Computer Science Foundation Exam

August 8, 2008

Section II A

DISCRETE STRUCTURES

KEY

Question #	Category	Max Score	Passing Score	Score
Q1	PRF (Induction)	25	15	
Q2	PRF (Sets)	15	10	
Total	---	40	35	

PART A

1) (25 pts) PRF (Induction)

Define a sequence of numbers as follows. Let W_i denote the i^{th} *W*-number. In particular, $W_0 = 2$, $W_1 = 1$, and $W_n = W_{n-1} + \frac{W_{n-2}}{2}$ for all integers $n > 1$.

For all positive integers *n*, prove that $\sum_{i=1}^{n} \frac{W_{i-1}}{W_i W_{i+1}} = 2 - \frac{2}{W_{n+1}}$.

Solution.

Base case: *n* = 1. LHS = $\sum_{i=1}^{1} \frac{W_{i-1}}{W_i W_{i+1}} = \frac{W_0}{W_1 W_2} = \frac{2}{1 \cdot 2} = 1$, RHS = $2 - \frac{2}{W_2} = 2 - \frac{2}{2} = 1$. Thus LHS = RHS and the base case is proven. (4 pts)

Inductive hypothesis: Assume for an arbitrary positive integer $n = k$ that $\sum_{i=1}^{k} \frac{W_{i-1}}{W_i W_{i+1}} = 2 - \frac{2}{W_{k+1}}$. (4 pts)

Inductive Step: Prove for $n = k + 1$ that $\sum_{i=1}^{k+1} \frac{W_{i-1}}{W_i W_{i+1}} = 2 - \frac{2}{W_{k+2}}$ (4 pts)

$$\sum_{i=1}^{k+1} \frac{W_{i-1}}{W_i W_{i+1}} = \left(\sum_{i=1}^{k} \frac{W_{i-1}}{W_i W_{i+1}} \right) + \frac{W_k}{W_{k+1} W_{k+2}} \text{ (2 pts)}$$

$$= 2 - \frac{2}{W_{k+1}} + \frac{W_k}{W_{k+1} W_{k+2}}, \text{ using the inductive hypothesis (3 pts)}$$

$$= 2 - \frac{2W_{k+2}}{W_{k+1} W_{k+2}} + \frac{W_k}{W_{k+1} W_{k+2}} \text{ (1 pt)}$$

$$= 2 - \left(\frac{2W_{k+2} - W_k}{W_{k+1} W_{k+2}} \right) \text{ (1 pt)}$$

$$= 2 - \frac{2W_{k+1}}{W_{k+1} W_{k+2}}, \text{ since } 2W_{k+2} = 2\left(W_{k+1} + \frac{W_k}{2} \right) = 2W_{k+1} + W_k. \text{ (3 pts)}$$

$$= 2 - \frac{2}{W_{k+2}}. \text{ (1 pt)}$$

The inductive step is complete. We have proven by induction that $\sum_{i=1}^{n} \frac{W_{i-1}}{W_i W_{i+1}} = 2 - \frac{2}{W_{n+1}}$ for all positive integers *n*. (2 pts)

2) (15 pts) PRF **(**Sets)

Let A, B and C be arbitrary sets taken from the positive integers.

Prove or disprove: If $A \cap B \cap C = \varnothing$, then $(A \subseteq \overline{B}) \vee (A \subseteq \overline{C})$.

Solution.

$A \cap B \cap C = \varnothing$ is the premise.
$\neg \exists x, x \in A \cap B \cap C$ by definition of the empty set.
$\forall x, \neg[x \in A \cap B \cap C]$ by negation of $\exists$.
$\forall x, \neg[x \in (A \cap B) \cap C]$ by the associative property of intersection.
$\forall x, \neg[(x \in A \cap B) \wedge (x \in C)]$ by the definition of intersection.
$\forall x, \neg(x \in A \cap B) \vee \neg(x \in C)$ by DeMorgan's Laws.
$\forall x, \neg[(x \in A) \wedge (x \in B)] \vee \neg(x \in C)$ by the definition of intersection.
$\forall x, [\neg(x \in A) \vee \neg(x \in B)] \vee \neg(x \in C)$ by DeMorgan's Laws.
$\forall x, \neg(x \in A) \vee \neg(x \in B) \vee \neg(x \in C)$ by the associative property of intersection.
$\forall x, (x \notin A) \vee (x \notin B) \vee (x \notin C)$ by negation of $\in$.
(6 pts)

For some arbitrary element y, assume $y \in A$.
$\neg(y \notin A)$ by double negation.
$(y \notin A) \vee (y \notin B) \vee (y \notin C)$ by application of $\forall$ in $\forall x, (x \notin A) \vee (x \notin B) \vee (x \notin C)$.
$(y \notin A) \vee [(y \notin B) \vee (y \notin C)]$ by the associative property of $\vee$.
$(y \notin B) \vee (y \notin C)$ by disjunctive syllogism.
$(y \in \overline{B}) \vee (y \in \overline{C})$ by definition of set complement.
(4 pts)

Case 1: $(y \in \overline{B})$.
$(y \in A) \rightarrow (y \in \overline{B})$.
$A \subseteq \overline{B}$ by definition of subset.
(2 pts)

Case 2: $(y \in \overline{C})$.
$(y \in A) \rightarrow (y \in \overline{C})$.
$A \subseteq \overline{C}$ by definition of subset.
(2 pts)

$(A \subseteq \overline{B}) \vee (A \subseteq \overline{C})$ by combining the two cases.
(1 pt)