

COP 3502 Recitation Sheet: Recurrence Relations Solutions

1) (Spring 2020) Use the iteration technique to solve the following recurrence relation in terms of n :

$$T(n) = 2T(n/2) + 1, \text{ for all integers } n > 1$$
$$T(1) = 1$$

Find a tight Big-Oh answer.

Solution

$$T(n) = 2T\left(\frac{n}{2}\right) + 1$$
$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + 1$$
$$T(n) = 2\left(2T\left(\frac{n}{4}\right) + 1\right) + 1$$
$$T(n) = 4T\left(\frac{n}{4}\right) + 2 + 1$$
$$T(n) = 4T\left(\frac{n}{4}\right) + 3$$
$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + 1$$
$$T(n) = 4\left(2T\left(\frac{n}{8}\right) + 1\right) + 3$$
$$T(n) = 8T\left(\frac{n}{8}\right) + 4 + 3$$
$$T(n) = 8T\left(\frac{n}{8}\right) + 7$$

Based on these three iterations, we see that after k iterations, the recurrence is

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + (2^k - 1)$$

Plug in the value of k such that $\frac{n}{2^k} = 1$ to this recurrence. This means that $2^k = n$. Substituting, we get:

$$T(n) = nT(1) + (n - 1)$$
$$T(n) = n + (n - 1)$$
$$T(n) = 2n - 1$$

It follows that $T(n) = O(n)$.

2) What is the closed form solution to the following recurrence relation? Please use the iteration technique, show all of your work and provide your final answer in Big-Oh notation.

$$T(1) = 1$$

$$T(n) = 2T(n/4) + 1$$

Solution

Iterate the recurrence three times:

$$T(n) = 2T\left(\frac{n}{4}\right) + 1 \quad (\text{one iteration})$$

$$T(n) = 2(2T\left(\frac{n}{16}\right) + 1) + 1$$

$$T(n) = 4T\left(\frac{n}{16}\right) + 3 \quad (\text{two iterations})$$

$$T(n) = 4(2T\left(\frac{n}{64}\right) + 1) + 3$$

$$T(n) = 8T\left(\frac{n}{64}\right) + 7 \quad (\text{three iterations})$$

Now, let's make a guess as to the form of the recurrence after iterating k times based on the first three iterations:

$$T(n) = 2^k T\left(\frac{n}{4^k}\right) + (2^k - 1)$$

Since we know $T(1)$, we want to plug in the value of k for which $\frac{n}{4^k} = 1$, in for k . Solving, we find that $n = 4^k$. Taking the square root of both sides, we find $\sqrt{n} = \sqrt{4^k} = \sqrt{2^{2k}} = (2^{2k})^{\frac{1}{2}} = 2^k$. Substituting for both 4^k and 2^k , in the right hand of the recurrence, we get:

$$T(n) = \sqrt{n}T\left(\frac{4^k}{4^k}\right) + (\sqrt{n} - 1) = \sqrt{n}T(1) + (\sqrt{n} - 1) = \sqrt{n} + \sqrt{n} - 1 \in \mathbf{O}(\sqrt{n})$$

3) Use the iteration technique to determine a close form solution for the recurrence relation $T(n)$ defined below. Note: due to the nature of this recurrence, it's possible to get an exact solution for $T(n)$, so please try to do that instead of just getting a Big-Oh bound.

$$T(n) = 2T(n - 1) + 2^n$$
$$T(1) = 2$$

Solution

Iterate as follows:

$$\begin{aligned}T(n) &= 2T(n - 1) + 2^n \\&= 2(2T(n - 2) + 2^{n-1}) + 2^n \\&= 4T(n - 2) + 2^n + 2^n \\&= 4T(n - 2) + 2(2^n) \\&= 4(2T(n - 3) + 2^{n-2}) + 2(2^n) \\&= 8T(n - 3) + 2^n + 2(2^n) \\&= 8T(n - 3) + 3(2^n)\end{aligned}$$

