Computer Science Foundation Exam

May 3, 2013

Section II A SOLUTION

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: ____

PID: ____

Question	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	10	PRF (Logic)	6	
3	10	PRF (Sets)	6	
4	15	NTH (Number Theory)	10	
ALL	50	---	32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) PRF (Induction)

Let T(n) be defined as follows:

$T(1) = 1$, $T(2) = 3$, and
$T(n) = T(n-1) + T(n-2) + 1$.

Use induction on n to show that $T(n) > 2^{n/2}$ for $n \geq 2$.

Basis Step:
$T(2) = 3 > 2^{1/1} = 2$, verifying the claim for n = 2.
$T(3) = 3+1+1 = 5 > 2^{3/2}$, because $4 = 2^2 > 2^{3/2}$, verifying the claim for n=3. **(2 pt)**

Inductive Hypothesis:
Assume the truth of the claim for n=2, …, k, for $k \geq 3$. **(2 pts)**

Inductive Step:
We will now use this to show the truth of the claim for n=k+1.

We are given: $T(k+1) = T(k) + T(k-1) + 1$ **(2 pts)**

Using the inductive hypothesis (which we can do since $k \geq 3$) , we have:
$T(k+1) > 2^{k/2} + 2^{(k-1)/2} + 1$ **(3 pts)**

The inequality is unaffected if we make the right-hand-side even smaller, so we drop the 1, and replace the first term by a copy of the (smaller) second term.

$T(k+1) > 2^{(k-1)/2} + 2^{(k-1)/2}$ **(2 pts)**
$T(k+1) > 2(2^{(k-1)/2})$
$T(k+1) > 2^{(k-1)/2+1}$
$T(k+1) > 2^{(k-1+2)/2}$
$T(k+1) > 2^{(k+1)/2}$ **(4 pts)**

This proves the inductive step. Thus, we can conclude that the given statement is true for all integers $n \geq 2$.

2) (10 pts) PRF (Logic)

You are given a set of four cards, each of which has a letter on the front side, and a number on the back side. The cards are placed on a table, some with the front side up, others with the front side down. The visible faces read “U, 3, K, 8”, and you are told that all these cards must obey the following rule:

“If the front side has a vowel, then the back side has an odd number.”

(a) Define two appropriate predicates and use them to express this rule using a quantified statement of predicate logic. Mention the universe of discourse.

We define V(x) = “Card x has a vowel on the front side” **(1 pt)**, and
O(x) = “Card x has an odd number on the back side” **(1 pt)**.

The rule is: $\forall x\, V(x) \rightarrow O(x)$, where the universe of discourse is the set of cards we were given. **(1 pt)**

(b) What is the logical negation of this rule? Simplify it so that no NOT operators are outside the scope of the quantifier.

Negating, we get: $\neg(\forall x\, V(x) \rightarrow O(x))$ (**1 pt)**

Using definition of implication: $\neg(\forall x\, \left(\neg V(x) \vee O(x)\right))$ (**1 pt)**

Moving the NOT inside the quantifier: $\exists x\, \left(\neg(\neg(V(x) \vee O(x)))\right)$ (**1 pt)**

By De Morgan’s laws: $\exists x\, \left(V(x) \wedge \neg O(x)\right)$ (**2 pts)**

(c) Which cards would you turn over to determine if the rule is violated? (**Hint:** Use your answer to (b))

Any card that violates the rule must obey the negation of the rule. Therefore we need to check for cards with a vowel on the front and an even number on the back. So we should turn over the U **(1 pt)**, and the 8 **(1 pt)**.

3) (10 pts) PRF (Sets)

Let A and B be sets. Use a set membership table to show that:

$$A = (A - B) \cup (A \cap B)$$

Solution

A	B	$A - B$	$A \cap B$	$(A - B) \cup (A \cap B)$
0	0	0	0	0
0	1	0	0	0
1	0	1	0	1
1	1	0	1	1

Since the first and last columns are exactly identical, the two sets they represent are equal.

2 pts each for the first two columns
3 pts each for the next two
3 pts for the last column
2 pts for correctly showing the first and last column to be identical.

4) (15 pts) PRF (Number Theory)

(a) (10 pts) Let a be an integer such that a ≡ 1 (mod 3). Prove that $a^3 \equiv 1$ (mod 9).

Solution
Since a ≡ 1 (mod 3), let a = 3n +1, for some integer n. (2 pts) Now, consider a^3:

$a^3 = (3n+1)^3$ (1 pt)
$= 27n^3 + 27n^2 + 9n + 1$ (4 pts)
$= 9(3n^3 + 3n^2 + n) + 1$ (2 pts)

Since n is an integer, we've expressed a^3 in the form 9c + 1 where c is an integer, it follows that $a^3 \equiv 1$ (mod 9). (1 pt)

(b) (5 pts) Using the result from (a) and the fact that if x ≡ 1 (mod m), x ≡ 1 (mod n), and gcd(m,n) =1, then x ≡ 1 mod (mn), prove that

$$666666666667^3 \equiv 1 \pmod{18}.$$

(Note: You may assume that an odd number cubed is odd.)

Solution
Note that 666666666667 ≡ 1 (mod 3) and 666666666667 ≡ 1 (mod 2). (1 pt)

Using the result from part a, we conclude that $666666666667^3 \equiv 1$ (mod 9). (1 pt)

Using the second fact and the note that an odd number cubed is odd, we arrive at the fact that

$666666666667^3 \equiv 1$ (mod 2). (1 pt)

Finally, using the result about two mod equations given in the question, setting x = 666666666667^3, m = 9 and n = 2, since gcd(9,2) = 1, we can conclude that

$666666666667^3 \equiv 1$ (mod 18), as desired. (2 pts)