Computer Science Foundation Exam

December 17, 2010

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

SOLUTION

Question #	Max Pts	Category	Passing	Score
1	15	PRF (Induction)	10	
2	10	PRF (Sets)	6	
3	15	PRF (Logic)	10	
ALL	40	---	26	

You must do all 3 problems in this section of the exam.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone. Credit cannot be given unless all work is shown and is readable. Be complete, yet concise, and above all <u>be neat</u>.

1) (15 pts) PRF **(**Induction)

Using mathematical induction, prove for all positive integers n, $\sum_{i=1}^{n} \frac{1}{i \cdot (i+1)} = \frac{n}{n+1}$.

LHS = $\frac{1}{1(1+1)} = \frac{1}{2}$**, RHS =** $\frac{1}{1+1} = \frac{1}{2}$**.**
The two sides are equal so the base case holds. (2 pts)

(2) Inductive hypothesis: Assume for an arbitrary positive integer n = k that

$$\sum_{i=1}^{k} \frac{1}{i \cdot (i+1)} = \frac{k}{k+1}.$$ **(2 pts)**

(3) Inductive step: Prove for n = k+1 that $\sum_{i=1}^{k+1} \frac{1}{i \cdot (i+1)} = \frac{k+1}{k+2}$**.(2 pts)**

$$\sum_{i=1}^{k+1} \frac{1}{i \cdot (i+1)} = \sum_{i=1}^{k} \frac{1}{i \cdot (i+1)} + \frac{1}{(k+1)(k+2)}$$ **(2 pts)**

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$ **(2 pts)**

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$ **(2 pts)**

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$ **(1 pts)**

$$= \frac{k+1}{k+2}$$ as desired. **(1 pt)**

Based on the logic of mathematical induction, this proves that the given assertion is true for all positive integers n. (1pt)

2) (10 pts) PRF **(**Sets)

Prove the following proposition about arbitrary chosen sets *A*, *B* and *C*:

$$(A \cup B) \cap C \subseteq A \cup (B \cap C)$$

(1) LHS, $(A \cup B) \cap C) = (A \cap C) \cup (B \cap C)$ **(based on set identity distributive law). (2 pts)**
(2) We now need to prove: $(A \cap C) \cup (B \cap C) \subseteq A \cup (B \cap C)$.
(3) So suppose that $x \in (A \cap C) \cup (B \cap C)$**. By the definition of union,** $x \in (A \cap C) \vee x \in (B \cap C)$ **is true. (2 pts)**
(4) if $x \in (A \cap C)$**, by the definition of intersection,** $x \in A \wedge x \in C$**. Therefore,** $x \in A$. **Based on the definition of union,** $x \in A \cup (B \cap C)$**. (2 pts)**
(5) if $x \in (B \cap C)$**, by the definition of union,** $x \in A \cup (B \cap C)$**. (2 pts).**
(6) Therefore we have $x \in (A \cap C) \cup (B \cap C) \rightarrow x \in A \cup (B \cap C)$**. Based on the definition of subset, we proved :** $(A \cap C) \cup (B \cap C) \subseteq A \cup (B \cap C)$. **(2 pts).**

3) (15 pts) (PRF) Logic

Prove the following logical expression is a tautology using the laws of logic equivalence and the definition of the conditional statement only. Show each step and state which rule is being used. (Note: You may combine both associative and commutative in a single step, so long as you do so properly.)

$((p \to q) \wedge (q \to r)) \to (p \to r)$

$((p \to q) \wedge (q \to r)) \to (p \to r)$	
$\equiv \neg((\neg p \vee q) \wedge (\neg q \vee r)) \vee (\neg p \vee r)$	**1) Definition of conditional statement**
$\equiv \neg(\neg p \vee q) \vee \neg(\neg q \vee r) \vee (\neg p \vee r)$	**2) De Morgan's Law**
$\equiv (\neg(\neg p) \wedge \neg q) \vee (\neg(\neg q) \wedge \neg r) \vee (\neg p \vee r)$	**3) De Morgan's Law**
$\equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r)$	**4) Double Negation**
$\equiv ((p \wedge \neg q) \vee \neg p) \vee ((q \wedge \neg r) \vee r)$	**5) Commutative and Associative Laws**
$\equiv ((p \vee \neg p) \wedge (\neg q \vee \neg p)) \vee ((q \vee r) \wedge (\neg r \vee r))$	**6) Distributive Law**
$\equiv (T \wedge (\neg q \vee \neg p)) \vee ((q \vee r) \wedge T)$	**7) Negation Law**
$\equiv (\neg q \vee \neg p) \vee (q \vee r)$	**8) Identity Law**
$\equiv (\neg q \vee q) \vee (\neg p \vee r)$	**9) Commutative and Associative Laws**
$\equiv T \vee (\neg p \vee r)$	**10) Negative Law**
$\equiv T$	**11) Domination Law**

Grading: 1 pt off for each mistake (for either a rule name or step itself), cap at 15.