Computer Science Foundation Exam

May 6, 2011

Section II B

DISCRETE STRUCTURES

SOLUTION

NO books, notes, or calculators may be used, and you must work entirely on your own.

Question	Max Pts	Category	Passing	Score
4	15	CTG (Counting)	10	
5	15	PRF (Relations)	10	
6	15	PRF (Functions)	10	
7	15	NTH (Number Theory)	10	
ALL	60	---	40	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

4) (15 pts) CTG (Counting)

(a) (8 pts) How many bit strings of length 8 do **not** contain 4 consecutive zeros? For example `11111111` and `11010001` are valid, but `00000000` and `11000010` are invalid.

(b) (7 pts) How many integers between 1 and 1,000,000 have sum of digits equal to 9? For example, `333` and `10,125` are valid, but `9,999` and `12,111,111` are invalid.

(a) We will find the total number of possible strings and subtract the total number of strings with exactly four, five, six, seven, and eight consecutive zeros to find the total number of strings without four consecutive zeros. To find the total number of strings with four consecutive zeros, consider that when a zero must be present there is only one choice for the character value. Also, if there are to be exactly 4 consecutive zeros, the numbers to the left and the right of that string must be 1's, so there is only one choice for those digits as well. In the following counts below, we delineate all the possibilities of strings. For example, the count labeled 1 1 1 1 1 2 2 2, corresponds to any string that starts with 4 0's and one 1.

4 consecutive zeros
1 1 1 1 1 2 2 2 (8)
1 1 1 1 1 1 2 2 (4)
2 1 1 1 1 1 1 2 (4)
2 2 1 1 1 1 1 1 (4)
2 2 2 1 1 1 1 1 (8)

5 consecutive zeros
1 1 1 1 1 1 2 2 (4)
1 1 1 1 1 1 1 2 (2)
2 1 1 1 1 1 1 1 (2)
2 2 1 1 1 1 1 1 (4)

6 consecutive zeros
1 1 1 1 1 1 1 2 (2)
1 1 1 1 1 1 1 1 (1)
2 1 1 1 1 1 1 1 (2)

7 consecutive zeros
1 1 1 1 1 1 1 1 (1)
1 1 1 1 1 1 1 1 (1)

8 consecutive zeros
1 1 1 1 1 1 1 1 (1)

The total number of possible strings is 2^8. From this, we subtract the count of strings above:

$$2^8 - 8 - 4 - 4 - 4 - 8 - 4 - 2 - 2 - 4 - 2 - 1 - 2 - 1 - 1 - 1 = 256 - 48 = 208.$$

Grading: 2 points for all bit strings of length 8, 2 points for subtraction idea, 4 points for all the specific cases above (give partial on this).

(b) Let the digits of the number be a, b, c, d, e and f. Thus, we want the number of solutions to a + b + c + d + e + f = 9, where a, b, c, d, e, and f are non-negative digits. This is identical to the combinations with repetition problem. Thus, the number of solutions is $\binom{9+6-1}{6-1} = \binom{14}{5}$.

Grading: 3 points for recognizing combo with repetition, 4 points for properly applying the formula

5) (15 pts) PRF (Relations)

Let R be a binary relation over the set of integers $\mathbb{Z}$, such that

$$(x, y) \in R \quad \text{if and only if} \quad (x^2 - y^2) \text{ is a multiple of 7}$$

Prove or disprove: R is an equivalence relation.

R is an equivalence relation.

1. Reflexivity

We must show that for all integers x, x R x. This amounts to showing that ($x^2 - x^2$) is a multiple of 7. Since $x^2 - x^2 = 0$, and 0 = 0 x 7, we've shown that R is reflexive. (3 points)

2. Symmetry

We must show that if (a, b) ∈ R, then (b, a) ∈ R, for an arbitrary choice of (a, b). (1 pt)

Let (a, b) ∈ R. Then, $a^2 - b^2 = 7n$ for some integer n. (2 pts)
Multiplying through by -1, we find that (2 pts)

$b^2 - a^2 = 7(-n)$, since n is an integer, -n is as well. It follows that (b, a) ∈ R as desired.

3. Transitivity

We must show that if (a, b) ∈ R, and (b, c) ∈ R, then (a, c) ∈ R. (1 pt)

Let (a, b) ∈ R and (b, c) ∈ R. Then $a^2 - b^2 = 7n$ and $b^2 - c^2 = 7m$ for some integers n and m. (2 pts)

Adding these two equations, we get

$a^2 - b^2 + b^2 - c^2 = 7n + 7m$. (2 pts)
$a^2 - c^2 = 7(n + m)$. Since n and m are integers, n + m is as well. It follows that (a, c) ∈ R. (2 pts)

Thus, R is an equivalence relation.

6) (15 pts) PRF (Functions)

Let $f: Q^{+} \to Q$ be the function where $f(m) = \frac{m+1}{m}$.

(a) (7 pts) What is the range of f?

(b) (8 pts) Prove that f is injective.

(a) The domain solely consists of positive rational numbers. It follows that the range must also be positive, since both m and m+1 must be positive quantities and a positive divided by a positive is positive. Furthermore, we can rewrite f(m) as 1 + 1/m, by dividing through by the denominator. With this representation, it's clear that if m can be an arbitrary positive rational number, then 1/m can be as well. Thus, the range is all rational numbers greater than 1.

3 points for realizing that it's only positive numbers.
4 points for recognizing that [0, 1] is not possible.

(b) To prove that f is injective, we simply must show that if f(x) = f(y) for two values x and y in the domain, then x = y. (1 pt)

$\boldsymbol{f(x) = f(y)}$
$\boldsymbol{\frac{x+1}{x} = \frac{y+1}{y}}$**, cross multiply since x ≠ 0 and y ≠ 0.** **(2 pts)**
$\boldsymbol{y(x+1) = x(y+1)}$ **(2 pts)**
$\boldsymbol{yx + y = xy + x}$ **(2 pts)**
$\boldsymbol{y = x}$ **, as desired.** **(1 pt)**

7) (15 pts) NTH (Number Theory)

(a) (10 pts) Find integers x and y such that $92x + 47y = 1$.

(b) (5 pts) Prove that if (a | c) and (b | d), then (ab | cd), for integers a, b, c and d.

(a)

92 = 1*47 + 45	**(1 pt)**
47 = 1*45 + 2	**(1 pt)**
45 = 22*2 + 1	**(1 pt)**
2 = 1*2	**(1 pt)**

Substitituing in:

45 - 22*2 = 1	
47 - 1*45 = 2	
45 - 22(47 - 1*45) = 1	**(2 pts)**
92 - 1*47 = 45	
(92-1*47) - (22(47 - 1*(92 - 1*47))) = 1	**(2 pts)**
92(23) - 47(45) = 1	**(1 pt)**
x = 23, y = 45	**(1 pt)**

Give 5 points out of 10 for a guess and check technique.

(b)

Since a|c and b|d, let c= an and d = bm, where n and m are integers. (1 pt)

So cd = (na)(mb).	**(2 pts)**
cd = (ab)(nm), since n and m are integers, so is nm.	**(1 pt)**
It follows by definition of divisibility that ab | cd.	**(1 pt)**