Computer Science Foundation Exam

August 13, 2010

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: SOLUTION

PID: ______________________________

In this section of the exam, there are three (3) problems.
You must do ALL of them.
They count for 40% of the Discrete Structures exam grade.
Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

Question #	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	15	PRF (Sets)	10	
3	10	PRF (Logic)	6	
ALL	40	---	26	

1) (15 pts) PRF **(**Induction)

Prove for all positive integers n, $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 3^n & 3^n - 1 \\ 0 & 1 \end{bmatrix}$.

Base case: n =1 LHS = $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, **RHS =** $\begin{bmatrix} 3^1 & 3^1 - 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$.
The two sides are equal so the base case holds. (2 pts)

Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$$\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 3^k & 3^k - 1 \\ 0 & 1 \end{bmatrix}.$$ **(2 pts)**

Inductive step: Prove for n = k+1 that $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 3^{k+1} & 3^{k+1} - 1 \\ 0 & 1 \end{bmatrix}$.**(2 pts)**

$$\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}^k$$ **(2 pts)**

$$= \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3^k & 3^k - 1 \\ 0 & 1 \end{bmatrix}$$ **(2 pts)**

$$= \begin{bmatrix} 3(3^k) + 2(0) & 3(3^k - 1) + 2(1) \\ 0(3^k) + 1(0) & 0(3^k - 1) + 1(1) \end{bmatrix}$$ **(2 pts)**

$$= \begin{bmatrix} 3^{k+1} & 3^{k+1} - 3 + 2 \\ 0 & 1 \end{bmatrix}$$ **(2 pts)**

$$= \begin{bmatrix} 3^{k+1} & 3^{k+1} - 1 \\ 0 & 1 \end{bmatrix}$$, as desired. **(1 pt)**

This proves that the given assertion is true for all positive integers n.

2) (15 pts) PRF **(**Sets)

Disprove the following three assertions about arbitrary chosen sets A, B and C with the use of a counterexample for each one. (You can use different counterexamples for all three parts.)

(a) If $A \subseteq B \cup C$, then either $A \subseteq B$ or $A \subseteq C$.
(b) If $(B - A) \subseteq (C - A)$, then $B \subseteq C$.
(c) If $(A \cup B) \subseteq (A \cap C)$, then $B = \varnothing$.

(a) Let $A = \{1, 2\}$, $B = \{1\}$ and $C = \{2\}$. In this situation, $B \cup C = \{1, 2\}$, so it is true that $A \subseteq B \cup C$. However, A is not a subset of either B or C in this example, since A contains one element (2) that B doesn't and one element (1) that C doesn't.

(b) The same counterexample set up for part (a) works here. To see this, note that $B - A = C - A = \varnothing$. Thus, it is true that $(B - A) \subseteq (C - A)$. But, it is certainly NOT the case that B is a subset of C, since B contains 1, which isn't an element of C.

(c) Let $A = B = C = \{1\}$. With these assignments, we have $(A \cup B) = (A \cap C) = \{1\}$, which makes $(A \cup B) \subseteq (A \cap C)$ true, but B is NOT the empty set.

Grading: 2 pts for specifying each set in question. 3 pts for explaining why the example is a counter-example to the assertion.

3) (10 pts) (PRF) Logic

Simplify the following logical expression as much as possible using the laws of logic only. Show each step and state which rule is being used. (Note: You may combine both associative and commutative in a single step, so long as you do so properly.)

$$p \lor [p \land [\neg(\neg r \lor \neg q) \lor (\neg r \land q)]]$$

$p \lor [p \land [\neg(\neg r \lor \neg q) \lor (\neg r \land q)]]$	$\leftrightarrow p \lor [p \land [(\neg\neg r \land \neg\neg q) \lor (\neg r \land q)]]$	DeMorgan's
	$\leftrightarrow p \lor [p \land [(r \land q) \lor (\neg r \land q)]]$	Double Negation
	$\leftrightarrow p \lor [p \land [(r \lor \neg r) \land q]]$	Distributive
	$\leftrightarrow p \lor [p \land [T \land q]]$	Inverse Law
	$\leftrightarrow p \lor [p \land q]$	Identity Law
	$\leftrightarrow p$	Absorption Law

Grading: 1 pt off for each mistake (for either a rule name or step itself), cap at 10.