x = 2ln^2 4$, taking the square root of both sides we find: $lnx = (ln4)\sqrt{2}$. Raising e to the power of both sides yields:

$$\boldsymbol{x = 4^{\sqrt{2}}}$$

We can verify this is correct by noting that by definition, $log_4 4^{\sqrt{2}} = \sqrt{2}$, and then by further noting that $(4^{\sqrt{2}})^{\sqrt{2}} = 4^2 = 16$, verifying that the original RHS, for $x = 4^{\sqrt{2}}$, is also equal to $\sqrt{2}$.

Grading: Holistic

8) (6 pts) Prove or disprove the following statement for all real numbers x and y:

$$\forall x \exists y\, [\, xy + 11 = 4x + 3y\,]$$

This is false. Consider the value of x = 3. (To disprove a for all x statement, we must just find one value of x for which the statement doesn't hold.) If we plug in x = 3, we get:

$$3y + 11 = 4(3) + 3y$$

This implies that:

$$11 = 12$$

But there is no value of y for which this is true.

Note: To discover this value of x, do the following: xy – 4x – 3y + 11 + 1 = 1 →
(x – 3)(y – 4) = 1
But if x = 3, LHS = 0 and the statement is false.

Grading: Holistic, but max 1/6 for any attempted proofs.

9) (6 pts) Jenna drives a motorboat downstream of a river with a current of 4 miles per hour for 2 hours. She decides to return, driving the motorboat upstream for another 2 hours, but only makes it two-thirds of the way back to her original starting point. How far (in miles) is she from her starting point when she stops going upstream?

Let r be rate Jenna drives the motorboat in still water.

In two hours, she travels $2(r + 4) = 2r + 8$ miles downstream.
In two hours coming back she travels $2(r – 4) = 2r – 8$ miles back upstream, so she is

$(2r+8) – (2r – 8) =$ **16 miles** from her original starting point.

Grading: Holistic, several ways to solve this.

10) (1 pt) On January 31, 1865, the House of Representatives passed the 13th amendment. How many amendments that were currently in effect had been previously passed by the House of Representatives, prior to the 13th amendment?

12 (Give to All)