Computer Science Foundation Exam

August 12, 2011

Section II A

DISCRETE STRUCTURES

SOLUTION

NO books, notes, or calculators may be used,
and you must work entirely on your own.

Question #	Max Pts	Category	Passing	Score
1	15	PRF	10	
2	10	PRF	6	
3	15	PRF	10	
ALL	40	---	26	

You must do all 3 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) PRF **(**Induction)

Using mathematical induction, prove the following statement P(n):

For all integers n > 1, $\sum_{i=1}^{n}\frac{1}{i^2} < 2-\frac{1}{n}$

(1) Base case: n =2, check P(2) is true.

LHS = $\sum_{i=1}^{2}\frac{1}{i^2} = 1+\frac{1}{4} = \frac{5}{4}$, **RHS =** $2-\frac{1}{2} = \frac{3}{2} = \frac{6}{4}$**.**

LHS < RHS, so the base case holds (or P(2) is true). (2 pts)

(2) Inductive hypothesis: Assume for an arbitrary positive integer n = k (k > 1) that

$\sum_{i=1}^{k}\frac{1}{i^2} < 2-\frac{1}{k}$. (or Assume P(k) is true) **(2 pts)**

(3) Inductive step: Prove for n = k+1 that $\sum_{i=1}^{k+1}\frac{1}{i^2} < 2-\frac{1}{k+1}$ **(2 pts)**

$\sum_{i=1}^{k+1}\frac{1}{i^2} = \sum_{i=1}^{k}\frac{1}{i^2} + \frac{1}{(k+1)^2}$ **(2 pts)**

$< 2-\frac{1}{k}+\frac{1}{(k+1)^2}$ **Using the induction hypothesis (2 pts)**

$= 2-\left(\frac{1}{k}-\frac{1}{(k+1)^2}\right)$

$= 2-\left(\frac{k^2+k+1}{k(k+1)^2}\right)$ **(1 pt)**

$= 2-\frac{k(k+1)}{k(k+1)^2}-\frac{1}{k(k+1)^2}$ **(1pt)**

$= 2-\frac{1}{(k+1)}-\frac{1}{k(k+1)^2}$ **(1pt)**

$< 2-\frac{1}{(k+1)}$ **as desired.** **(1pt)**

Based on the logic of mathematical induction, this proves that the given assertion is true for all integers n>1. (1pt)

2) (10 pts) PRF **(**Sets)

Prove the following proposition about arbitrary chosen sets *A*, *B* and *C*:

$$(A\text{–B}) - C \subseteq A\text{–}C$$

(1) So suppose that $x \in (A-B)-C$**. By the definition of difference,** $x \in (A-B) \wedge x \notin C$ **is true. (2 pts)**
(2) By the definition of difference, $(x \in A \wedge x \notin B) \wedge x \notin C$**is true. (1 pt)**
(3) Which is equivalent to $(x \in A \wedge x \notin C) \wedge x \notin B$ **is true. (1 pt)**
(4) We can infer $(x \in A \wedge x \notin C)$ **is true. (2 pts).**
(5) By definition of difference, $x \in (A-C)$ **is true. (2pts)**
(6) Therefore we have $x \in (A-B)-C \rightarrow x \in A-C$**. Based on the definition of subset, we proved:** $(A\text{–B}) - C \subseteq A\text{–}C$. **(2 pts).**

3) (15 pts) (PRF) Logic

Prove the following logical expression is a tautology using the laws of logic equivalence and the definition of conditional statement only. Show each step and state which rule is being used. (Note: You may combine both associative and commutative in a single step, so long as you do so properly.)

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$$

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$$
$$\equiv \neg((p \vee q) \wedge (\neg p \vee r)) \vee (q \vee r)$$
$$\equiv \neg(p \vee q) \vee \neg(\neg p \vee r) \vee (q \vee r)$$
$$\equiv (\neg p \wedge \neg q) \vee (\neg\neg p \wedge \neg r) \vee (q \vee r)$$
$$\equiv (\neg p \wedge \neg q) \vee (p \wedge \neg r) \vee (q \vee r)$$
$$\equiv ((\neg p \wedge \neg q) \vee q) \vee ((p \wedge \neg r) \vee r)$$
$$\equiv ((\neg p \vee q) \wedge (\neg q \vee q)) \vee ((p \vee r) \wedge (\neg r \vee r))$$
$$\equiv ((\neg p \vee q) \wedge T) \vee ((p \vee r) \wedge T)$$
$$\equiv (\neg p \vee q) \vee (p \vee r)$$
$$\equiv (\neg p \vee p) \vee (q \vee r)$$
$$\equiv T \vee (q \vee r)$$
$$\equiv T$$

1) **Definition of conditional statement**
2) **De Morgan's Law**
3) **De Morgan's Law**
4) **Double Negation**
5) **Commutative and Associative Laws**
6) **Distributive Law**
7) **Negation Law**
8) **Identity Law**
9) **Commutative and Associative Laws**
10) **Negative Law**
11) **Domination Law**

Grading: 1 pt off for each mistake (for either a rule name or step itself), cap at 15.