Computer Science Foundation Exam

May 3, 2013

Section II B SOLUTION

DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,
and you must work entirely on your own.**

Question	Max Pts	Category	Passing	Score
1	15	CTG (Counting)	10	
2	10	PRB (Probability)	6	
3	15	PRF (Relations)	10	
4	10	PRF (Functions)	6	
ALL	50		32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) CTG (Counting)

(a) (5 pts) A company has 1000 computers divided into two distinct groups of sizes 100 computers and 900 computers, respectively. Let the first group be A and the second group be B. We connect the computers within each group by picking a designated computer (a_0 for A, b_0 for B) and connecting it to all others in the group. Finally, we connect each computer in group A to each computer in group B. Determine the number of direct connections between computers.

(b) (10 pts) You are assigning IP addresses to computers in a portion of your local area network. You are only allowed to assign the very last byte (8 bits) of the IP address, since the first three bytes are fixed. You would like for no two of the IP addresses to differ in only one bit to reduce transmission errors. Prove that it's impossible for you to assign IP addresses to 129 computers. (Hint: group the IP addresses in different groups based on the first seven bits of the portion of the address you are assigning.) Note: the greedy algorithm of iterating through the addresses in numerical order and adding them if the rule isn't broken will successfully allocate 128 IP addresses that satisfy the rule.

Note: Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for any of the questions.

<u>Solution (a)</u>
Within group A, we have 100 – 1 = 99 connections from a_0 to the other 99 computers. **(1 pt)**

Within group B, we have 900 – 1 = 899 connections from b_0 to the other 899 computers. **(1 pt)**

Between the two groups, we have 100 x 900 = 90000 connections, since for each of the 100 computer in group A, we create 900 connections to the different computers in group B. **(2 pts)**

Adding, we have a total of 99 + 899 + 90000 = 90998 total connections. **(1 pt)**

<u>Solution (b)</u>
Consider the 2^7 = 128 different groups of two IP addresses which share the first 7 bits in common. **(2 pts)** The first of these groups would contain 00000000 and 00000001. The next would contain 00000010 and 0000011. **(2 pts – for any sort of explanation of the grouping)** It is impossible to assign both IP addresses in a single group, because they differ in only 1 bit. **(2 pts)** Thus, the maximum number of IP address you could possibly assign is 128, the number of groups, since we can choose a maximum of 1 IP address per group. **(4 pts)** Another way to look at it uses the Pigeonhole Principle. If we choose 129 IP addresses from these 128 groups, we are guaranteed to choose 2 IP addresses from the same group, since 129 > 128. But, this would contradict our goal of not choosing 2 IP addresses that differ in a single bit.

Note: You may need to adjust the grading if students solve this a different way.

2) (10 pts) PRB (Probability)

(a) (5 pts) The probability a thumb drive you buy is defective is .002. Assume that the probability one thumb drive is defective does not affect the probability of another one being defective. Given that you've bought two thumb drives, what is the probability that both are defective?

(b) (5 pts) Luckily, both of the thumb drives you've bought work! Each has 1 GB of memory available for you to save files. You have four files you would like to save on the two thumb drives of the following sizes: .7 GB, .4 GB, .3 GB and .3 GB. Unfortunately, you've left your nimwit brother to copy the files from your desktop to the two thumb drives. For each of the four files, he randomly chooses one of the two thumb drives to copy the file, not checking if the copy was successful. (Assume a copy successful as long as the requisite space is available.) What is the probability that all four files get copied successfully? Leave your answer as a fraction in lowest terms.

