Computer Science Foundation Exam

May 8, 2009
Section II A
DISCRETE STRUCTURES

SOLUTIONS

In this section of the exam, there are three (3) problems.
You must do all of them.
They count for 40% of the Discrete Structures exam grade.
Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone.

Credit cannot be given when your results are unreadable.

Question #	Category	Score	Max Score
1	PRF (Induction)		15
2	PRF (Direct Proof)		15
3	PRF (Logic)		10
ALL	---		40

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer ***all*** of Part A and ***all*** of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: **Work all of the following problems (1, 2, 3 and 4).**

1) (15 pts) (PRF) Induction

Prove for all positive integers n that $\begin{pmatrix} c & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} c^n & \frac{c^n-1}{c-1} \\ 0 & 1 \end{pmatrix}$, where c is any real number not equal to 1.

(Hint: Remember that a matrix raised to a power is repeated matrix multiplication and that for 2x2 matrices, multiplication is defined as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Base case: n = 1. LHS = $\begin{pmatrix} c & 1 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} c & 1 \\ 0 & 1 \end{pmatrix}$, RHS = $\begin{pmatrix} c^1 & \frac{c^1-1}{c-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c & 1 \\ 0 & 1 \end{pmatrix}$ **(2 pts)**

Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$\begin{pmatrix} c & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} c^k & \frac{c^k-1}{c-1} \\ 0 & 1 \end{pmatrix}$ **(2 pts)**

Inductive step: Prove for n = k+1 that

$\begin{pmatrix} c & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} c^{k+1} & \frac{c^{k+1}-1}{c-1} \\ 0 & 1 \end{pmatrix}$ **(2 pts)**

$\begin{pmatrix} c & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} c & 1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} c & 1 \\ 0 & 1 \end{pmatrix}$ **(2 pts)**

$= \begin{pmatrix} c^k & \frac{c^k-1}{c-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & 1 \\ 0 & 1 \end{pmatrix}$, using the IH **(2 pts)**

$= \begin{pmatrix} c^{k+1} & c^k + \frac{c^k-1}{c-1} \\ 0 & 1 \end{pmatrix}$, multiplying out **(2 pts)**

$= \begin{pmatrix} c^{k+1} & \frac{c^k(c-1)+c^k-1}{c-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c^{k+1} & \frac{c^{k+1}-c^k+c^k-1}{c-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c^{k+1} & \frac{c^{k+1}-1}{c-1} \\ 0 & 1 \end{pmatrix}$

(5 pts total)

2) (15 pts) (PRF) Sets

Let A and B be arbitrary sets taken from the universe of integers.

(a) Show that the following statement is false by providing a single counter example:
if $A \times B \subseteq B \times A$, then $A = B$.

(b) Show that if we restrict A and B to both be non-empty sets, then the proposition above is true.

(a) Let A = {} and B = {1}. In this example, A x B = {} and B x A = {}, so that the first is a subset of the second. But, clearly, in the example, A and B are not equal sets.

(Grading: 3 pts for a valid counter example, 2 pts for explaining it.)

(b) If both A and B are non-empty, then we know that A x B and B x A must also be non-empty. Thus, there does exist some element (x, y) in A x B. **(2 pts)** Consider all such elements (x, y) in A x B. By definition, it is necessarily the case that x $\in$ A and y $\in$ B. Since $A \times B \subseteq B \times A$, it follows that (x, y) is in B x A.**(3 pts)** But, by definition, this means that x $\in$ B and y $\in$ A. **(3 pts)** As we go through each element in A and each element in B in this fashion, we find that each element in A must be contained in B and vice versa, which means that the two sets are equal. **(2 pts)**

3) (10 pts) (PRF) Logic

Using the law of logic and implication, verify the following argument:
(Note: For the purposes of this problem, ~ represents the unary not operator)

$(p \wedge q) \rightarrow r$
$s \rightarrow \sim r$
$s \wedge p$

$\therefore \sim q$

1) $s \wedge p$ Premise
2) s Conjunctive Simplification with #1
3) $s \rightarrow \sim r$ Premise
4) $\sim r$ Modus Ponens with #2, #3
5) $(p \wedge q) \rightarrow r$ Premise
6) $\sim(p \wedge q)$ Modus Tollens with #4, #5
7) $\sim p \vee \sim q$ DeMorgan's Law
8) p Conjuctive Simplification with #1
9) $\sim q$ Disjunctive Syllogism

Grading: 1 pt off per mistake, only take off 3 points total if the reasons are missing but the steps are correct.

Computer Science Foundation Exam

May 8, 2009
Section II B
DISCRETE STRUCTURES

SOLUTIONS

In this section of the exam, there are four (4) problems.
You must do ALL of them.
Each counts for 15% of the Discrete Structures exam grade.
Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

Question #	Category	Score	Max Score
4	CTG (Counting)		15
5	PRF (Direct)		15
6	PRF (Direct)		15
7	NTH (Number Theory)		15
ALL	---		60

PART B: Work ALL 4 of the problems 4 – 7.

4) (CTG) Counting

(a) (7 pts) How many permutations of MISSISSIPPI have no consecutive vowels?

(b) (8 pts) An ascending number is a number where each of the digits are distinct and are contained in the number in ascending order. For example, 145 and 24679 are ascending numbers and no ascending numbers start with the digit 0. How many ascending numbers are there total?

