Computer Science Foundation Exam

December 16, 2011

Section II B

DISCRETE STRUCTURES

SOLUTION

NO books, notes, or calculators may be used,
and you must work entirely on your own.

Question	Max Pts	Category	Passing	Score
1	15	CTG (Counting)	10	
2	15	PRF (Relations)	10	
3	15	PRF (Functions)	10	
4	15	NTH (Number Theory)	10	
ALL	60	---	40	

You must do all 4 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) CTG (Counting)

(a) (9 pts) How many different permutations without consecutive vowels are there by reordering the letters of the word "FLORIDA"?
(b) (6 pts) How many license plates consisting of three letters followed by three digits contain no letter or digit twice? (For the purposes of this question there are 10 digits and 26 letters.)

Note: Please leave your answers in factorials, permutations, combinations and powers. Do not calculate out the actual numerical value for either question.

(a) There are 4 consonants. Use these as separators: __ F __ L __ R __ D __. We can permute the 3 vowels amongst the 5 blank locations, with at most one vowel per blank. This can be done in $_5P_3$ ways. Similarly, we can permute the consonants in 4! ways for EACH arrangement of the vowels. Thus, the total number of permutations desired is the product of these two, $_5P_3$ x 4! = 1440. (Grading: 3 pts for the separator idea, 3 pts for placing the vowels, 2 points for arranging the consonants, 1 pt for multiplying the two. Note – if the student tries a significantly different approach, please attempt to give partial credit separate to this criteria.)

(b) Just use the multiplication principle to get 26 x 25 x 24 x 10 x 9 x 8 = 11232000. (Grading: 2 pts to realize that there's multiplication, 2 pts for 26,25,24 and 2 pts for 10, 9, 8.)

2) (15 pts) PRF (Relations)

Let **R** be the relation defined on the set of integers **Z** where $(a, b) \in$ **R** if and only if $6|(a-b)$.
(a) (12 pts) Prove that **R** is an equivalence relation.
(b) (3 pts) Find the equivalence set $[5]_{\mathbf{R}}$.

(a)

To show that R is reflexive, we must show that (a, a) ∈ R, for all integers a. (1 pt) Since a – a = 0, and 6 | 0 by definition of divisibility, R is reflexive. (2 pts)

To show that R is symmetric, we must show that if that (a, b) ∈ R, then (b, a) ∈ R, for all integers a and b. (1 pt) If (a, b) ∈ R, then (a – b) = 6c, for some integer c. (1 pt) Multiply both sides of this equation by -1 to obtain

-(a – b) = (-1)(6c) (1 pt)
b – a = 6(-c) (1 pt)

Since c is an integer, so is c. It follows that (b, a) ∈ R, as desired.

To show that R is transitive, we must show that if (a, b) ∈ R and (b, c) ∈ R, then (a, c) ∈ R, for all integers a, b and c. (1 pt) Using the given information we have:

a – b = 6X, for some integer X (1 pt)
b – c = 6Y, for some integer Y (1 pt)

Adding these equations we get

a – c = 6X + 6Y (1 pt)
a – c = 6(X + Y) (1 pt)

Since X and Y are integers, X + Y is also and we've expressed a – c as 6 times an integer. It follows that 6 | (a – c) and (a, c) ∈ R as desired.

(b) $[5]_R = \{\ldots, -13, -7, -1, 5, 11, 17, \ldots\}$**, or more formally,**
$[5]_R = \{ x \mid x = 5 + 6n, \text{ where } n \in Z \}$

Grading: 1 pt for listing 5, 1 pt for listing at least one number bigger than 5 and smaller than 5, 1 pt for indicating that the list continues towards infinity in both directions.

3) (15 pts) PRF (Functions)

(a) (5 pts) Prove that the function $f(x) = (x-1)^2$, where the domain is the real numbers, is NOT one-to-one.

(b) (10 pts) Let f and g be functions such that f: A → B and g: B → C, where f is onto and g is onto. Prove that the composition function, $g \circ f$, must be onto.

(a) f(0) = 1 and f(2) = 1, so the function is NOT one-to-one, since more than one input value map to the same output value. 2 pts for each pair (any pair will do) and 1 pt for the explanation

(b)

To prove g ∘ f is onto, we need to prove that for all elements c ∈ C, there exists an element a such that g ∘ f (a) = c. (1 pt)

Let c be an arbitrary element from the set C. (1 pt)

Since g is onto, we know there exists an element b such that g(b) = c. (2 pts)

Similarly, for this element b, since f is onto, we know there exists an element a such that f(a) = b. (2 pts)

Substituting, we have g(f(a)) = c. (2 pts)

Thus, we have shown that for an arbitrarily chosen element c, we can always find an element a such that g ∘ f (a) = c. (2 pts)

4) (15 pts) NTH (Number Theory)

(8 pts) Prove that if *n* is an odd positive integer, then $n^2 \equiv 1 (\mathrm{mod}\, 8)$.

(7 pts) Find the GCD of 496 and 384 using the Euclidean Algorithm and find integers x and y such that 496x + 384y = GCD(496, 384).

(a) First, we prove the result that the expression n(n+1) is even, for all integers n. If n is even itself, it's clear that n(n+1) is even also, because n is a factor of the expression. Alternatively, if n is odd, then we can express n = 2a + 1, for some integer a. Then n(n+1) = (2a + 1)(2a + 1 + 1) = (2a + 1)(2a + 2) = (2a + 1)2(a + 1). Once again, this expression also has a factor of 2, (and all of the other terms are integers since a is an integer), thus, in this case, n(n+1) is even as well. (1 pt)

Now, let's consider an arbitrary odd integer n. Let n = 2x + 1, for some integer x. (1 pt) Then we have:

$(2x + 1)^2 = 4x^2 + 4x + 1$ **(1 pts)**
$= 4(x^2 + x) + 1$ **(1 pt)**
$= 4x(x + 1) + 1$ **(1 pt)**
= 4(2n') + 1, since we derived that expressions of the form x(x+1) are even.(2 pts)
≡ 1 mod 8, by definition of mod (1 pt)

(b) 496 = 1 x 384 + 112 (1 pt)
384 = 3 x 112 + 48 (1 pt)
112 = 2 x 48 + 16 (1 pt)
48 = 3 x 16

112 – 2 x 48 = 16 (1 pt)
112 – 2(384 – 3x112) = 16
112 – 2 x 384 + 6x112 = 16
7x112 – 2x384 = 16 (1 pt)
7(496 – 384) – 2x384 = 16
7 x 496 – 9 x 384 = 16 (1 pt)

Thus, one solution is x = 7, y = -9. (1 pt)