Computer Science Foundation Exam

December 16, 2011

Section II A

DISCRETE STRUCTURES

SOLUTION

NO books, notes, or calculators may be used,
and you must work entirely on your own.

Question	Max Pts	Category	Passing	Score
1	15		10	
2	10		6	
3	15		10	
ALL	40	---	27	

You must do all 3 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all be neat.

1) (15 pts) PRF (Induction)

Prove, using mathematical induction, that for all non-negative integers n, $10 \mid (9^{n+1} + 13^{2n})$.

Base case: n = 0. Plug into the expression to get $9^{0+1} + 13^{2(0)} = 9 + 1 = 10$. Since 10 | 10, the base case holds. (2 pts)

Inductive Hypothesis (IH): Assume for an arbitrary non-negative integer n = k, that $10 \mid (9^{k+1} + 13^{2k})$. Equivalently, there is some integer c such that $10c = 9^{k+1} + 13^{2k}$. (2 pts)

Inductive Step: Prove for n = k+1 that $10 \mid (9^{k+1+1} + 13^{2(k+1)})$. (2 pts)

$9^{k+1+1} + 13^{2(k+1)} = 9^{k+2} + 13^{2k+2}$ (1 pt)
$= 9^1 9^{k+1} + 13^2 13^{2k}$, since $a^{b+c} = a^b a^c$. (1 pt)
$= 9(9^{k+1}) + 169(13^{2k})$ (1 pt)
$= 9(9^{k+1}) + (160 + 9)(13^{2k})$ (1 pt)
$= 9(9^{k+1}) + 9(13^{2k}) + 160(13^{2k})$ (1 pt)
$= 9(9^{k+1} + 13^{2k}) + 160(13^{2k})$ (1 pt)
$= 9(10c) + 160(13^{2k})$, using the integer c defined in the IH (2 pts)
$= 10(9c + 16(13^{2k}))$ (1 pt)

Since 9c, 16 and 13^{2k} are all integers, it follows that the expression above is divisible by 10, proving the inductive hypothesis.

2) (10 pts) PRF (Logic)

A boolean expression in 3 conjunctive normal form (3 CNF) includes clauses that are each connected with an and. Each clause has three literals that are connected with an or. For example, the following expression with three variables, x_1, x_2, and x_3 is a boolean expression in 3 CNF with 4 clauses:

$$(x_1 \lor x_2 \lor \overline{x_3}) \land (x_2 \lor x_3 \lor \overline{x_1}) \land (x_1 \lor \overline{x_2} \lor \overline{x_3}) \land (x_3 \lor \overline{x_1} \lor \overline{x_2})$$

If a variable can not be repeated in a clause, with proof, what is the fewest number of clauses necessary to create a 3 CNF expression that is impossible to make true, no matter what you set each of the variables to. Note: Since each variable in a clause is different, the minimum number of variables you can use is 3. Give a boolean expression in 3 CNF with this many clauses that can not be made true, no matter what each boolean variable is set to.

The fewest number of clauses is 8, using 3 separate variables. To the four clauses written above, add the following four:

$$(x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_3 \lor \overline{x_2}) \land (x_2 \lor \overline{x_1} \lor \overline{x_3}) \land (\overline{x_3} \lor \overline{x_1} \lor \overline{x_2})$$

The reason this full expression with 8 clauses is impossible to satisfy is that it covers all 8 possibilities of the three boolean variables in the expression. No matter what truth assignment you give to x_1, x_2, and x_3, one of these eight clauses will have all three set to false, thus making the entire expression false.

To prove that no fewer clauses are possible, consider any expression with 3 variables and 7 clauses. At least one of the eight possibilities written above will be missing. Write down this possibility and set each of the variables in it to false. By default, each of the clauses that are actually in the expression will have at least one variable set to true.

Note: This result is contingent upon the condition that no variable appear more than once in a clause, which forces each clause to have three different variables.

Grading: Correct numerical answer(8) – 1 pt
Correct example expression – 3 pts
Justification that this expression can never be true – 4 pts
Justification that any 7 clauses under the rules can be satisfied – 2 pts

3) (15 pts) PRF (Sets)

Derive the inclusion-exclusion principle for three sets using the inclusion-exclusion principle for two sets A and B, listed below:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(Note: Even if you've never seen this before, you should be able to use this rule above and set theory rules to solve the question. In particular, you are solving for $|A \cup B \cup C|$.)

Applying the Inclusion-Exclusion Principle to sets *A* and (*B* ∪ *C*), we have

$|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$ (3 pts)

We can use the principle again with sets B and C ($|B \cup C| = |B| + |C| - |B \cap C|$) to obtain:

$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)|$ (3 pts)

Now, use the distributive law on the set $A \cap (B \cup C)$:

$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$. (3 pts)

Finally, we can apply the Inclusion-Exclusion Principle to the sets $(A \cap B)$ and $(A \cap C)$:

$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - (|(A \cap B)| + |(A \cap C)| - |(A \cap B) \cap (A \cap C)|)$ - (3 pts)

$= |A| + |B| + |C| - |B \cap C| - |(A \cap B)| - |(A \cap C)| + |(A \cap B) \cap (A \cap C)|$ - (1 pt)

$\underline{= |A| + |B| + |C| - |B \cap C| - |(A \cap B)| - |(A \cap C)| + |A \cap B \cap C|}$ (2 pts)

The last step follows since $A \cap A = A$ and intersection is associative and commutative.