Computer Science Foundation Exam

December 14, 2012

Section II A

DISCRETE STRUCTURES

SOLUTION

NO books, notes, or calculators may be used, and you must work entirely on your own.

Question	Max Pts	Category	Passing	Score
1	15		10	
2	10		6	
3	15		10	
ALL	40	---	27	

You must do all 3 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) PRF (Induction)

Recall that the Fibonacci sequence is defined as follows: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for all integers $n > 1$. Use induction to prove the following matrix exponentiation statement for all positive integers n:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Solution

Base Case: n = 1. Using the given definition of the Fibonacci Numbers, we find that $F_0 = 0$, $F_1 = 1$, and $F_2 = 1$. Thus, in evaluating the claim for n = 1, we find:

LHS = $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and RHS = $\begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Thus, the claim is true for n = 1. **(2 pts)**

Inductive Hypothesis: Assume for an arbitrary positive integer n = k that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}. \textbf{(2 pts)}$$

Inductive Step: Prove for n = k+1 that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}$. **(2 pts)**

$$\begin{aligned}
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^k, \text{ by definition of exponents } \textbf{(2 pts)} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}, \text{ using the inductive hypothesis. } \textbf{(2 pts)} \\
&= \begin{pmatrix} F_{k+1} + F_k & F_k + F_{k-1} \\ F_{k+1} & F_k \end{pmatrix}, \text{ carrying out the matrix multiply } \textbf{(3 pts)} \\
&= \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}, \text{ using the definition of Fibonacci numbers. } \textbf{(2 pts)}
\end{aligned}$$

2) (10 pts) PRF (Logic)

Fill in the truth table below to prove that the following logical expression is a tautology.

$$(\bar{q} \wedge r) \to ((p \vee q) \to p)$$

Solution

p	**q**	**r**	$\bar{q} \wedge r$	$p \vee q$	$((p \vee q) \to p)$	$(\bar{q} \wedge r) \to ((p \vee q) \to p)$
F	**F**	**F**	F	F	T	T
F	**F**	**T**	T	F	T	T
F	**T**	**F**	F	T	F	T
F	**T**	**T**	F	T	F	T
T	**F**	**F**	F	T	T	T
T	**F**	**T**	T	T	T	T
T	**T**	**F**	F	T	T	T
T	**T**	**T**	F	T	T	T

Grading: 3 pts for first column, 3 pts for second column, 3 pts for third column, 1 pt for last column. Give 3 pts if a column is completely correct, 2 pts if it's a majority correct and 1 pt if there are 1-4 entries correct, for the first three columns. Only award the point for the last column if it's completely correct.

3) (15 pts) PRF (Sets)

Let A, B and C be finite sets of integers. Prove or disprove the following assertions:

a) If $C \subseteq A \cap B$, then $A - B \subseteq A - C$.
b) If $A \cap B \subseteq C$, then $(A - C) \cup (B - C) \cup (A \cap B) = A \cup B$.

Solution (a)
This assertion is true. In order to prove it, we must show that for an arbitrarily chosen element x, if $x \in A - B$, then $x \in A - C$. **(2 pts)**

if $x \in A - B$, by definition of set difference, $x \in A$ and $x \notin B$. **(1 pt)**

We are given that $C \subseteq A \cap B$. Since $A \cap B \subseteq B$ by the definition of set intersection and the subset relation is transitive, we can ascertain that $C \subseteq B$. **(3 pts)**

But, since $x \notin B$, it follows that $x \notin C$, because if $x \in C$ it would contradict the assertion that $C \subseteq B$. **(3 pts)**

(Another way to derive this is to note that $C \subseteq B$ means that for any arbitrary element x, that if $x \in C$, then $x \in B$. The contrapositive of this statement is: if $x \notin B$, then $x \notin C$. Finally, since we know that $x \notin B$ already, applying the statement we just derived yields that $x \notin C$.)

Finally, since we've shown for our arbitrarily chosen element x that $\in A$ and $x \notin C$, it follows by definition of set difference that $x \in A - C$. **(1 pt)**

Solution (b)
This assertion is false. In order to see this, consider the following counter-example:

A = {1}
B = {2}
C = {1, 2} **(3 pts for the counter example)**

In this example, we have $A \cap B = \emptyset$, thus it's vacuously true that $A \cap B \subseteq C$.

But, notice that each of the sets $(A - C)$, $(B - C)$, and $(A \cap B)$ are empty. Thus, the left-hand side of the statement we are trying to prove evaluates to the empty set. But, $A \cup B$ = {1, 2}, which is not empty. **(2 pts for showing it's a counter example)**

Thus, in this particular example, the statement does not hold. Thus, it's not true for all finite sets of integers, A, B and C.