Spring 2021 Honors COT3100 Exam #2 Solutions

Date: 3/24/2021

1) (10 pts) Tubas weigh 39 pounds and trombones weigh 15 pounds. A high school has weighed all of its tubas and trombones and they weigh exactly 2313 pounds.

(a) Give one possible number of tubas and trombones (both non-negative integers) that satisfy the given information.

We want an integer solution (x, y) to the equation 39x + 15y = 2313. By inspection, we can see that the gcd(39, 15) = 3, so we can divide this equation by 3 to get:

13x + 5y = 771

Get one solution to 13x + 5y = 1 via the Extended Euclidean Algorithm:

13 = 2 x 5 + 3
5 = 1 x 3 + 2
3 = 1 x 2 + 1

3 - 1 x 2 = 1
3 - (5 - 3) = 1
3 - 5 + 3 = 1
2 x 3 - 1 x 5 = 1
2(13 - 2 x 5) - 1 x 5 = 1
2 x 13 - 5 x 5 = 1

Multiply this through by 771:
(2 x 771) x 13 + (-5 x 771) x 5 = 771

It follows that one possible solution to the adjusted equation is (1542, -3855).
All possible solutions are of the form { (1542 - 5n, -3855 + 13n) | n ∈ Z }

To get a valid solution, plug in a value of n which makes -3855 + 13n ≥ 0. By inspection, a value of 300 will suffice.

This means that the school could have 1542 - 5(300) = **42 tubas** and -3855 + 13(300) = **45 trombones.** (Naturally, many other answers are valid here.

(b) With proof, determine the number of different ordered pairs of non-negative integers (x, y) satisfy the given information such that the school has x tubas and y trombones.

Inspection won't suffice here. The minimum value for n for which -3855 + 13n ≥ 0 is n = $\left\lceil \frac{3855}{13} \right\rceil = 297$. The maximum value for n is the largest value for which 1542 - 5n ≥ 0. Thus the maximum value of n is n = $\left\lfloor \frac{1542}{5} \right\rfloor = 308$. There are 12 integers in between 297 and 308, inclusive, thus, there are 12 possible ordered pairs of solutions (x, y) which satisfy the given information.

2) (10 pts) Let X equal the least common multiple of 840,000,000 and 99,000,000.

(a) Express X in prime factorized form.

$840000000 = 84 \times 10^7 = 4 \times 21 \times 2^7 \times 5^7 = 2^2 \times 3 \times 7 \times 2^7 \times 5^7 = 2^9 \times 3 \times 5^7 \times 7$
$99000000 = 99 \times 10^6 = 9 \times 11 \times 2^6 \times 5^6 = 2^6 \times 3^2 \times 5^6 \times 11$

Using the LCM formula given in class for two integers with their prime factorizations, we have

Lcm(840000000, 99000000) = $LCM(2^9 \times 3 \times 5^7 \times 7, 2^6 \times 3^2 \times 5^6 \times 11) =$ **$2^9 \times 3^2 \times 5^7 \times 7 \times 11$**

(b) Find the number of divisors of X.

Using the formula from class, the # of divisors of X is

$(9 + 1)(2 + 1)(7 + 1)(1 + 1)(1 + 1) = 10 \times 3 \times 8 \times 2 \times 2 =$ **960**

3) (10 pts) A fun trick to square a number ending in 5 is as follows:

1) Take the part preceding the digit 5 and call this part X.
2) Multiply X by (X+1) and write this down.
3) Add the digits 25 to the end of it.

For example, if we try to calculate 115^2, we get X = 11, and X(X+1) = 132, so the value of 115^2 = 13225, completing the trick.

Prove that this trick always works!

The value of the number X5 is 10X + 5, since the digits of X are shifted to the left by one place. Let's calculate this value squared:

$(10X + 5)^2 = 100X^2 + 100X + 25 = 100X(X+1) + 25$

We can see that the product X(X+1) is multiplied by 100, which means it's shifted by two places. Then, the last two places are filled in by the 25, which is precisely what the trick tells us to do!

4) (12 pts) Recall that the Fibonacci numbers are defined as follows:

$F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, for all $n \geq 2$.

Using induction on n, prove the following summation for all positive integers n:

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$$

Base case: n = 1, LHS = $\sum_{i=1}^{1} F_i^2 = F_1^2 = 1$, RHS = $F_1 F_2 = 1(1) = 1$
Thus, the given equation is true for n = 1

Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$$\sum_{i=1}^{k} F_i^2 = F_k F_{k+1}$$

