Spring 2021 COT 3100H Exam #2 Review

Exam Format - In Class

Several Free Response Questions, 100 points total

Exam AID

All printed course notes.

Exam Format - Online

Exam will be split into three parts. Here are the times the parts will be posted, the due time and the late due times:

Part	Released	Due Time	Late Due Time
Num Theory	10:20 am	10:50 am	11:00 am
Induction	10:50 am	11:20 am	11:30 am
Counting/Recitation	11:20 am	11:50 am	12:00 pm

Please just print out a copy of the Exam 1 Reference Sheet and any of your course notes and use these only. I am using an honor system to ensure that you do not cheat, as well as the timings above. (The latter don't prevent cheating, but they make it unlikely that cheating can earn you extra points.)

You can only submit files of the following formats: .pdf, .doc, .docx and .txt for each part. I would like for you to submit ONLY 1 file per each part of the exam.

Outline of Intro to Discrete Exam #2 Topics

I. Number Theory

- a. Division Algorithm
- b. Euclid's Algorithm
- c. Extended Euclid's Algorithm
- d. Full Solution to $ax+by = c$ for integers given a,b,c.
- e. Finding modular inverses
- f. Divisibility proofs
- g. Pi notation
- h. Fundamental Thm of Algebra
- i. Least Common Multiple (LCM)
- j. Connection between LCM and GCD
- k. Idea behind proofs relating lcm, gcd with 3 integers.
- l. Calculating # of divisors of an integer.
- m. Calculating the sum of divisors of an integer.
- n. Calculating the number of times prime p divides into n!
- o. Proof that $\sqrt{2}$ is irrational.

II-A. Arith Geo Series, Sums, Matrices

- a. Arithmetic Series - solving for terms, sums, etc.
- b. Geometric Series - solving for terms, sums, etc.
- c. How to recursively define sequences
- d. Definition of summation notation
- e. Summation Rules
- f. Index shift idea
- g. Telescopic sum idea
- h. Idea of bounding sums with integrals
- i. Matrix Addition, Subtraction, Multiplication

II-B. Mathematical Induction

- **a. Base Case**
- **b. Inductive Hypothesis**
- **c. Inductive Step**
- **d. Summation Rules**
- **e. Not all induction problems use summations**
- **f. How to deal with inequalities**
- **g. Strong Induction**
- **h. Divisibility Problems**
- **i. Matrix Exponentiation Problems**
- **j. Problems with recursively defined sequences**
- **k. Problems with Harmonic numbers**
- **l. Unorthodox Examples - NIM, Nuggets, Trominos**

III-A. Counting

- **a. Addition, Subtraction, Multiplication, Division**
- **b. Permutations unique objects**
- **c. Permutations with repeated objects**
- **d. Combinations**
- **e. Inclusion-Exclusion Principle**
- **f. Combinatorial vs. Algebraic Proof**
- **g. Super Letter Idea**
- **h. Spacing Idea (for disallowing consecutive items)**
- **i. Combinations with Repetition (CwR)**
- **j. Lower Bound Requirements for CwR**
- **k. Upper Bound Requirements for CwR**
- **l. Inequality for CwR**

III-B. Recitation Material

- **a. Factorization**
- **b. Averages**

What to Study

1) Read the notes.

2) Practice problems posted online in my archive.

3) Look over written lectures and past homework solutions.

Sample Questions (from old exams)

1) Use induction to prove that $64 \mid (3^{2n} - 8n - 1)$ for all integers $n \geq 0$.

2) Let c be an integer such that $3 \mid c$. Prove that $(c+1)^3 \equiv 1 \pmod{9}$.

3) Prove the following inequality for all positive integers n:

$$\sum_{i=1}^{2^n} \log_2 i \leq (n-1)2^n + 1$$

(Hint : Remember that $\log_2(2^x - y) \leq x$ when x is a positive integer and y is a non-negative integer such that $y < 2^x$.)

4) Let a and b be integers such that $7 \mid (2a + 3b)$. Prove that $7 \mid (13a + 16b)$.

5) Determine 59^{-1} mod 203. Please express your answer as an integer in between 0 and 202. In order to earn full credit you must use the Extended Euclidean Algorithm.

6) Let $a = 2^3 3^5 5^2 7^1$ and $b = 3^3 5^6 7^2$. How many divisors does a have? How many divisors does b have? Express the greatest common divisor of a and b in prime factorized form. Express the least common multiple of a and b in prime factorized form.

7) Using induction on n, prove for all non-negative integers n that $9 \mid (2^{2n} + 6n - 1)$.

8) Using induction on n, prove for all positive integers n that $\sum_{i=1}^{n} i^2 \leq n^3$.

9) Let $H_n = \sum_{i=1}^{n} \frac{1}{i}$. Using induction on n, prove for all positive integers n that

$$\sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2} H_n - \frac{n(n-1)}{4}$$

10) Define the sequence t_n as follows: $t_0 = 7$, $t_1 = 10$, $t_n = 3t_{n-1} - 2t_{n-2}$, for all integers $n \geq 2$. Prove, using strong induction on n with 2 base cases, that for all non-negative integers n,

$$t_n = 4 + 3(2^n).$$

11) Use the Extended Euclidean Algorithm to determine 43^{-1} mod 95. Points will be awarded for the process only and not the answer. To receive credit, you must use the process shown in class.

12) There are 10 quizzes in a class which comprise the whole grade for the class. The i^{th} quiz is out of 10i points and each point in the class has equal value. (Note: the quizzes do

NOT have equal value…) After the first 8 quizzes, Tamara has an 80% in the class. Note that all scores on quizzes are an integer number of points. What is the highest integer percentage average she can receive for the 10 quizzes? If she achieved this average exactly, but didn't ace (earn all of the points) either the 9th or 10th quiz, what is the sum of the digits of her scores on the last two quizzes? Also, given this information, are both quiz scores uniquely determined, or are there multiple pairs of scores for quiz 9 and 10 that satisfy all the given constraints. Prove your answer.

13) What is the sum of divisors of the integer 34.3 million? Please give your answer in prime factorized form. (Note: 5 points for any reasonable expression that is correct, 5 points to take that expression and represent it in prime factorized form.) **Note: In order to earn full credit, you must show each step as if you didn't use a calculator. (Namely, you may use the calculator to check basic arithmetic, but each step must be something I could imagine a reasonable student doing without a calculator. In short, I am specifically testing skills I went over that allow you to solve this problem completely without a calculator.)**

14) Using induction on n, prove the following inequality for all positive integers n:

$$\sum_{i=1}^{2^n-1} log_2 i \geq (n-2)2^n + 2$$

15) Define the sequence t_n for positive integers n as follows:

$$t_1 = t_2 = t_3 = 1$$
$$t_n = t_{n-1} + t_{n-2} + t_{n-3}, for\ n > 3$$

Using strong induction on n with four base cases, prove the following for all positive integers n.

if $4 \mid n$, then $t_n \equiv 3\ mod\ 4$
if $4 \nmid n$, then $t_n \equiv 1\ mod\ 4$

Note: In your proof you will have four cases, but three of those four cases turn out to be equivalent, so it's enough to break your proof into two cases: where k+1 is divisible by 4 and where k+1 is NOT divisible by 4.

16) Prove, using strong induction with four base cases, that for all positive integers n ≥ 24, that n can be expressed as the sum of perfect squares, not including 1. (Note: perfect squares are numbers that can be represented as an integer squared. The first few perfect squares except for 1 are 4, 9, 16, and 25.) **<u>Also, prove that 24 is the smallest integer for which such a claim can be made.</u>**