<u>Spring 2021 Honors COT3100 Exam #1 Solutions</u>

Date: 2/8/2021

1) (10 pts) Complete the following truth table. Please write T for true and F for false.

<u>Solution</u>

p	q	r	$\overline{p \vee r}$	$\bar{q} \wedge r$	$\overline{p \vee r} \vee (\bar{q} \wedge r)$
F	F	F	T	F	T
F	F	T	F	T	T
F	T	F	T	F	T
F	T	T	F	F	F
T	F	F	F	F	F
T	F	T	F	T	T
T	T	F	F	F	F
T	T	T	F	F	F

Grading: 1 pt per row, 2 pt bonus for getting everything correct.

2) (15 pts) Using the laws of logic, show that the following logical expression is a tautology:

$$((q \to (p \wedge r)) \wedge \bar{p}) \to \bar{q}$$

<u>Solution</u>

$\big((q \to (p \wedge r)) \wedge \bar{p}\big) \to \bar{q}$	$\leftrightarrow$	
$\big((\bar{q} \vee (p \wedge r)) \wedge \bar{p}\big) \to \bar{q}$	$\leftrightarrow$	Implication Identity
$\big(\bar{p} \wedge (\bar{q} \vee (p \wedge r))\big) \to \bar{q}$	$\leftrightarrow$	Commutative Law (not necessary)
$\big((\bar{p} \wedge \bar{q}) \vee (\bar{p} \wedge (p \wedge r))\big) \to \bar{q}$	$\leftrightarrow$	Distributive Law
$\big((\bar{p} \wedge \bar{q}) \vee ((\bar{p} \wedge p) \wedge r)\big) \to \bar{q}$	$\leftrightarrow$	Associative Law
$\big((\bar{p} \wedge \bar{q}) \vee (F \wedge r)\big) \to \bar{q}$	$\leftrightarrow$	Inverse Law
$\big((\bar{p} \wedge \bar{q}) \vee (r \wedge F)\big) \to \bar{q}$	$\leftrightarrow$	Commutative Law (not necessary)
$((\bar{p} \wedge \bar{q}) \vee F) \to \bar{q}$	$\leftrightarrow$	Domination Law
$((\bar{p} \wedge \bar{q})) \to \bar{q}$	$\leftrightarrow$	Identity Law
$\overline{\bar{p} \wedge \bar{q}} \vee \bar{q}$	$\leftrightarrow$	Implication Identity
$(\bar{\bar{p}} \vee \bar{\bar{q}}) \vee \bar{q}$	$\leftrightarrow$	DeMorgan's Law
$(p \vee q) \vee \bar{q}$	$\leftrightarrow$	Double Negation
$p \vee (q \vee \bar{q})$	$\leftrightarrow$	Associative Law
$p \vee T$	$\leftrightarrow$	Inverse Law
T		Domination Law

Grading: If close, take off 1 pt per error, if not, award 1 pt per valid step. If all reasons are missing -5, If a reason is incorrect -1 for first, then -1 for the next two, etc.

3) (10 pts) Use the rules of implication to make the following argument:

$$p \to q$$
$$r \to s$$
$$\bar{q} \vee \bar{s}$$
$$t \to p$$
$$u \to r$$
$$u$$

$$\therefore \bar{t}$$

Solution

Step	**Reason**
1. $p \to q$	Premise
2. $r \to s$	Premise
3. $\bar{q} \vee \bar{s}$	Premise
4. $\bar{p} \vee \bar{r}$	Rule of Destructive Dilemma with 1,2,3
5. u	Premise
6. $u \to r$	Premise
7. r	Modus Ponens with 5, 6
8. $\bar{p}$	Disjunctive Syllogism with 4, 7
9. $t \to p$	Premise
10. $\bar{t}$	Modus Tollens with 8, 9

Grading: If close, take off 1 pt per error, if not, award 1 pt per valid step. If all reasons are missing -3, If a reason is incorrect -1 for first, then -1 for the next 3 or so.