(a) Place each of the consonants out with gaps in between them:

_ M _ S _ S _ S _ S _ P _ P _

We can choose 4 of these 8 locations to place the Is in ${}_8C_4$ ways. **(3 pts)**

Independently, we can permute the consonants in $\frac{7!}{4!2!}$ ways. **(3 pts)** Thus, the total number of permutations is $\binom{8}{4}\frac{7!}{4!2!}$ to arrange the letters with the given restriction. **(1 pt).**

(b) For each digit, 1 through nine, we can either choose to place it in the number or not. Once these choices are made, the number if fixed. Thus, there are 2^9 possible choices. **(6 pts)**

But, one of those possibilities, not taking the digits for all 9 digits, results in an "empty" number. Thus, we don't count this one and have a total of $2^9 - 1$ ascending numbers. **(2 pts)**

5) (PRF) Relations

(a) (5 pts) Consider a relation defined by people who know each other. For example, (Arup Guha, Ben Douglass) would be in the relation because the first person listed in the ordered pair knows the second person listed. We assume that a person knows their self and that if person A knows person B, then person B knows person A. (For example, I may say that I know Dan Marino, but if he doesn't know me, then I really don't know him.) Even with these assumptions, this relation is neither an equivalence relation, nor a partial ordering relation. Prove these assertions using real-life examples. (Note: A partial ordering relation is one that is reflexive, anti-symmetric and transitive.)

(b) (10 pts) Let R be a non-empty relation on a set A. Prove that if R is symmetric and transitive, that it is NOT irreflexive.

(a) This relation is NOT transitive. Since it's not transitive, it is neither an equivalence relation nor a partial-ordering relation. For example, I (Arup) know David White (a friend from high school). David works for the United Nations in Kuala Lumpur, Malaysia and knows his boss there. But, I do not know David's boss and he does not know me! (This turns out to be true for most of my friends' bosses.) **(5 pts – decide partial…)**

(b) Since R is non-empty, we can assume that there exists an element, (x,y), in R. **(2 pts)** If $x = y$, then the relation is NOT irreflexive and we are done. **(2 pts)**

Otherwise, assume that $x \neq y$. Since R is symmetric, it follows that $(y,x) \in R$. **(2 pts)**

But, since R is transitive, and $(x,y) \in R$ and $(y,x) \in R$, it follows that $(x,x) \in R$, which shows that R is not irreflexive in this case either. **(4 pts)**

6) (15 pts) (PRF) Functions

Let f be a function with f: A → B and g be a function with g: B → C.

(a) (7 pts) Show by example that it is possible for $g \circ f(x)$ to be surjective while $f(x)$ is not.

(b) (8 pts) Prove that if $g \circ f(x)$ is surjective, then $g(x)$ must be as well.

(a) Let A = {1,2,3}, B = {1,2} and C = {1}. Let f(1) = 1, f(2) = 1 and f(3) = 1. Let g(1) = 1 and g(2) = 1. In this example, g(f(1)) = g(f(2)) = g(f(3)) = 1 and g(f(x)) is surjective, since 1 is covered. But, f(x) is NOT, because there is no value for which f(x) equals 2. **(4 pts for completely describing the example, 3 pts for the explanation as to why g(f(x)) is surjective but f(x) is not.)**

(b) We must show that g(x) is surjective. Namely, we must prove that for each y ∈ C, there exists an x such that g(x) = y. **(2 pts)** We know that g(f(x)) is surjective. Thus, for any arbitrarily chosen y, there exists some value a such that g(f(a)) = y. **(2 pts)** But, it is definitely the case that f(a) ∈ B**.(2 pts)** Furthermore, just set x = f(a), and we have found the element x in B such that g(x) = y. **(2 pts) (Give partial credit for picture proofs.)**

7) (NTH) Number Theory

(a) (5 pts) In the proof that there are an infinite number of prime numbers, a number is constructed by multiplying a set of primes and adding one to that product. (As an example if we use 2, 3 and 5, we get 2x3x5 + 1 = 31.) However, this construction does not always yield a prime. Give an example of a number that is constructed by multiplying together two or more distinct primes and adding one that is NOT a prime number.

(b) (10 pts) Prove that the equation $a^2 + b^2 = 10023$ has no integer solutions for a and b. (In your proof, consider the value of all perfect squares mod 4.)

(a) 2x7 + 1 = 15, where 2 and 7 are prime, but 15 is not. **(5 pts, all or nothing)**

(b) Investigating perfect squares mod 4, split up all values into even and odd squares:

For even squares, we have $x^2 = (2a)^2 = 4a^2 \equiv 0 \bmod 4$ **(3 pts)**
For odd squares, we have $x^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 4(a^2 + a) + 1 \equiv 1 \bmod 4$. **(3 pts)**

Thus, we find that perfect squares are always equivalent to either 0 or 1 mod 4.

Thus, when we evaluate $a^2 + b^2$ mod 4, we must obtain either 0, 1 or 2. (0 if both a and b are even, 1 if exactly one of them is even, and 2 if both are odd.) **(2 pts)**

But, $10023 \equiv 3 \bmod 4$. Thus, the two sides of the equation are definitively not equivalent mod 4. This means there can not be any solutions to the equation. **(2 pts)**