Inductive step: Prove for n = k + 1 that

$$\sum_{i=1}^{k+1} F_i^2 = F_{k+1} F_{k+2}$$

$$\sum_{i=1}^{k+1} F_i^2 = \left(\sum_{i=1}^{k} F_i^2\right) + F_{k+1}^2$$

$$= F_k F_{k+1} + F_{k+1}^2, \text{ via the inductive hypothesis}$$

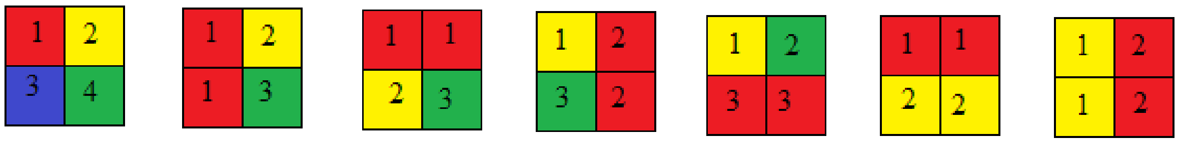
$$= F_{k+1}(F_k + F_{k+1})$$

$$= F_{k+1}(F_{k+2}), \text{ as desired.}$$

This completes the proof of the inductive step. It follows that for all positive integers n,

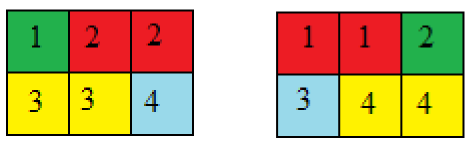
$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$$

5) (18 pts) Consider tiling a 2 x n hallway with tiles that are either size 1 x 1 or 1 x 2. Using strong induction on n with 2 base cases, prove that the tiling can be done in $\frac{3}{4}(3^n) + \frac{1}{4}(-1)^n$ ways, for all positive integers n. Two tilings are considered different if any square in one tiling is covered with either a different type of tile or a 1 x 2 tile in a different position between the two tilings. To provide some help, here are all the tilings for a 2 x 2 hallway:



Each individual tile is shown in a different color. For the printed version, since the colors don't show up well, each square that is part of the same tile is labeled with the same number. The first tiling has 4 separate 1 x 1 tiles. The following four tilings have exactly one 1 x 2 tile, oriented in 4 different ways. The last two tilings have two 1 x 2 tiles, both horizontal or both vertical.

The added restriction to the problem is that for all tiles that are horizontally placed, another horizontally placed tile is not allowed above or below it that is offset by exactly one square either way. Thus, for this problem, the two following tilings of a 2 x 3 hallway are not allowed:



Hint: To build all possible tilings of a 2 x (k+1) hallway, build off of all valid tilings of hallways of sizes 2 x k and 2 x (k - 1) respectively. This is only possible due to the restriction given.

In your solution, there's no need to reproduce the drawings above, just reference the drawings as needed.

Base case: n = 1, there are 2 possible tilings, one using 2 1 x 1 tiles and one using a single 1 x 2 tile. Now, evaluate the given formula for n = 1: $\frac{3}{4}(3^1) + \frac{1}{4}(-1)^1 = \frac{9}{4} - \frac{1}{4} = 2$

n = 2, there are the 7 possible tilings illustrated in the problem description. Now, evalulate the given formula for n = 2: $\frac{3}{4}(3^2) + \frac{1}{4}(-1)^2 = \frac{27}{4} + \frac{1}{4} = 7$.

Thus, the given formula correct calculates the number of tiles for n = 1 and n = 2.

Inductive hypothesis: Assume for all positive integers n ≤ k, where k is a positive integer 2 or greater that a 2 x n hallway can be tiled in $\frac{3}{4}(3^n) + \frac{1}{4}(-1)^n$ ways adhering to the given restriction with 1 x 1 and 1 x 2 tiles.

Inductive step: Prove for n = k+1 that a 2 x (k+1) hallway can be tiled in $\frac{3}{4}(3^{k+1}) + \frac{1}{4}(-1)^{k+1}$ ways adhering to the given restriction with 1 x 1 and 1 x 2 tiles.

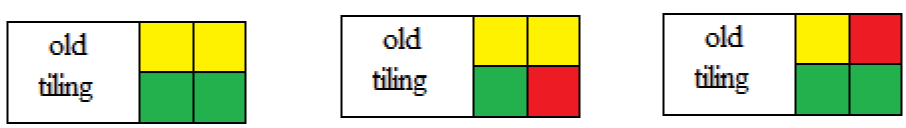
Break down the tilings into two groups:

(a) Those with NO horizontal 1 x 2 tiles in the last column
(b) Those with at least one horizontal 1 x 2 tile in the last column.

For group (a), we can build off all possible tilings of 2 x k sized hallways and tack onto those tilings either of these columns as the last column:

This means, using the inductive hypothesis that these tilings contribute to $2(\frac{3}{4}(3^{k}) + \frac{1}{4}(-1)^{k})$ tilings of the 2 x (k+1) hallways, since we've constructed two tilings for each tiling of a 2 x k hallway.