4) (10 pts) Prove or disprove the following assertion for all sets A, B, C and D:

$$(A \cup B) \times (C \cap D) \subseteq (A \times D) \cup (B \times C)$$

Solution

The claim is true.

We prove one set is a subset of another by taking an arbitrary element of the first set and proving it belongs to the second set.

Let (x, y) be an arbitrary element of $(A \cup B) \times (C \cap D)$.

It follows from the definition of Cartesian Product that $x \in (A \cup B)$ and $y \in (C \cap D)$.

It follows from the definition of intersection that $y \in C$ and $y \in D$.

Then, there are two cases we must split our work into, based on the definition of union:

(1) $x \in A$
(2) $x \in B$

Case 1

In this case, since we know that $x \in A$ and $y \in D$, by definition of Cartesian Product, $(x, y) \in A \times D$. Then, by definition of union, we ascertain that $(x, y) \in (A \times D) \cup (B \times C)$, as desired.

Case 2

In this case, since we know that $x \in B$ and $y \in C$, by definition of Cartesian Product, $(x, y) \in B \times C$. Then, by definition of union, we ascertain that $(x, y) \in (A \times D) \cup (B \times C)$, as desired.

Since we've proven the claim in both possible cases, the proof is complete and we can ascertain in all cases that that $(x, y) \in (A \times D) \cup (B \times C)$, as desired, and the original statement is proven true.

Grading: 2 pts taking an arbitrary ordered pair in the first set.
1 pt breaking down requirements on x
1 pt breaking down requirements on y
2 pts splitting into two cases
2 pts for proof when x is in A
2 pts for proof when x is in B

5) (10 pts) Prove or disprove the following assertion for all sets A and B:

$$if A - B \subseteq C - D, then\ A \subseteq (C \cup D)$$

Solution

This is false.

Consider the following counter-example:

A = {1}
B = {1}
C = {}
D = {}

In this example A - B = C - D = {}, so the if clause holds true, since the empty set is a subset of itself. But, in this example, $C \cup D$={} and A = {1}, thus, A is NOT a subset of $C \cup D$, since the latter set does not contain 1, which is an element of A.

Grading: 0/10 for any proof

3 pts for saying it's false.
3 pts for specifying any counter-example
2 pts for any valid counter-example
2 pts for a brief explanation as to why it's a counter-example

6) (10 pts) In the string orchestra, each student plays at least one of the following three instruments: violin, viola, and cello. The number of students who play the violin, plus the number of students who play the viola, plus the number of students who play the cello is 45. (Note that a student who plays two of the three instruments is counted twice in this count and a student who plays all 3 is counted 3 times in this count.) The number of students who play either the violin or viola or both is 17, and the number of students who play all three instruments is 5. How many students are in either (or both) of the following groups?

(a) students who play both the violin and viola
(b) students who play the cello

Solution

Let A be the set of students who play the violin.
Let B be the set of students who play the viola.
Let C be the set of students who play the cello.

Using the given information, we have:

$|A| + |B| + |C| = 45$
$|A \cup B| = 17$
$|A \cap B \cap C| = 5$

The desired quantity is:

$$\begin{aligned}|(A \cap B) \cup C| &= |A \cap B| + |C| - |(A \cap B) \cap C| \\ &= |A| + |B| - |A \cup B| + |C| - |A \cap B \cap C| \\ &= |A| + |B| + |C| - |A \cup B| - |A \cap B \cap C| \\ &= 45 - 17 - 5 \\ &= 23\end{aligned}$$

The first step applies the Inclusion-Exclusion Principle to the two sets in question.
The second set applies it to the sets A and B, but solving for the intersection cardinality.
Then, we regroup and substitute the given values.

Grading: 2 pts for assigning sets to the items in the problem
3 pts (1 pt each) for writing down the given information
4 pts for the work with the I/E, getting down to the third line
1 pt for substituting the given values and simplifying to 23.

