Computer Science Foundation Exam

December 12, 2014

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

SOLUTION

Question	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	10	PRF (Logic)	6	
3	15	PRF (Sets)	10	
4	10	NTH (Number Theory)	6	
ALL	50		32	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.

1) (15 pts) PRF (Induction)

Use mathematical induction on n to prove that, for every positive integer n,

$$133|(11^{(n+1)} + 12^{(2n-1)})$$

Let P(n) represent the statement "for every positive integer n $133|(11^{(n+1)} + 12^{(2n-1)})$ "

Basis Case: Let's check P(1) is true. Since $11^{1+1} + 12^{2-1} = 121 + 12 = 133$, therefore $133|(11^{1+1} + 12^{2-1})$. So P(1) is ture. **(2 pts)**

Induction Hypothesis:
Assume that P(k) is true for any positive integer k, which is $133|\left(11^{(k+1)} + 12^{(2k-1)}\right)$. Thus, there is an integer m such that $11^{k+1} + 12^{2k-1} = 133 \cdot m$. **(2 pts)**

Inductive Step:

(1) We want to prove that P(k+1) is true, which is $133|\left(11^{(k+1+1)} + 12^{(2(k+1)-1)}\right)$. **(2 pts)**

(2) $11^{k+1+1} + 12^{2(k+1)-1} = 11^{k+1} + 12^{2k+1}$

$= 11 \cdot 11^{k+1} + 144 \cdot 12^{2k-1}$ **(2 pts)**

$= 11\,(11^{k+1} + 12^{2k-1}) + 133 \cdot 12^{2k-1}$ **(2 pts)**

$= 133 \cdot m + 133 \cdot 12^{2k-1}$,using Inductive Hypothesis **(2 pts)**

$= 133 \cdot (m + 12^{2k-1})$ **(2 pts)**

By the logic of induction, P(n) is true for all positive integer n. **(1 pts)**

2) (10 pts) PRF (Logic)

Use the laws of logic to show that the following logical expression is a tautology:

$$\big((p \to q) \land p\big) \to q$$

Note: a tautology is an expression equivalent to True. Label the use of each law.

$\big((p \to q) \land p\big) \to q$	
$\equiv ((\neg p \lor q) \land p) \to q$	Definition of Implication
$\equiv ((\neg p \land p) \lor (q \land p)) \to q$	Distributive Law
$\equiv (F \lor (q \land p)) \to q$	Negation Law
$\equiv (q \land p) \to q$	Identity Law
$\equiv \neg(q \land p) \lor q$	Definition of Implication
$\equiv (\neg q \lor \neg p) \lor q$	De Morgan's Law
$\equiv (\neg q \lor q) \lor \neg p$	Commutative and Associative Laws
$\equiv T \lor \neg p$	Negation Law
$\equiv T$	Domination Law

Grading: 1 pt off per incorrect step. 2 pts for proper reason. Deduct whole points only as you see fit.

3) (15 pts) PRF (Sets)

Prove the following statement is true:

Let A, B and C be three finite sets. Prove that $(A - B) - C \subseteq A - (B - C)$.

(Note: $-$ denotes differences of two sets.)

We first find an equivalent statement to prove:

(1) Based on the definition of "difference", the proposed statement is equivalent to $(A \cap \bar{B}) \cap \bar{C} \subseteq A \cap \overline{(B \cap \bar{C})}$. **(2 pts)**
(2) Based on the De Morgan's law and Double Negation, we can change the statement to $(A \cap \bar{B}) \cap \bar{C} \subseteq A \cap (\bar{B} \cup C)$. **(2 pts)**
(3) Based on the Associative law, we can change the statement to $(A \cap \bar{B}) \cap \bar{C} \subseteq (A \cap \bar{B}) \cup (A \cap C)$. **(2 pts)**

Now we use direct proof to prove this statement:

(4) Let x be an arbitrary element such that $x \in (A \cap \bar{B}) \cap \bar{C}$. By the definition of intersection, $x \in (A \cap \bar{B})$ and $x \in \bar{C}$. **(2 pts)**
(5) Therefore, we can see $x \in (A \cap \bar{B})$ is true. **(2 pts)**
(6) In this case, we can infer the following statement is true: $x \in (A \cap \bar{B}) \lor x \in (A \cap C)$. Formally, this is the rule of Disjunctive Amplification. **(2pts)**
(7) Based on the definition of Union, we can say that $x \in (A \cap \bar{B}) \cup (A \cap C)$ is true. **(1 pt)**
(8) Based on the definition of subset, we have proved that $(A \cap \bar{B}) \cap \bar{C} \subseteq (A \cap \bar{B}) \cup (A \cap C)$. **(1 pt)**
(9) Therefore $(A - B) - C \subseteq A - (B - C)$ is true. **(1 pt)**

Grading note: there are many ways to do this problem. Map the grading accordingly. Note that even a set table can be used. After listing out the eight rows, you simply have to note that each row for which the expression on the right contains the element, the expression on the left does also, and quickly explain why this is a subset relationship. In essence, a set table lays out a proof by cases in a systematic way.

4) (10 pts) NTH (Number Theory)

Prove that if a is an odd integer, then $a^2 \equiv 1(mod\ 8)$.

(1) Since a is an odd integer, then there is an integer k, such that a = 2k+1. **(1 pt)**
(2) Therefore $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4 \cdot k \cdot (k+1) + 1$. **(1 pt)**
(3) If k is odd, there is an integer m such that k=2m+1. **(1pt)**
In this case, $a^2 = 4 \cdot (2m+1) \cdot (2m+1+1) + 1 = 4 \cdot (2m+2) \cdot (2m+1) + 1 = 8 \cdot (m+1) \cdot (2m+1) + 1$. **(2 pts)**
Based on the definition of congruency, therefore, $a^2 \equiv 1(mod\ 8)$. **(1 pt)**
(4) If k is even, there is an integer n such that k=2n. **(1pt)**
In this case, $a^2 = 4 \cdot (2n) \cdot (2n+1) + 1 = 8 \cdot (n) \cdot (2n+1) + 1$. **(2 pts)**
Based on the definition of congruency, therefore, $a^2 \equiv 1(mod\ 8)$. **(1pt)**
(5) In conclusion, $a^2 \equiv 1(mod\ 8)$ if a is an odd integer.

Grading note: one can simply argue that the product of two consecutive integers k(k+1) is always even because if k is even the product is even and if k is odd, then k+1 is even, making the product even. From there, substitute (2m) for k(k+1) and this will take care of steps 3 and 4.