Computer Science Foundation Exam

August 8, 2014

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used,
and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	10	PRF (Logic)	6	
3	10	PRF (Sets)	6	
4	15	NTH (Number Theory)	10	
ALL	50		32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) PRF (Induction)

Use mathematical induction on n to prove that $5^n + 2(11^n)$ is divisible by 3 for all non-negative integers n.

Base case: For n = 0, $5^n + 2(11^n) = 5^0 + 2(11^0) = 1 + 2 = 3$. Since 3 | 3, the expression is divisible by 3 for n = 0 and the base case is true. (2 pts)

Inductive hypothesis: Assume for an arbitrary non-negative integer n = k that $3 \mid (5^k + 2(11^k))$. Thus, there exists an integer c such that $5^k + 2(11^k) = 3c$. (2 pts)

Inductive step: Prove for n = k + 1 that $3 \mid (5^{k+1} + 2(11^{k+1}))$. (2 pts)

$5^{k+1} + 2(11^{k+1}) = 5(5^k) + 11(2(11^k))$ (2 pts)
$= 5(5^k) + (5 + 6)(2(11^k))$ (2 pts)
$= 5(5^k) + 5(2(11^k)) + 6(2(11^k))$
$= 5[5^k + 2(11^k)] + 6(2(11^k))$ (1 pt)
$= 5(3c) + 6(2(11^k))$, using the inductive hypothesis (2 pts)
$= 3[5c + 4(11^k)]$, proving that the given expression is divisible by 3. (2 pts)

2) (10 pts) PRF (Logic)

Use the laws of logic to show that the two following logical expressions below are equivalent.

(1) $(p \wedge (q \vee \bar{q})) \wedge (\overline{p \wedge q} \vee p)$
(2) p

Label the use of each law. Note that $\bar{x}$ means not(x).

$$(p \wedge (q \vee \bar{q})) \wedge (\overline{p \wedge q} \vee p) \leftrightarrow$$
$$(p \wedge T) \wedge (\overline{p \wedge q} \vee p) \leftrightarrow, Inverse\ Law$$
$$p \wedge (\overline{p \wedge q} \vee p) \leftrightarrow, Identity\ Law$$
$$p \wedge \left((\bar{p} \vee \bar{q}) \vee p\right) \leftrightarrow, DeMorgan's\ Law$$
$$p \wedge \left((\bar{p} \vee p) \vee q\right) \leftrightarrow, Associative\ \&\ Commutative\ Laws$$
$$p \wedge (T \vee q) \leftrightarrow, Inverse\ Law$$
$$p \wedge T \leftrightarrow, Domination\ Law$$
$$p, Identity\ Law$$

Grading: 1 pt off per incorrect step, 3 pts total for proper reasons. Deduct whole points only as you see fit.

3) (10 pts) PRF (Sets)

(a) (5 pts) The following statement is false:

Let A, B and C be finite sets of integers. If $A \subset B$, then $A - C \subset B - C$.

(Note: ⊂ denotes proper subset.)

Specify a counter-example with a list of the values of A, B and C that disprove this statement.

A = {1}, B = {1, 2} and C = {1, 2}. For this example, we have A – C = B – C = ∅, since C contains all elements in A and B. It follows that the conclusion is false, since B – C does not contain at least one element that is NOT in A – C, but the premise is true, since B contains all items of A, and it contains the element 2, which is not in A.

Grading: All or nothing – 5 pts if the counter-example is clearly specified and valid, 0 pts otherwise. No explanation is necessary.

(b) (5 pts) If we change the question to read: If $A \subset B$ and $C \subseteq A$, then $A - C \subset B - C$, the assertion is true. Explain intuitively why adding this restriction makes the statement true.

Adding this restriction guarantees that any element that is in B but not A (there must be at lest one of these, call one of them x), is NOT in C. Thus, B – C must contain this element (x) while A – C can not.

Grading: Use your judgment to decide how clear their explanation is. Make a sliding scale based on correctness and clarify and grade consistently. Assign an integer number of points out of 5. No formalism is needed to get full credit.

4) (15 pts) NTH (Number Theory)

Using the Extended Euclidean Algorithm, determine the unique integer x, such that $0 \leq x < 317$ and $233x \equiv 1 \bmod 317$.

317 = 1 x 233 + 84
233 = 2 x 84 + 65
84 = 1 x 65 + 19
65 = 3 x 19 + 8
19 = 2 x 8 + 3
8 = 2 x 3 + 2
3 = 1 x 2 + 1 (5 pts – 1 pt off per mistake)

3 – 1 x 2 = 1
3 – (8 – 2 x 3) = 1
3 x 3 – 1 x 8 = 1
3(19 – 2 x 8) – 1 x 8 = 1
3 x 19 – 7 x 8 = 1
3 x 19 – 7(65 – 3 x 19) = 1
24 x 19 – 7 x 65 = 1
24(84 – 65) – 7 x 65 = 1
24 x 84 – 31 x 65 = 1
24 x 84 – 31(233 – 2 x 84) = 1
86 x 84 – 31 x 233 = 1
86(317 – 233) – 31 x 233 = 1
86 x 317 – 117 x 233 = 1 (9 pts – 1 pt off per mistake)

Thus, $x \equiv -117 \bmod 317$. It follows that $x = -117 + 317 = 200$. (1 pt)

Note: In grading this, do cascading mistakes. Basically, if there is one mistake and other steps are wrong because of that mistake, but correct based on their mistaken step, don't count the latter line as a mistake.