In general, after k iterations, we'll have:

$$= 2^k T(n - k) + k(2^n)$$

Plug in $k = n-1$ to obtain

$$\begin{aligned}T(n) &= 2^{n-1} T(n - (n - 1)) + (n - 1)(2^n) \\&= 2^{n-1} T(1) + (n - 1)(2^n) \\&= 2^{n-1}(2) + (n - 1)(2^n) \\&= 2^n + (n - 1)(2^n) \\&= \mathbf{n2^n}\end{aligned}$$

4) Using the iteration technique, find a Big-Oh bound for the following recurrence relation, in terms of n :

$$T(n) = 2T\left(\frac{n}{2}\right) + n^2, \text{ for } n > 1$$

$$T(1) = 1$$

Solution

$$T(n) = 2T\left(\frac{n}{2}\right) + n^2$$

$$T(n) = 2\left[2T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right] + n^2$$

$$T(n) = 4T\left(\frac{n}{4}\right) + 2 \times \frac{n^2}{4} + n^2$$

$$T(n) = 4T\left(\frac{n}{4}\right) + \frac{3n^2}{2}$$

$$T(n) = 4\left[2T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2\right] + \frac{3n^2}{2}$$

$$T(n) = 8T\left(\frac{n}{8}\right) + 4 \times \frac{n^2}{16} + \frac{3n^2}{2}$$

$$T(n) = 8T\left(\frac{n}{8}\right) + \frac{7n^2}{4}$$

In general, after k steps, we see that our formula will iterate to:

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + \frac{(2^k - 1)n^2}{2^{k-1}}$$

Plug in $\frac{n}{2^k} = 1$, so let $n = 2^k$ and $k = \log_2 n$ to obtain:

$$T(n) = nT(1) + \frac{(n-1)n^2}{n/2}$$

$$T(n) = n + 2n(n-1)$$

$$T(n) = 2n^2 - n = \mathbf{O(n^2)}$$

5) Use the iteration technique to determine a Big-Oh solution for the following recurrence relation:

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2, T(1) = 1$$

Solution

$$\begin{aligned}T(n) &= 4T\left(\frac{n}{2}\right) + n^2 \\T(n) &= 4\left(4T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right) + n^2 \\T(n) &= 16T\left(\frac{n}{4}\right) + 4\left(\frac{n^2}{4}\right) + n^2 \\T(n) &= 16T\left(\frac{n}{4}\right) + 2n^2 \\T(n) &= 16\left(4T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2\right) + 2n^2 \\T(n) &= 64T\left(\frac{n}{8}\right) + 16\left(\frac{n^2}{16}\right) + 2n^2 \\T(n) &= 64T\left(\frac{n}{8}\right) + 3n^2\end{aligned}$$

After k iterations, we guess the form of our recurrence to be:

$$T(n) = 4^k T\left(\frac{n}{2^k}\right) + kn^2$$

We plug in a value of k such that $\frac{n}{2^k} = 1$. Namely, $n = 2^k$ and $k = \log_2 n$. Note that $4^k = (2^2)^k = (2^k)^2 = n^2$:

$$T(n) = n^2 T(1) + (\log_2 n)n^2$$

Plugging in $T(1) = 1$, we have

$$T(n) = n^2 + (\log_2 n)n^2 = \mathbf{O(n^2 \lg n)}$$

6) (Spring 2026) Determine a closed form solution to the following recurrence relation, in terms of n . (Your solution must be an exact function in terms of n , not a Big-Oh bound.)

$$T(n) = 3T(n - 1) + 3^n, \text{ for integers } n > 1$$
$$T(1) = 12$$

Use the iteration technique:

$$\begin{aligned} T(n) &= 3T(n - 1) + 3^n \\ &= 3(3T(n - 2) + 3^{n-1}) + 3^n \\ &= 9T(n - 2) + 3^n + 3^n \\ &= 9T(n - 2) + 2(3^n) \\ &= 9(3T(n - 3) + 3^{n-2}) + 2(3^n) \\ &= 27T(n - 3) + 3^n + 2(3^n) \\ &= 27T(n - 3) + 3(3^n) \end{aligned}$$