7) (10 pts) Let r and s be the roots of the equation $3x^2 - 12x + 5 = 0$. What is the quadratic equation with leading coefficient of 1 with the roots r^3 and s^3? **(Note: Of the two coefficients not equal to 1, one will be a non-integer fraction and the other one will be an integer.)**

Solution

r + s = -(-12)/3 = 4.
rs = 5/3

Now, we must find the following two quantities:

$r^3 + s^3$ and r^3s^3.

To find the first:

$$4^3 = 64 = (r+s)^3 = r^3 + 3r^2s + 3rs^2 + s^3 = (r^3 + s^3) + 3rs(r+s)$$

Substituting, for r+s and rs, we have:

$$64 = (r^3 + s^3) + 3(\frac{5}{3})(4)$$
$$64 = (r^3 + s^3) + 20$$
$$(r^3 + s^3) = 44$$

To find the second, we have:

$$r^3s^3 = (rs)^3 = (\frac{5}{3})^3 = \frac{125}{27}$$

It follows that the desired quadratic equation is $x^2 - 44x + \frac{125}{27} = 0$.

Grading: 2 pts for writing down equations for r and s.
2 pts for identifying what to solve for
4 pts for solving for the sum
1 pt for solving for the product
1 pt for writing the quadratic down

8) (10 pts) A river runs East-West with the current flowing west. Kayla starts canoeing from her house, going east, against the current. After 80 minutes, Kayla reached a nice waterfall and decided to turn around and starts canoeing west. She passes her house and eventually stops after another 160 minutes. It turns out that when she finished the trip, she has twice as far from her house as she was when she was at the waterfall. If the entire trip was 16 miles long, what is her rate (in miles per hour) of rowing in still water, and what is the rate of the river current (in miles per hour)?

Solution

Let D be the distance from Kayla's house to the waterfall in miles. Then, her return trip was the length 3D because she was 2D from her house and had to travel back to her house (so D + 2D = 3D) and the whole trip was 4D miles. Since 4D = 16, D = 4 miles.

Thus, in 4/3 hours (this is 80 minutes converted to hours), she traveled 4 miles, meaning that her average speed going to the waterfall was 3 miles per hour.

In 8/3 hours (this is 160 minutes converted to hours), she traveled 12 miles, meaning that her average speed after turning around was 9/2 miles per hour.

Let r1 = her rate of rowing in still water
Let r2 = the rate of the current of the river

Using the given information, we have:

r1 - r2 = 3
r1 + r2 = 9/2

Adding the two equations, we find:

2r1 = 15/2, so r1 = **15/4 miles per hour (The rate Kayla rows at)**

Subtracting the two equations (subtract top from bottom) we find:

2r2 = 3/2, so r2 = **3/4 miles per hour (The rate of the current)**

Grading: 3 points for figuring out that the distance from her house to the waterfall is 4 miles
2 pts for figuring out her average speed to the waterfall
2 pts for figuring out her average speed after the waterfall
3 pts to set up the system and solve it

9) (10 pts) Let x, y and z satisfy the following equations:

$$4^x = 8,\ 8^y = 32,\ y - x = log_{64}z$$

What are the values of x, y and z? Express all answers as fractions in lowest terms or integers.

Solution

By definition of log and changing bases, we have $x = log_4 8 = \frac{log_2 8}{log_2 4} = \frac{3}{2}$.

By definition of log and changing bases, we have $y = log_8 32 = \frac{log_2 32}{log_2 8} = \frac{5}{3}$.

Then, substitute:

$$\frac{5}{3} - \frac{3}{2} = log_{64}z$$

$$\frac{10-9}{6} = log_{64}z$$

$$\frac{1}{6} = log_{64}z$$

By definition of log we have $z = 64^{\frac{1}{6}} = (2^6)^{\frac{1}{6}} = 2^1 = 2$

Grading: 3 pts for x, 3 pts for y, 4 pts for z.

10) (5 pts) What color does the commonly used food dye Yellow 5 make food?

YELLOW (Give to all)