Computer Science Foundation Exam

August 9, 2013

Section II A

DISCRETE STRUCTURES

SOLUTION

NO books, notes, or calculators may be used, and you must work entirely on your own.

Question	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	10	PRF (Logic)	6	
3	15	PRF (Sets)	10	
4	10	NTH (Number Theory)	6	
ALL	50	---	32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) PRF (Induction)

Use mathematical induction to prove the following statement is true for integers $n \geq 0$:

$$3 \mid n^3 - n.$$

<u>Solution</u>

Base Case: $n = 0$.
$n^3 - n = (0)^3 - 0 = 0$
$3 \mid 0 = 0$

Because 3 divides evenly into $n^3 - n$ when n is 0 the base case is shown to be true.

1 pt for using the correct base value
2 pts for showing the base case holds.

Inductive Hypothesis: Assume for an arbitrary positive integer n = k that $3 \mid k^3 - k$ **(2 pts)**

Inductive Step: Prove for n = k+1 that $3 \mid (k+1)^3 - (k+1)$. **(2 pts)**

$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)$ **(1 pt)**
$= k^3 + 3k^2 + 3k + 1 - k - 1$
$= k^3 - k + 3k^2 + 3k$
$= (k^3 - k) + 3(k^2 + k)$ **(1 pt)**

$3 \mid k^3 - k$ by the inductive hypothesis **(1 pt)**
$3 \mid 3(k^2 + k)$ definition of divides **(1 pt)**

If $3 \mid a$ and $3 \mid b$, then $3 \mid (a + b)$, by the properties of divisibility. **(1 pt)**
Thus, $3 \mid (k+1)^3 - (k+1)$ **(3 pts)**

2) (10 pts) PRF (Logic)

Use the Rules of Inference and the Law of Contraposition to determine if the following argument is valid. Show each step and state which rule is being used.

$p \vee q$
$\neg r \rightarrow \neg p$
$r \rightarrow s$
$\neg q$
----
s

Solution

Step	**Rule**	
1. $p \vee q$	**Premise**	**(1 pt)**
2. $\neg q$	**Premise**	**(1 pt)**
3. p	**Disjunctive Syllogism (1, 2)**	**(1 pt)**
4. $\neg r \rightarrow \neg p$	**Premise**	**(1 pt)**
5. $p \rightarrow r$	**Contraposition (4)**	**(1 pt)**
6. r	**Modus ponens (3, 5)**	**(2 pts)**
7. $r \rightarrow s$	**Premise**	**(1 pt)**
8. s	**Modus Ponens (6, 7)**	**(2 pts)**

3) (15 pts) PRF (Sets)

Let A and B be finite sets of integers. Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$ by:

a) completing a membership table AND
b) giving a proof using logical equivalence. (Note: this means showing that

$\{ x \mid x \in \overline{A \cap B} \}$ and $\{ x \mid x \in \bar{A} \cup \bar{B} \}$

are describing the same set from logical first principles.

Solution (a)

A	B	$\bar{A}$	$\bar{B}$	$A \cap B$	$\overline{A \cap B}$	$\bar{A} \cup \bar{B}$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

Award 1 point per column for a total of 5 points.

Solution (b)

$\overline{A \cap B} = \{x \mid x \in \overline{A \cap B}\}$ **(1 pt)**
$= \{x \mid x \notin A \cap B\}$ **(1 pt)**
$= \{x \mid \neg(x \in A \cap B)\}$ **(1 pt)**
$= \{x \mid \neg(x \in A \wedge x \in B)\}$ **(1 pt)**
$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$ **(1 pt)**
$= \{x \mid x \notin A \vee x \notin B\}$ **(1 pt)**
$= \{x \mid x \in \bar{A} \vee x \in \bar{B}\}$ **(1 pt)**
$= \{x \mid x \in \bar{A} \cup \bar{B}\}$ **(1 pt)**
$= \bar{A} \cup \bar{B}$. **(1 pt)**

4) (10 pts) NTH (Number Theory)

(a) (3 pts) Determine the number of integers in the set {1, 2, 3, …, 15} that are relatively prime with respect to 15. Note: In order for two integers a and b to be relatively prime with respect to each other, gcd(a, b) = 1. Simply list all of these integers and then give the answer to the question. Put a box around your final answer.

The following integers in the given set are relatively prime with 15:

1, 2, 4, 7, 8, 11, 13, 14.

<u>Thus, there are 8 integers in the set relatively prime with 15.</u> (Grading: 3 pts for a correct answer, 2 pts if off by 1, 1 pt if off by 2, 0 pts otherwise, the list of numbers isn't necessary.)

(b) (7 pts) Prove that for any integer n that can be represented as the product of two distinct primes p and q, that the number of integers in the set {1, 2, 3, …, n} that are relatively prime to n is (p – 1)(q – 1). (Hint: Use the inclusion-exclusion principle.)

There are n integers in the given set.

Of these, p of them, q, 2q, 3q, …, pq share a common factor with q and should not be included in our final count. (2 pts)

Of these, q of them, p, 2p, 3p, …, pq share a common factor with p and should not be included in our final count. (2 pts)

Of these 1 of them, pq is divisible by both p and q, sharing a factor with both. (1 pt)

Thus, using the subtraction principle and the inclusion-exclusion principle, we determine the total number of values in the list that don't share a common factor with n are:

n – p – q + 1 = pq – p – q + 1 = (p – 1)(q – 1). (2 pts)