For group (b), we can place either one or two horizontal tiles in the last two columns, and **due to the given restriction, are guaranteed that if we only use one horizontal tile, the other two squares MUST HAVE single tiles**. Thus, we can create 3 possible tilings out of every old tiling of 2 x (k-1) hallways as follows:



This means, using the inductive hypothesis that these tilings contribute to $3(\frac{3}{4}(3^{k-1}) + \frac{1}{4}(-1)^{k-1})$ more valid tilings of the 2 x (k+1) hallway, since we've constructed three tilings for each tiling of a 2 x (k-1) hallway. All of these tilings are distinct because they differ in the placement of at least one 1 x 2 tile. They are complete, because of the restriction; no other tilings are allowed.

Thus, the total number of tilings of the 2 x (k+1) hallway is

$$2(\frac{3}{4}(3^{k}) + \frac{1}{4}(-1)^{k}) + 3(\frac{3}{4}(3^{k-1}) + \frac{1}{4}(-1)^{k-1}) =$$

$$\frac{3^{k+1}}{2} + \frac{(-1)^{k}}{2} + \frac{3^{k+1}}{4} - \frac{3(-1)^{k}}{4} =$$

$$\frac{3}{4} \times 3^{k+1} - \frac{1}{4} \times (-1)^{k} =$$

$\frac{3}{4} \times 3^{k+1} + \frac{1}{4} \times (-1)^{k+1}$, as desired.

6) (12 pts) Permutations - please leave your answers in factorials, combinations, powers, etc.

(a) How many permutations are there of the letters in the word CHATTANOOGA?

Using the permutation formula, we get $\frac{11!}{3!2!2!}$, permutations of 1 C, 1 H, 1 N, 1 G, 2 Ns, 2 Os and 3 As.

(b) How many of the permutations of the letters in CHATTANOOGA do NOT have any consecutive vowels?

Use the six consonants as separators: __ C __ H __ T __ T __ N __ G __ .

The 5 vowels can be placed in the 7 slots in $\binom{7}{5}$ ways. Furthermore, those vowels can be permuted in $\frac{5!}{3!2!}$ ways. Finally, the consonants can be permuted in $\frac{6!}{2!}$ ways. Since each of these choices are independent of one another and each of vowel slots can be paired with each ordering of vowels and each ordering of consonants, it follows that the total number of permutations with no consecutive vowels is $\binom{7}{5} \times \frac{5!}{3!2!} \times \frac{6!}{2!} = 21 \times 10 \times 360 = 75{,}600$ ways.

(c) How many of the permutations of the letters in CHATTANOOGA contain the substring "HAT"?

Create the super-letter "HAT". The remaining letters are C, T, A, N, O, O, G and A. There are a total of 9 letters, with 2 As and 2 Os. It follows that the total number of permutations with the substring HAT is $\frac{9!}{2!2!}$.

7) (12 pts) How many solutions are there to the equation

$$a + b + c + d + e + f = 40$$

where a, b, c, d, e and f are non-negative integers that adhere to the following restrictions:

$$a \geq 3, \quad b \geq 4, \quad c \leq 12 \text{ and } e \geq 5?$$

Let a = 3+a'. b=4+b' and e=5+e', which means that the restrictions on a, b and e are that they are non-negative. Substitute into the equation:

$(3+a') + (4 + b') + c + d + (5 + e') + f = 40$

$a' + b' + c + d + e' + f = 28$

We want the number of non-negative integer solutions to this equation with $c \leq 12$. To obtain this, simply count all the non-negative integer solutions and subtract out the solutions where $c > 12$.

There are $\binom{28+6-1}{6-1} = \binom{33}{5}$ total solutions, using the combinations with repetition formula woth n = 28 and r = 6.

To find the ones where $c > 12$, let $c = 13 + c'$ and substitute:

$a' + b' + c' + 13 + d + e' + f = 28$

$a' + b' + c' + d + e' + f = 15$

The restriction here is simply that all variables are non-negative integers.

Again, using the combinations with repetition formula, we get a total of $\binom{15+6-1}{6-1} = \binom{20}{5}$ solutions where $c > 12$.

It follows that the desired number of solutions to the given equation is

$$\binom{\mathbf{33}}{\mathbf{5}} - \binom{\mathbf{20}}{\mathbf{5}}$$

8) (12 pts) Integers x and y with $x > y > 0$ satisfy $x + y + xy = (2^6)(3^7) - 1$. How many different ordered pairs (x, y) satisfy all of these requirements?

Add 1 to both sides of the equation to yield:

$1 + x + y + xy = 2^6 3^7 - 1 + 1$
$(x + 1)(y + 1) = 2^6 3^7$

The value on the right hand side has 7 x 8 = 56 divisors.
Thus, there are 28 pairs of divisors that multiply to 56. One of these pairs is $x + 1 = 2^6 3^7$ and $y + 1 = 1$, but this pair doesn't count because it sets $y = 0$ and the restriction disallows that setting. It follows that there are **27 ordered pairs** (x, y) that satisfy the equation. (Note: it's guaranteed, that because $2^6 3^7$, that one divisor in the pair will be less than the square root of this value and one greater than it, and since $x > y$, for each pair we can determine which value is x and which is y.)

9) (4 pts) On what planet is the Ingenuity Mars helicopter currently located?

Mars