Computer Science Foundation Exam

May 4, 2012

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used,
and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	15	PRF (Logic)	10	
3	10	PRF (Sets)	6	
ALL	40	---	26	

You must do all 3 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) PRF (Induction)

Prove, using mathematical induction, that for all positive integers n, we have

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \le \log(n) + 1$$

where $\log(n)$ denotes the natural logarithm. You may use that $\log(n+1) - \log(n) \ge \frac{1}{n+1}$ holds for all positive integers n.

<u>Solution</u>

Base case: n = 1. The left-hand side of this inequality evaluated at n = 1 is 1.
The right hand side of this inequality evaluated at n = 1 is log(1) + 1 = 1.
Thus, the inequality holds for n = 1. (2 pts)

Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \le \log(k) + 1 \quad \textbf{(2 pts)}$$

Inductive step: Prove for n = k + 1 that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+1} \le \log(k+1) + 1 \quad \textbf{(2 pts)}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k+1} = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\mathrm{k}}\right) + \frac{1}{\mathrm{k+1}} \qquad \textbf{(2 pts)}$$

$$\le \log(k) + 1 + \frac{1}{\mathrm{k+1}}, \text{ using I. H.} \qquad \textbf{(3 pts)}$$

$$= \left(\log(k) + \frac{1}{k+1}\right) + 1 \qquad \textbf{(0 pts)}$$

$$\le \log(k+1) + 1, \text{ since using the inequality given it follows that}$$
$$\log(k+1) \ge \log(k) + \frac{1}{k+1} \quad \textbf{(4 pts)}$$

This completes proving the inductive step. It follows that the original assertion holds for all positive integers n.

2) (15 pts) PRF (Logic)

Prove that the following logical expression is a tautology using the laws of logic equivalence and the definition of the conditional statement only. Show each step and state which rule is being used. Note: You may combine both associative and commutative laws in a single step, so long as you do this properly.

$$\overline{q \vee r} \rightarrow \overline{(\bar{p} \vee q) \wedge (q \vee r)}$$

<u>Solution</u>

$\overline{q \vee r} \rightarrow \overline{(\bar{p} \vee q) \wedge (q \vee r)}$		
$(\overline{\overline{q \vee r}}) \vee \overline{(\bar{p} \vee q) \wedge (q \vee r)}$	defn of implication	**(3 pts)**
$(q \vee r) \vee \overline{(\bar{p} \vee q) \wedge (q \vee r)}$	Double Negation	**(2 pts)**
$(q \vee r) \vee (\overline{(\bar{p} \vee q)} \vee \overline{(q \vee r)})$	DeMorgan's Law	**(3 pts)**
$((q \vee r) \vee \overline{(q \vee r)}) \vee (\overline{\bar{p} \vee q})$	Commutative Law	**(2 pts)**
$\mathrm{T} \vee (\overline{\bar{p} \vee q})$	Inverse Law (using $p = q \vee r$)	**(3 pts)**
T	Domination Law	**(2 pts)**

Grading: Give positive points for steps in the correct direction, roughly the steps are outlined above. Students may use more steps and rules. In this case, award points based on the relative progress of their intermediate steps as compared to what is shown above.

3) (10 pts) PRF (Sets)

Let *A*, *B*, and *C* be sets. Show that

$$(A - B) - C = (A - C) - (B - C).$$

By definition,

$\boldsymbol{LHS} = (A - B) - C = (A \ \cap \ \bar{B}) - C = \big((A \cap \bar{B}) \cap \bar{C}\big) = A \cap \bar{B} \cap \bar{C}$,

using def. of set difference.

$\boldsymbol{RHS} = (A - C) - (B - C)$	
$= (A \cap \bar{C}) - (B \cap \bar{C}))$	defn of set difference
$= ((A \cap \bar{C}) \cap \overline{(B \cap \bar{C})})$	defn of set difference
$= \Big((A \cap \bar{C}) \cap \big(\bar{B} \cup \bar{\bar{C}}\big)\Big)$	De Morgan's Law
$= (A \cap \bar{C}) \cap (\bar{B} \cup C)$	Double Negation
$= \big((A \cap \bar{C}) \cap \bar{B}\big) \cup ((A \cap \bar{C}) \cap C)$	Distributive Law
$= (A \cap \bar{B} \cap \bar{C}) \cup (A \cap (\bar{C} \cap C))$	Commutative/Associative Laws
$= (A \cap \bar{B} \cap \bar{C}) \cup (A \cap \emptyset)$	Inverse Law
$= (A \cap \bar{B} \cap \bar{C}) \cup \emptyset$	Domination Law
$= (A \cap \bar{B} \cap \bar{C})$	Identity Law

Since the two sides are equivalent to the same set, the two sides must be equal.

Grading: LHS = 2 pts – 1 pt per step, RHS = 8 pts – 1 pt per step in the right direction