Computer Science Foundation Exam

August 9, 2013

Section II B SOLUTION

DISCRETE STRUCTURES

NO books, notes, or calculators may be used,
and you must work entirely on your own.

Question	Max Pts	Category	Passing	Score
1	15	CTG (Counting)	10	
2	15	PRB (Probability)	10	
3	10	PRF (Functions)	6	
4	10	PRF (Relations)	6	
ALL	50		32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.

1) (15 pts) CTG (Counting)

Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for any of the questions.

(a) (3 pts) A typical RGB color value is stored using 24 bits. How many possible colors can be represented in this manner?

Each of the 24 bits can be one of two values, and each bit is independent of the rest, thus, there are 2^{24} possible colors that can be represented. (3 pts for the correct answer, 1pt for any incorrect answer that incorporates both 2 and 24, 0 otherwise.)

(b) (5 pts) You need to assign voter ID numbers within the range [3000000, 3999999] with the restriction that you can't use an even number and you can't use a number whose sum of digits is even. How many distinct ID numbers are you allowed to assign?

The last digit must be odd. There are 5 choices for this digit. All of these numbers are of the form 3xxxxxo, where x is any digit and o is an odd digit. Of these values, we must only count the ones that have an even sum amongst all the digits labeled with an x. Assuming that there are 10 choices for each of the first four of these digits, then this sum must be some fixed parity, always making 5 possible choices to fill in the last x to obtain an even sum of all the digits labeled with an x. Using the multiplication principle, we get our final answer to be:

5 x 10 x 10 x 10 x 10 x 5 = 250,000 (3 pts for the answer, 2 pts for the explanation, give partial credit as you see fit.)

(c) (7 pts) A shop sells five brands of computers. If you want to buy exactly 12 computers, how many different combinations of computers can be bought, if we assume that computers of the same brand are identical?

This is a combination with repetition, since we are allowed to get more than one copy of each brand of computer. We are buying 12 computers, chosen from 5 separate brands. Using the formula derived from finding the number of non-negative integer solutions to

a + b + c + d + e = 12,

we get a total of $\binom{12+5-1}{5-1} = \binom{16}{4}$ different combinations. (Note: ${}_{16}C_{12}$ is another correct response.)

Grading: 4 points for the recognition of combinations with repetition, 3 points for correctly plugging into the formula.

2) (15 pts) PRB (Probability)

A factory manufactured 1000000 cell phones in 2012, of which 20,000 were defective. In the factory there are two assembly lines, A and B, responsible for manufacturing all of the phones. Assembly line A manufactured 200,000 phones total and assembly line B manufactured 12,000 defective cell phones. Determine the following probabilities:

(a) (4 pts) Given that a cell phone was manufactured in assembly line A, what is the probability that it is defective? Give your answer as a percentage.

We can infer that 20,000 – 12,000 = 8,000 defective phones were made in assembly line A. (2 pts) Thus, the desired probability is $\frac{8000}{200000} = \frac{4}{100} = 4\%$ (2 pts)

(b) (4 pts) Given that a cell phone was manufactured in assembly line B, what is the probability that it is defective? Give your answer as a percentage.

We can infer that 1,000,000 – 200,000 = 800,000 phones were made in assembly line B. (2 pts) Thus, the desired probability is $\frac{12000}{800000} = \frac{3}{200} = 1.5\%$ (2 pts)

A packet gets sent without corruption over a local area network 95% of the time.

(c) (7 pts) What is the probability that of 10 packets sent, exactly 8 are delivered uncorrupted? Please leave your answers in factorials, permutations, combinations and powers.

Since we are repeating a trial with a fixed probability of success multiple times, we have a binomial distribution. In this problem, the probability of success, p = .95, the number of trials n = 10 and the number of successes we want is k = 8. Our probability is

$$\binom{10}{8} .95^8 .05^2$$

Grading: 4 points for identifying that this is a binomial distribution, 2 points for plugging into the formula. (Note: $_{10}C_2$ can also be used for the combination.)

3) (10 pts) PRF (Functions)

(a) Consider the function f(x) = | 3x – 6 | – | 2x + 4 | within the domain x ∈ [-a, a], where a is a positive integer. What is the maximum of a such that f(x) is a injection. Prove your answer.

The critical points for evaluating functions with absolute values are when the value of the inside of the absolute value equals 0. Thus, for this function, the critical values are as follows:

3x – 6 = 0 **2x + 4 = 0**
x = 2 **x = -2** **(4 pts)**

Thus, we can see that f(x) will be a single continuous linear function over the interval [-2,2]. In particular, in this domain, f(x) = – (3x – 6) – (2x + 4) = 2 – 5x. We can show that this function is a injection simply by finding the inverse function within this domain, $f^{-1}(x) = \frac{2-x}{5}$. (Note the range of this function would be [-8, 12].) (3 pts)

Now, to prove that a = 2 is the maximal answer, we will show that within the domain [-3, 3], the function is not an injection. Note that f(3) = | 3(3) – 6 | – | 2(3) + 4 | = 3 – 10 = -7. In addition, we have f(1.8) = | 3(1.8) – 6 | – | 2(1.8) + 4 | = .6 – 7.6 = -7. Since f(3) = f(1.8), the function is NOT an injection on the domain [-3, 3] and can't be an injection on any domain [-a, a] for positive integers a > 3, due to this same counter-example. (3 pts)

4) (10 pts) PRF (Relations)

Consider the following relation defined over the universe of UCF students, each of whom have taken at least one official UCF class:

R = { (x, y) | x and y have taken an official UCF class together. }

Determine, with proof, whether or not R is (i) reflexive, (ii) irreflexive, (iii) symmetric, (iv) anti-symmetric, and (v) transitive.

(i) Yes, for all students a, (a,a) $\in$ R, since all students who have been in at least one UCF class have taken that particular class with themselves.

(ii) No, since there exists at least one UCF student in our domain, there does exist an ordered pair of the form (a,a) that is part of the relation, proving it's not irreflexive.

(iii) Yes, for all students and b, if (a, b) $\in$ R, this means that students a and b have taken a class together. But, this implies that students b and a have also taken a class together, thus, (b, a) $\in$ R, proving the relation to be symmetric.

(iv) No, since there are pairs of distinct students who have taken a UCF class together. (We can find such a pair in any class with more than one student.) In this situation, for distinct students a and b, we have both (a, b) $\in$ R and (b, a) $\in$ R, proving the relation to not be anti-symmetric.

(v) No, since there are situations where Adam has taken a class with Bob, and Bob has taken a class with Mary, but Adam has never taken a class with Mary. Thus, it's entirely possible (and probable), that for some group of three students, a, b and c, we have (a, b) $\in$ R, (b, c) $\in$ R and (a, c) $\notin$ R. (As an example, my sister and I took a class together in college, but I never took a class with her roommate Marisola, with whom my sister took physics.)

Grading: 1 pt for each answer, 1 pt for each justification