In general, after k iterations, we have:

$$T(n) = 3^k T(n - k) + k(3^n)$$

Since $T(1)$ is known, substitute $k = n - 1$ in the general form above to yield:

$$\begin{aligned} T(n) &= 3^{n-1} T(n - (n - 1)) + (n - 1)(3^n) \\ T(n) &= 3^{n-1} T(1) + (n - 1)(3^n) \\ T(n) &= 12(3^{n-1}) + (n - 1)(3^n) \\ T(n) &= 4(3^n) + (n - 1)(3^n) \\ T(n) &= (n - 1 + 4)(3^n) \\ \mathbf{T(n)} &= \mathbf{(n + 3)(3^n)} \end{aligned}$$

7) (Spring 2025) Use the iteration technique to find **an exact closed-form** solution to the recurrence relation defined below for all positive integers n :

$$T(1) = 1$$

$$T(n) = 2T(n - 1) + 5, \text{ for all integers } n \geq 2$$

Please explicitly show the work for the first three iterations before attempting to find the form for an arbitrary iteration, followed by arriving at the closed form. Hint: Your answer should be of the form $T(n) = a(b^n) + c$, where a , b , and c are all integers.

Here are the first three iterations:

$T(n) = 2T(n - 1) + 5$	// Iteration #1	Grading: 1 pt
$T(n) = 2(2T(n - 2) + 5) + 5$		
$T(n) = 4T(n - 2) + (10 + 5)$		
$T(n) = 4T(n - 2) + 15$	// Iteration #2	Grading: 1 pt
$T(n) = 4(2T(n - 3) + 5) + 15$		
$T(n) = 8T(n - 3) + (20 + 15)$		
$T(n) = 8T(n - 3) + 35$	// Iteration #3	Grading: 2 pts

After k iterations, we have:

$T(n) = 2^k T(n - k) + 5(2^k - 1)$	Grading: 2 pts
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Since we know $T(1)$, plug in $k = n - 1$ into this formula:

$T(n) = 2^{n-1} T(n - (n - 1)) + 5(2^{(n-1)} - 1)$	
$= 2^{n-1} T(1) + 5(2^{n-1}) - 5$	Grading: 1 pt
$= 2^{n-1} + 5(2^{n-1}) - 5$	
$= 6(2^{n-1}) - 5$	Grading: 1 pt
$= (3)(2)(2^{n-1}) - 5$	
$= 3(2^n) - 5$	Grading: 1 pt

Note: 1 pt is allocated to factor out the 2 from the 6 and include it in the exponent.

8) (Summer 2024) Using the iteration technique, determine the Big-Oh solution to the recurrence relation below, in terms of n .

$$T(n) = 2T\left(\frac{n}{2}\right) + n^3, \text{ for } n > 1$$

$$T(1) = 1$$

Let's work out three iterations:

$$T(n) = 2T\left(\frac{n}{2}\right) + n^3 \quad (\text{Iteration 1})$$

$$T(n) = 2\left(2T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^3\right) + n^3$$

$$T(n) = 4T\left(\frac{n}{4}\right) + n^3 + \frac{n^3}{4} \quad (\text{Iteration 2 – some students might take more steps to get here})$$

$$T(n) = 4\left(2T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^3\right) + n^3 + \frac{n^3}{4}$$

$$T(n) = 8T\left(\frac{n}{8}\right) + \left(n^3 + \frac{n^3}{4} + \frac{n^3}{16}\right) \quad (\text{Iteration 3})$$

After k iterations, we have the following:

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1} \frac{n^3}{4^i}$$

Plug in $\frac{n}{2^k} = 1$, $n = 2^k$ and substitute:

$$T(n) \leq nT(1) + n^3 \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i = n + n^3 \left(\frac{1}{1 - \frac{1}{4}}\right) = n + \frac{4}{3}n^3 = O(n^3)$$

