Computer Science Foundation Exam

August 10th, 2012

Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used,
and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	CTG (Counting)	10	
2	15	PRF (Relations)	10	
3	15	PRF (Functions)	10	
4	15	NTH (Number Theory)	10	
ALL	60	---	40	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) CTG (Counting)

(a) (8 pts) A shelf holds 12 books in a row. How many ways are there to choose 5 books so that no two adjacent books are chosen.

(b) (2 pts) How many permutations of "ATTTCCCAAGGG" are there?

(c) (5 pts) How many permutations of "ATTTCCCAAGGG" are there such that All "C"s and "G"s are all appear together consecutively?

Note: Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for either question

<u>Solution</u>

(a) Let the books be selected from left to right. Let the number of books not selected to the left of the first book be x_1, the number of books between the first and second book selected by x_2, and so on. Since 7 books aren't selected, we arrive at the equation: $\sum_{i=1}^{6} x_i = 7$. **(4 pts)** In addition, for each x_i, $2 \le i \le 5$, we must have $x_i > 0$. Our goal is to determine the number of non-negative integer solutions to this equation. By subtracting 1 from x_2, x_3, x_4, and x_5, which we know we can do since these are positive, we arrive at the equation $\sum_{i=1}^{6} x_i = 3$. **(2 pts)** Now, we must find the number of non-negative solutions to this new equation. This is just a combination with repetition problem with 3 items to choose out of 6. The total number of ways to select the books is $\binom{3+6-1}{3} = \binom{8}{3} = \binom{8}{5}$. **(2 pts)**

(b) $\frac{12!}{3! \times 3! \times 3! \times 3!}$, since we are simply calculating permutations with repeated symbols or C(12,3)*C(9,3)*C(6,3) , since we can choose 3 spots out of 12 for our As, 3 spots out of 9 remaining for our Cs, 3 spots out of 6 remaining for our Gs, and our Ts are forced into place. **(2 pts)**

(c) Create a super letter with all Cs and Gs. Now, there are 7 letters to permute with 3 repetitions of A and T, which can be permuted in $\frac{7!}{3!3!}$ ways. **(2 pts)** Then, within the superletter, the Cs and Gs can be permuted in $\frac{6!}{3!3!}$ ways. **(2 pts)** Multiplying we get $\frac{7!6!}{3!3!3!3!}$ total ways to permute the letters. **(1 pt)**

2) 2) (15 pts) PRF (Relations)

(a) (12 pts) Suppose **A** is the set composed of all ordered pairs of integers. Let **R** be the relation defined on **A** where $(a,b)\mathbf{R}(c,d)$ means that $a + b = c + d$. Prove that **R** is an equivalence relation.
(b) (3 pts) Find $[(3,3)]_{\mathbf{R}}$.

<u>Solution</u>

(a)

R is reflexive. To see this, note that for an arbitrary pair of integers (a,b), a + b = a + b. Thus, ((a,b),(a,b)) ∈ R, for any pair of integers (a,b). **(4 pts)**

R is symmetric. For two arbitrary pairs of integers (a,b) and (c,b), if ((a,b),(c,d)) ∈ R, then a + b = c + d. We can see that c + d = a + b. Therefore ((c,d),(a,b)) ∈ R. **(4 pts)**

R is transitive. For three arbitrary pairs of intergers (a,b), (c,d) and (e,f). If ((a,b),(c,d)) ∈ R and ((c,d),(e,f)) ∈ R, then a + b = c + d and c +d = e + f. Therefore a + b = e + f. So ((a,b),(e,f)) ∈ R. **(4 pts)**

(b) $[(3,3)]_{\mathbf{R}}=\{(a,b) \mid a+b=6, a \in \mathbf{Z} \text{ and } b \in \mathbf{Z}\}$ **(3 pts)**

3) (15 pts) PRF (Functions)

(a) (10 pts) Let *f* be a function from the set *A* to the set *B*. Let *S* and *T* be subsets of A. Show that $f(S \cup T) = f(S) \cup f(T)$.

(b) (5 pts) Let $f(x) = x - 1$ and $g(x) = x^2 + 1$, where the domain for both functions is the set of real numbers. Determine $f \circ g$ and $g \circ f$.

<u>Solution</u>

(a)

(part i) $f(S \cup T) \subseteq f(S) \cup f(T)$

Suppose $b \in f(S \cup T)$, thus $b = f(a)$ for some $a \in S \cup T$. **(2 pts)**
Either $a \in S$, in which $b \in f(S)$, or $a \in T$, in which $b \in f(T)$. **(2 pts)**
Thus $b \in f(S) \cup f(T)$. **(1 pts)**

(part ii) $f(S \cup T) \supseteq f(S) \cup f(T)$

Suppose $b \in f(S) \cup f(T)$, then either $b \in f(S)$ or $b \in f(T)$. **(2 pts)**
Therefore, either that $b = f(a)$ for some $a \in S$, or $b = f(a)$ for some $a \in T$. **(2 pts)**
In either case, $b = f(a)$ for some $a \in S \cup T$, so $b \in f(S \cup T)$. **(1 pts)**

(b)
$f \circ g(x) = f(g(x)) = f(x^2 + 1) = x^2 + 1 - 1 = x^2$ **(2 pts)**
$g \circ f(x) = g(f(x)) = f(x - 1) = (x - 1)^2 + 1 = x^2 - 2x + 1 + 1 = x^2 + 2x + 2$ **(3 pts)**

4) (15 pts) NTH (Number Theory)

(a) (10 pts) Prove that for any two integers *x* and *y*, if $13 | (3x+4y)$, then $13 | (7x+5y)$.
(b) (5 pts) Use Euclidean Algorithm to find the greatest common divisor of 252 and 198.

<u>Solution</u>

(a)

1) Since 13|(3*x*+4*y*), there exists an integer *k,* such that

$$3x+4y=13k$$
$$3x=13k-4y \quad \textbf{(2 pts)}$$
$$x=\frac{13k-4y}{3}$$

2) Therefore,

$$\begin{aligned}&7x+5y\\&=7\left(\frac{13k-4y}{3}\right)+5y\\&=\frac{13\cdot 7\cdot k-28y+15y}{3}\quad \textbf{(2 pts)}\\&=\frac{13\cdot 7k-13y}{3}\\&=13\cdot\left(\frac{7k-y}{3}\right)\end{aligned}$$

3) In this case, to show that 13|(7*x*+5*y*), we need to show that 3|(7*k*-*y*). **(2 pts)**
4) Since $3x+4y=13k$, we can see that $3x+3y+y=6k+7k$. **(2 pts)**
5) Therefore $7k-y=3x+3y-6k=3(x+y-3k)$. So 3|(7*k*-*y*). **(2 pts)**

Alternate Solution: 7x + 5y = 13x + 13y – (6x + 8y) (4 pts)
= 13(x+y) – 2(3x + 4y) (2 pts)
= 13(x+y) – 2(13c), for some integer c, since 13 | (3x+4y) (2pts)
= 13(x + y – 2c), proving that 13 | (7x + 5y) (2 pts)

(b)
GCD (198,252)= GCD (252,198).
252=1*198+54 **(1 pt)**
198=3*54+36 **(1 pt)**
54=1*36+18 **(1 pt)**
36=2*18 **(1 pts)**
GCD(198,252)=18 **(1 pt)**