# Computer Science Foundation Exam

## May 2, 2014

## Section II B

## DISCRETE STRUCTURES

**NO books, notes, or calculators may be used, and you must work entirely on your own.**

## SOLUTION

| Question | Max Pts | Category | Passing | Score |
|---|---|---|---|---|
| 1 | 15 | CTG (Counting) | 10 | |
| 2 | 15 | PRB (Probability) | 10 | |
| 3 | 10 | PRF (Functions) | 6 | |
| 4 | 10 | PRF (Relations) | 6 | |
| ALL | 50 | | 32 | |

**You must do all 4 problems in this section of the exam.**

**Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.**

**1)** (15 pts) CTG (Counting)

Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for any of the questions.

A classic identity in combinatorics is $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$. In this question, you'll prove the identity. (Note: Recall that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.)

(a) (5 pts) Use a counting argument or the definition of combinations to prove that $\binom{n}{k} = \binom{n}{n-k}$.

The combinatorial argument is as follows: Each unique subset of k items corresponds to the subset of n – k items that weren't chosen in the original k. (Grading: 3 pts) This mapping is one-to-one since each different subset of size k maps to a different subset of size n – k and vice versa. (Grading: 2 pts)

Algebraically, we have: RHS = $\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k} = LHS$. (Grading: 2 pts, 2 pts, 1 pt)

(b) (5 pts) Let set A contain n objects and set B contain n objects. How many ways can we choose k objects out of set A and n – k objects out of set B?

Since each choice is independent, we multiply the number of selections from A with the number of selections from B. By definition, both are combinations, so our final answer is $\binom{n}{k}\binom{n}{n-k}$. (Grading: 2 pts for each part, 1 pt for multiplying.)

(c) (5 pts) Now, consider choosing n total objects from sets A and B together. This can be done in $\binom{2n}{n}$ ways since we are choosing n objects out of 2n total objects. Another way of counting these same sets is to consider taking 0 items from A and n items from B, or 1 item from A and n – 1 items from B, or 2 items from A and n – 2 items from B, …, or n items from A and 0 items from B. Using your result from (b), write a summation representing the number of ways to choose n of the items from both sets by splitting up the counting in this manner. Simplify the summation using the identity proved in part (a) and conclude your proof of the original identity.

The process described simply indicates that we can express $\binom{2n}{n}$ as a sum of terms of the form in part (b) where k ranges through all possible values. Specifically, we must choose in between 0 and n items inclusive, from set A. Thus, we have the following:

$\binom{2n}{n} = \sum_{k=0}^{n}[\binom{n}{k}\binom{n}{n-k}] = \sum_{k=0}^{n}[\binom{n}{k}\binom{n}{k}] = \sum_{k=0}^{n}\binom{n}{k}^2$ , as desired. (Grading: sum = 3 pts, substitution for n choose n – k is 2 pts.)

**2)** (15 pts) PRB (Probability)

Two factories, A and B produce flat screen TVs. Factory A produces 10,000 of these TVs a year, with 400 of them being defective while Factory B produces 30,000 of these TVs a year, with 600 of them being defective. Each of the following questions will use this scenario. Answer each question as a **fraction in lowest terms.**

(a) (5 pts) If we randomly select a TV out of the 40,000 the two factories produced this year, what is the chance that it's defective?

The total number of defective TVs produced is 400 + 600 = 1000. Thus, the probability a randomly selected TV out of the 40,000 is defective is $\frac{1000}{40000} = \frac{1}{40}$. (Grading: 2 pts for numerator, 2 pts for denominator, 1 pt for reducing to lowest terms.)

(b) (5 pts) Given that the TV we select is defective, what is the probability it came from factory A?

There are 1000 defective TVs, which is our sample space. Of these, 400 came from factory A. Thus, the desired probability is $\frac{400}{1000} = \frac{2}{5}$. (Grading: 2 pts for numerator, 2 pts for denominator, 1 pt for reducing to lowest terms. Note: There are other ways to solve this, like Bayes' Law. Please grade accordingly.)

(c) (5 pts) If we randomly select a TV made in factory A and another TV made in factory B, what is the probability that exactly one of the TVs is defective?

We can simply add the disjoint probabilities that the TV from factory A is bad while the TV from factory B is good plus the probability that the TV from factory B is bad and the TV from factory A is good. Since each choice is independent, we get the following sum:

$$\frac{400}{10000} \times \frac{29400}{30000} + \frac{9600}{10000} \times \frac{600}{30000} = \frac{1}{25} \times \frac{49}{50} + \frac{24}{25} \times \frac{1}{50} = \frac{73}{1250}$$

Grading: 2 pts for each term, 1 pt for adding and simplifying.

**3)** (10 pts) PRF (Functions)

Let $f(x) = 2x^2 - 16x + 6$ with a domain of $x \in (-\infty, 4]$. Determine $f^{-1}(x)$ and its domain and range.

Let y = f(x) and swap x and y:

$$x = 2y^2 - 16y + 6$$
$$x = 2(y^2 - 8y) + 6$$
$$x = 2(y^2 - 8y + 16) - 32 + 6$$
$$x = 2(y - 4)^2 - 26$$
$$x + 26 = 2(y - 4)^2$$
$$(y - 4)^2 = \frac{x + 26}{2}$$
$$y - 4 = \pm\sqrt{\frac{x + 26}{2}}$$
$$y = 4 \pm \sqrt{\frac{x + 26}{2}}$$

To determine whether to use the plus or minus sign, note that the original domain was real values less than or equal to 4. Thus, the range of the inverse function must equal this domain. Clearly, to make all y values above less than or equal to 4, we have to choose the negative sign. Thus, our final answer is

$$y = 4 - \sqrt{\frac{x + 26}{2}}$$

The range, as previously stated is $(-\infty, 4]$. The domain is [-26, ∞), since everything under the square root must be non-negative.

<u>Grading: 1 pt swapping x and y, 1 pt for factoring out 2, 4 pts for completing the square, 2 pts for choosing the minus sign, 1 pt for the domain, 1 pt for the range.</u>

**4)** (10 pts) PRF (Relations)

Let W be the set of women in the world. Assume that one woman can wear another woman's clothes if and only if their dress sizes (positive integers) differ by no more than 2. Define a relation R over W as follows:

R = {(a, b) | a can wear b's clothes}

Determine, with proof, whether or not R satisfies the following properties:

(a) reflexive
(b) symmetric
(c) transitive

Note: For the purposes of this problem assume the valid dress sizes are positive even integers less than or equal to 24.

R is reflexive, for any woman w, she may wear her own clothes, since her dress size is within 0 of her dress size! (Grading: 3 pts – 1 pt for saying it's reflexive and 2 pts for the explanation)

R is symmetric. If $w_1$ can wear $w_2$'s clothes, it means that the difference of their dress sizes is 2 or fewer. This infers that $w_2$ can also wear $w_1$ clothes, since this difference is the same.
(Grading: 3 pts – 1 pt for saying it's symmetric and 2 pts for the explanation)

R is NOT transitive. We may have a situation where $w_1$ wears size 14, $w_2$ wears size 12 and $w_3$ wears size 10. In this specific example, we see that $w_1$ and $w_2$ can share clothes since $|14 – 12| \leq 2$, so $(w_1, w_2) \in R$, and $w_2$ and $w_3$ can share clothes since $|12 – 10| \leq 2$, so $(w_2, w_3) \in R$, but $w_1$ and $w_3$ can't share clothes because $|14 – 10| > 2$. (Grading: 4 pts – 2 pts for saying it's NOT transitive and 2 pts for the explanation)