Grading:

- 1 pt first iteration**
- 2 pts second iteration**
- 2 pts third iteration**
- 2 pts guess**
- 1 pt substitute $n = 2^k$**
- 2 pts work to final answer**

9) (Summer 2023) Use the iteration technique to determine an **exact closed-form solution** for the recurrence relation, $T(N)$, described below. (**Note: Be very careful with what occurs towards the end of the iteration, in the general case.**)

$$T(N) = (N + 1)T(N - 1) \text{ (for } N > 1)$$

$$T(1) = 1$$

$$T(N) = (N + 1)T(N - 1)$$

Grading: 1pt

$$T(N - 1) = (N - 1 + 1)T(N - 1 - 1)$$

$$T(N - 1) = NT(N - 2)$$

$$T(N) = (N + 1)(N)T(N - 2)$$

Grading: 2pts

$$T(N - 2) = (N - 2 + 1)T(N - 2 - 1)$$

$$T(N - 2) = (N - 1)T(N - 3)$$

$$T(N) = (N + 1)(N)(N - 1)T(N - 3)$$

Grading: 2pts

General Form after k iterations

$$T(N) = T(N - k) \prod_{i=1}^k (N + 2 - i)$$

Grading: 2pts

The recursion stops when $N - k = 1$; $k = N - 1$

Plugging in N for k we get

$$T(N) = T(1) \prod_{i=1}^{N-1} (N + 2 - i)$$

Grading: 1pt

$$T(N) = 1 (N + 1)(N)(N - 1) \dots (3)$$

$$T(N) = (N + 1)!/2$$

Grading: 2 pts

10) (Summer 2022) Using the iteration technique, determine a closed-form solution for the following recurrence relation in terms of n . Note: Your answer should be **EXACT** and not a Big-Oh bound.

$$T(0) = 1$$

$$T(n) = 4T(n - 1) + 2^n, \text{ for integers } n > 0$$

Using the iteration technique for three iterations, we get:

$$T(n) = 4T(n - 1) + 2^n \quad \text{Grading: 1 pt}$$

$$T(n) = 4(4T(n - 2) + 2^{n-1}) + 2^n$$

$$T(n) = 16T(n - 2) + (2^2)2^{n-1} + 2^n$$

$$T(n) = 16T(n - 2) + [2^{n+1} + 2^n] \quad \text{Grading: 2 pts}$$

$$T(n) = 16(4T(n - 3) + 2^{n-2}) + [2^{n+1} + 2^n]$$

$$T(n) = 64T(n - 3) + (2^4)2^{n-2} + [2^{n+1} + 2^n]$$

$$T(n) = 64T(n - 3) + [2^{n+2} + 2^{n+1} + 2^n] \quad \text{Grading: 2 pts}$$

From here we see that after k iterations, the recurrence is:

$$T(n) = 4^k T(n - k) + \sum_{i=0}^{k-1} 2^{n+i} \quad \text{Grading: 2 pts}$$

Since we know $T(0)$, let's plug in $k = n$ into the formula above to yield:

$$T(n) = 4^n T(n - n) + \sum_{i=0}^{n-1} 2^{n+i} \quad \text{Grading: 1 pt}$$

$$T(n) = 4^n T(0) + \sum_{i=0}^{n-1} 2^n 2^i$$

$$T(n) = 2^{2n} + 2^n \sum_{i=0}^{n-1} 2^i$$

$$T(n) = 2^n (2^n) + 2^n \sum_{i=0}^{n-1} 2^i$$

$$T(n) = 2^n [2^n + \sum_{i=0}^{n-1} 2^i]$$

$$T(n) = 2^n [\sum_{i=0}^n 2^i]$$

$$T(n) = 2^n \left(\frac{2^{n+1} - 1}{2 - 1} \right)$$

$$T(n) = 2^n (2^{n+1} - 1) \quad \text{Grading: 2 pts}$$