Computer Science Foundation Exam

August 13, 2010

Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: SOLUTION

PID: ______________________________

In this section of the exam, there are four (4) problems.
You must do ALL of them.
Each counts for 15% of the Discrete Structures exam grade.
Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

Question	Max Pts	Category	Passing	Score
4	15	CTG (Counting)	10	
5	15	PRF (Relations)	10	
6	15	PRF (Functions)	10	
7	15	NTH (Number Theory)	10	
ALL	60	---	40	

4) (CTG) Counting (15 pts)

(a) (5 pts) Round robin play is being set up for a chess tournament. There are 160 participants broken up into ten groups of 16. Within each group, each pair of participants plays each other exactly once. How many total chess games will be played in the round robin portion of this tournament?

(b) (10 pts) Determine the number of non-negative integer solutions to the equation

$$a + b + c + d + e = 70.$$

(a) With 16 teams in a group, there are $\binom{16}{2}$ games to be played in the round robin format, since there are that many ways to choose 2 teams out of 16. For each choice, there is one game. Since there are 10 such pools, the total number of round robin games is $10 \times \binom{16}{2} = \frac{10\times16\times15}{2} = 1200$. (Grading: 3 pts for 16 choose 2, 2 pts for multiplying by 10. No need to simplify to 1200.)

(b) This is a combinations with repetition question without any special circumstances. There are 5 distinct objects and we are choosing exactly 70 of them. The number of ways to do this is $\binom{70+5-1}{5-1} = \binom{74}{4} = \binom{74}{70}$.

(Grading: Recognizing combinations with repetition is worth 3 pts. Stating both 70 and 5 (or 4 depending on how they conceive of the problem) is worth 1 pt each. Applying the formula correctly is worth 5 pts.

5) (PRF) Relations (15 pts)

Let R be a relation over the positive integers defined as follows:

R = {(a,b) | gcd(a,b) > 1 but $a \nmid b$ and $b \nmid a$ }

(a) (5 pts) In laymen's terms, describe how to determine whether or not two positive integers are related via R.

(b) (10 pts) Determine whether or not R satisfies the following properties. Give a brief justification for each of your answers.

(i) reflexive
(ii) irreflexive
(iii) symmetric
(iv) anti-symmetric
(v) transitive

(a) Two positive integers are related to one another via R if they share a common factor, but neither itself is a factor of the other. (Use your discretion in grading this. Feel free to give partial credit if you feel it's deserved.)

(b) (i) No, (a,a) $\notin$ R because a | a.
(ii) Yes, the proof above shows that there are no positive integers a such that (a,a) $\in$ R.
(iii) Yes, by definition, this relation is symmetric. If a and b are related, they share a common factor but not have one number be a factor of the other. If this is the case then b and a must ALSO share a common factor (the same one), and neither is a factor of the other.
(iv) No, note that (6, 15) $\in$ R and (15, 6) $\in$ R.
(v) No, note that (6, 15) $\in$ R and (15, 35) $\in$ R, but (6, 35) $\notin$ R.

Grading: 1 pt for each answer (yes or no), 1 pt for each justification.

6) (PRF) Functions (15 pts)

(a) (7 pts) Explain why the function $f(x) = x^2 - 8x + 12$ with a domain of the real numbers is not an injective function. Determine a restriction on the domain of the form $[a, \infty)$, where a is a real number that would make this function an injection. Attempt to minimize a. Namely, for your answer a, it should be the case that a domain $[a - \varepsilon, \infty)$, where $\varepsilon > 0$ would make the function not injective.

(b) (8 pts) For the injective function described by f(x) in part (a) with the domain you've determined for that function, find $f^{-1}(x)$.

(a) $f(x) = x^2 - 8x + 12 = (x - 4)^2 - 4$. (This can be seen by completing the square.) Thus, the function is NOT injective with a domain of the real numbers because $f(3) = f(5) = -3$. We can restrict the domain as follows: $x \in [4, \infty)$. The form above shows that x=4 corresponds to the x coordinate of the vertex of the parabola formed by f(x). A quick examination shows that this is a strictly increasing function on this restricted domain, hence the function is injective. Any larger domain won't work because $f(4 - \varepsilon) = f(4 + \varepsilon)$ for all positive ε.

(Grading: 4 points for showing the function is NOT injective with a counter-example. 2 points for stating the adjusted domain, 1 pt for any reasonable explanation.)

(b) Begin by replacing f(x) with x and x with $f^{-1}(x)$ in the original equation.

$\boldsymbol{x = (f^{-1}(x) - 4)^2 - 4}$ **(2 pts)**
$\boldsymbol{x + 4 = (f^{-1}(x) - 4)^2}$ **(2 pts)**
$\boldsymbol{\sqrt{x+4} = f^{-1}(x) - 4}$ **, we take the positive square root, since the domain of the original function only contains positives. (3 pts)**

$\boldsymbol{f^{-1}(x) = \sqrt{x+4} - 4}$ **. (1 pt)**

7) (NTH) Number Theory (15 pts)

(a) (5 pts) Two integers are defined as "relatively prime" to one another if they share no common factors. (For example, 12 and 35 are relatively prime but 18 and 33 are not.) Determine how many positive integers less than 24 are relatively prime to 24.

(b) (10 pts) Find at least one integer solution (for x and y) to the equation 117x + 48y = 3.

(a) Here is a list of the positive integers less than 24 that are relatively prime to it: 1, 5, 7, 11, 13, 17, 19, and 23. Thus, there are 8 such integers. (1 point off for each mistake, cap at 5.)

(b) Divide the equation through by 3 to obtain:

39x + 16y = 1

Use the Extended Euclidean Algorithm:

39 = 2 x 16 + 7
16 = 2 x 7 + 2
7 = 3 x 2 + 1 (3 pts – 1 pt per line)

7 – 3 x 2 = 1 **(6 pts – 1 pt off for each error, cap at 6)**
7 – 3(16 – 2 x 7) = 1
7 – 3 x 16 + 6 x 7 = 1
7 x7 – 3 x 16 = 1
7(39 – 2 x 16) – 3 x 16 = 1
7 x 39 – 14 x 16 – 3 x 16 = 1
7 x 39 – 17 x 16 = 1

One possible solution is x = 7 and y = -17. (1 pt)