Solution (a)
Since both events are independent, we can multiply the probabilities of each thumb drive being defective to obtain .002 x .002 = .000004. **(5 pts, pretty much all or nothing, since there's only one step to solve the problem, give partial if you deem it necessary)**

Solution (b)
Label the four files as follows: A (.7 GB), B (.4 GB), C (.3 GB) and D(.3 GB)

Without loss of generality, assume that file A gets copied to the first thumb drive. Then we have 8 possible arrangements for the files on the first thumb drive, each equally likely:

1) A
2) A, B
3) A, C
4) A, D
5) A, B, C
6) A, B, D
7) A, C, D
8) A, B, C, D **(2 pts)**

Naturally, the second thumb drive will always contain the rest of the files not listed for the files on the first thumb drive. Note that the sizes of B, C and D are 1.0 GB. Thus, we are guaranteed that the second thumb drive will have its files properly copied. Thus, we simply need to count how many of these 8 arrangements are valid. The only valid arrangements are #1 (.7 GB), #3 (1.0 GB) and #4 (1.0 GB). **(2 pts)** Thus the corresponding probability is $\frac{3}{8}$. **(1 pt)**

3) (10 pts) PRF (Functions)

Consider the function $f_n(x) = y$ such that $30x \equiv y \pmod{n}$ and $0 \le y < n$, over the set $\{0, 1, 2, ..., n-1\}$ for any positive integer $n > 1$.

(a) (6 pts) Is the function $f_{65}(x)$ a injection? Prove your answer.

(b) (4 pts) Determine one value for n (no proof necessary) for which the function $f_n(x)$ is a bijection.

<u>Solution (a)</u>
No. Consider $f_{65}(0)$ and $f_{65}(13)$.

$f_{65}(0) = 30(0) \equiv 0 \pmod{65}$, thus $f_{65}(0) = 0$.
$f_{65}(13) = 30(13) \equiv 390 \pmod{65}$, thus $f_{65}(0) = 0$, since $390 \equiv 0 \pmod{65}$.

Remember, for a function to be injective, if $f(x) = f(y)$, then $x = y$ must be true. But in this instance, $f(0) = f(13)$, but clearly 0 and 13 aren't equal.

Grading: 2 pts for choosing two specific values to plug into $f_{65}(x)$. 4 pts for calculating the corresponding output values and explaining why this shows that the function isn't injective

<u>Solution (b)</u>
n = 7 suffices. Basically n can be any value that doesn't share any common factors with 30. (Equivalently, n can be any integer without 2, 3 or 5 in its prime factorization.) If n doesn't share a common factor with 30, then we can prove that 30(0), 30(1), 30(2), …, 30(n – 1) are all inequivalent mod n.

Grading: 4 pts all or nothing

4) (15 pts) PRF (Relations)

Let R be a relation over the positive integers greater than 1 defined as follows: R = { (x, y) | gcd(x,y) > 1, namely, x and y share a common factor.} Determine, with proof, whether or not R satisfies the following properties: (i) reflexive, (ii) irreflexive, (iii) symmetric, (iv) anti-symmetric, (v) transitive.

__**Solution**__

(i) reflexive: Yes, consider any value of a > 1. gcd(a, a) = a, by definition, since a divides into both terms. Since a > 1, for all a, (a, a) ∈ R, proving that R is reflexive. **(1 pt for answer, 2 pts for proof)**

(ii) irreflexive: No. (2,2) ∈ R, showing that R is NOT irreflexive. **(1 pt for answer, 2 pts for proof)**

(iii) symmetric: Yes. If (a,b) is in R, then gcd(a, b) > 1. Similarly, gcd(b, a) > 1, since gcd itself is a symmetric function. It follows that (b, a) is in R, as desired. **(1 pt for answer, 2 pts for proof)**

(iv) anti-symmetric: No. (2, 4) ∈ R and (4 , 2) ∈ R. But, 2 and 4 are not equal. This proves that R isn't anti-symmetric. **(1 pt for answer, 2 pts for proof)**

(v) R is not transitive. Note that (2, 6) ∈ R, (6, 9) ∈ R, but (2, 9) ∉ R. Thus, we've found an instance where (a, b) ∈ R, (b, c) ∈ R but (a, c) ∉ R, proving that R isn't symmetric. **(1 pt for answer, 2 pts for proof)**