Computer Science Foundation Exam

May 7, 2010

Section II B

DISCRETE STRUCTURES SOLUTIONS

NO books, notes, or calculators may be used,
and you must work entirely on your own.

Name: ______________________________

PID: ______________________________

In this section of the exam, there are four (4) problems.
You must do ALL of them.
Each counts for 15% of the Discrete Structures exam grade.
Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

Question	Max Pts	Category	Passing	Score
4	15	CTG (Counting)	10	
5	15	PRF (Relations)	10	
6	15	PRF (Functions)	10	
7	15	NTH (Number Theory)	10	
ALL	60	---	40	

4) (CTG) Counting (15 pts)

(a) (5 pts) How many permutations are there of the following string "VERYEASYEXAM"?

V	E	R	Y	A	S	X	M
1	**3**	**1**	**2**	**2**	**1**	**1**	**1**

Number of distinct permutations = $\frac{12!}{3!2!2!}$**, grading – 1 pt numerator, 3 pts den, 1 pt divide**

(b) (10 pts) There are N users and M servers with $M \geq N$. Each user can send a request to any of the servers. Determine the number of situations in which at least one collision occurs, i.e., there is at least one pair of users that send the request to the same server.

Total number of ways for users to select servers: M^N **(4 pts – 1 pt each, 2 pts exponent)**

Number of ways in which <u>no</u> collisions occur: $\binom{M}{N} N!$ **(# of way to choose N servers out of M, ordered in N! ways)**

(4 pts – can also write as $_MP_N$.)

Number of situations where a collision occurs: $M^N - \binom{M}{N} N!$ **(2 pts sub)**

5) (PRF) Relations (15 pts)

Let R be the relation defined on Z where $a\ R\ b$ means that $a + b^2 \equiv 0$ (mod 2).
(a)Prove that R is an equivalence relation. (12 pts)

We must prove 3 things:
i) $a\ R\ a$ (R is reflexive) (3 pts)
ii) $a\ R\ b \Rightarrow b\ R\ a$ (R is symmetric) (5 pts)
iii) $a\ R\ b$ and $b\ R\ c \Rightarrow a\ R\ c$ (R is transitive) (4 pts)

i) $a\ R\ a$

$$\begin{aligned} & a + a^2 && \text{// 1 pt} \\ &= a(a + 1) && \text{// 1 pt} \\ &= 2p \text{ [for some } p \in Z\text{]} && \text{// 1 pt} \\ &\equiv 0 \text{ (mod 2)} \end{aligned}$$

Thus $a\ R\ a$

ii) $a\ R\ b \Rightarrow b\ R\ a$

$$\begin{aligned} & a + b^2 \equiv 0 \text{ (mod 2)} && \text{// 1 pt} \\ & a + b^2 + b + a^2 \equiv b + a^2 \text{ (mod 2)} && \text{// 1 pt} \\ & b + a^2 \equiv a^2 - a + b - b^2 \text{ (mod 2)} && \text{// 1 pt} \\ & b + a^2 \equiv a(a - 1) - b(b - 1) \text{ (mod 2)} && \text{// 1 pt} \\ & b + a^2 \equiv 2p - 2q \text{ (mod 2) [for some } p, q \in Z\text{]} && \text{// 1 pt} \\ & b + a^2 \equiv 0 \text{ (mod 2)} \end{aligned}$$

Thus $a\ R\ b \Rightarrow b\ R\ a$

iii) $a\ R\ b$ and $b\ R\ c \Rightarrow a\ R\ c$

$$\begin{aligned} & a + b^2 \equiv 0 \text{ (mod 2)} \\ & b + c^2 \equiv 0 \text{ (mod 2)} && \text{// 1 pt for writing out both} \\ & a + b^2 + b + c^2 \equiv 0 \text{ (mod 2)} && \text{// 1 for adding} \\ & a + b(b + 1) + c^2 \equiv 0 \text{ (mod 2)} && \text{// 1 pt} \\ & a + 2p + c^2 \equiv 0 \text{ (mod 2) [for some } p \in Z\text{]} && \text{// 1 pt} \\ & a + c^2 \equiv 0 \text{ (mod 2)} \end{aligned}$$

Thus $a\ R\ b$ and $b\ R\ c \Rightarrow a\ R\ c$

(b) Find the equivalence class [-13]. (3 pts)

[-13] = {…, -5, -3, -1, 1, 3, 5, 7, …} = {$2p + 1 \mid p \in Z$} = The set of all odd integers (3 pts all or nothing)

6) (PRF) Functions (15 pts)

Let f and g be functions such that f: A→B, g: B→C, where A, B and C are finite sets.

(a) Disprove the following statement with a single counter-example:

if f is surjective and g is injective, then g°f is injective.

A = {1,2}
B = {3}
C = {4} **// 3 pts – 1 pt for listing each set**

f(1) = f(2) = 3 **// 4 pts – 2 pts for listing each function.**
g(3) = 4

g°f (1) = 4 **// 3 pts for showing why g(f(x)) isn't injective**
g°f (2) = 4

⇨ **g°f is not injective, since there exist 2 distinct elements which map to the same element.**

(b) Under what conditions is the statement above true? (Specifically, deduce something about the relationship between |A|, |B| and |C| for the situations when the statement holds.)

The statement is true if |A| = |B| ≤ |C|. (3 pts for the first part, 2 pts for the second part)

Proof (not required by the problem statement):
Let f be surjective, g be injective, and |A| = |B| ≤ |C|. Our goal is to show g°f is injective. Assume the contrary is true, that is g°f is not injective. Then there exist two distinct elements $x, y \in A$ **such that g°f (*x*) = g°f (*y*). This must mean that there are 2 elements in A which map to the same element in B, since we know g to be injective. But this is impossible, because if 2 elements in A map to the same element in B and |A| = |B|, then f will not be surjective, since by the pigeonhole principle there would be some element of B which would be underrepresented. Thus the assumption must have been false, and g°f is therefore injective.**

7) (NTH) Number Theory (15 pts)

(a) (10 pts) Prove that the sum of five consecutive positive integers is divisible by 5.

Let $x \in Z$. Then

$x + (x+1) + (x+2) + (x+3) + (x+4)$	**// 5 pts**
$= 5x + (1+2+3+4)$	**// 2 pts**
$= 5x + 10$	**// 2 pts**
$\equiv 0 \pmod 5$	**// 1 pt**

(b) (5 pts) Compute the GCD(126,33).

126 = 3*33 + 27
33 = 1*27 + 6
27 = 4*6 + 3
6 = 2*3

GCD(126,33) = 3.

Grading: 1 pt each step, 1 pt answer