Computer Science Foundation Exam

December 19, 2008

Section II B

DISCRETE STRUCTURES

KEY

Question	Max Pts	Category	Passing	Score
4	15	CTG (Counting)	10	
5	15	PRF (Relations)	10	
6	15	PRF (Functions)	10	
7	15	NTH (Number Theory)	10	
ALL	60	---	40	

PART B

4) (CTG) Counting (15 pts)

(a) (5 pts) A ternary string is a string composed of the symbols 0, 1, and 2. How many ternary strings of length 9 contain 5 zeroes, 2 ones, and 2 twos? (For example, one such string is 010010202.)

(b) (5 pts) Ten professors are planning to sit in a row of seats at a conference. Professor Sutton would rather not sit next to Professor Merriman, because he knows Professor Merriman has a cold. How many ways are there for the ten professors to be seated so that Professor Sutton is not seated next to Professor Merriman?

(c) (5 pts) In a popular computer game, during each turn, the player has the choice of moving North, East, South, or West. How many paths of 8 moves can be created in which each move is either North, East, South, or West? For example, one such path is North, East, East, South, South, West, South, West. Two paths are counted as the same only if each "move" in order is identical in both paths. For example, East, North, East, South, South, West, South, West should count differently than the previously mentioned path.

Solution.

(a) (5 pts)
Each ternary string can be created in three steps. First, from the 9 available places, choose 5 places for the zeroes; there are C(9,5) ways to do this. Second, from the 4 remaining places, choose 2 places for the ones; there are C(4,2) ways to do this. Third, from the 2 remaining places, choose 2 places for the twos; there are C(2,2) ways to do this. By the Product Rule, the total number of strings is $C(9,5) \cdot C(4,2) \cdot C(2,2) =$ **756. (Grading: 1 pt for each component, 1 pt for multiplying, 1 pt for the reasoning.)**

Alternate solution: This is a permutation question with 9 total symbols, with the first repeating 5 times and the other two repeating twice, thus, there are $\frac{9!}{5!2!2!} = 756$ strings.

(Grading: 1 pt for the numerator, 2 pts for the denominator, 1 pt for dividing, 1 pt for the explanation.)

(b) (5 pts)
First, we consider the complementary problem of seating the ten professors so that Professor Sutton is seated next to Professor Merriman. Consider Professors Sutton and Merriman as one unit and the remaining 8 professors each as separate units. There are 9! ways to arrange these 9 units, and then there are 2 ways to arrange Professors Sutton and Merriman within their unit. Thus, by the Product Rule there are $2 \cdot 9!$ ways to solve the complementary problem. **(3 pts)**

Now we consider the total number of ways to seat the ten professors without restriction. There are 10! ways to do so. Therefore, the number of ways to seat the ten professors such that Professor Sutton is not seated next to Professor Merriman is **10! - 2·9!** . **(2 pts)**

Alternate solution: Placing the first nine professors can be done in 9! ways. **(2 pts)** Now consider placing Professor Sutton. Let the X's in the diagram below represent the 9 seated professors and let the underscores represent possible placements for Professor Sutton.

__ X __ X __ X __ X __ X __ X __ X __ X __ X __

Of these 10 possible locations for Professor Sutton, only 8 of them can be used **(2 pts)**, since two of them will necessarily be adjacent to Professor Merriman. Thus, the total number of possible seating arrangements is **8(9!). (1 pt)** (Note: 10! – 2(9!) = 9!(10 – 2) = 8(9!).)

(c) (5 pts)
There are four choices for each "move" made. Since each choice is independent of the previous one, we simply multiply the number of possibilities at each juncture, obtaining $\mathbf{4^8}$ possible paths. **(Grading: 2 pts for each part of the answer, 1 pt for the explanation.)**

5) (PRF) Relations

Let $A = \{10, 11, 12, 13, 14, 20, 21, 22, 23, 24\}$. Define the following relation R on A:

$R = \{(x, y) \mid$ the sum of the digits in x equals the sum of the digits in $y\}$.

(a) (9 pts) Show that R is an equivalence relation.

(b) (6 pts) Find the partition A/R.

Solution.

(a) (9 pts)

R is reflexive, because for any $x \in A$, the sum of the digits in x is equal to the sum of the digits in x, so $(x, x) \in R$. **(3 pts)**

R is symmetric, because if $(x, y) \in R$, then the sum of the digits in x equals the sum of the digits in y, by the definition of R. But this means that the sum of the digits in y equals the sum of the digits in x as well, that is, $(y, x) \in R$. **(3 pts)**

To show that R is transitive, let $(x, y) \in R$ and $(y, z) \in R$. We need to show that $(x, z) \in R$. By the definition of R, $(x, y) \in R$ implies that the sum of the digits in x equals the sum of the digits in y, and $(y, z) \in R$ implies that the sum of the digits in y equals the sum of the digits in z. From these two equalities we know that the sum of the digits in x equals the sum of the digits in z, that is, $(x, z) \in R$. **(3 pts)**

(b) **(6 pts)**

$A/R = \{ \{10\},$
$\{11, 20\},$
$\{12, 21\},$
$\{13, 22\},$
$\{14, 23\},$
$\{24\} \}.$

(1 pt for each of the six equivalence classes)

6) (PRF) Functions

(15 pts) Let $f(x) = \dfrac{3x}{x+6}$, for all real $x \geq 1$. Find $f^{-1}(x)$ and state the domain and range of $f^{-1}(x)$.

Solution.

Solve for *x* in the given function.

$$f(x) = \frac{3x}{x+6} = \frac{3}{1+\frac{6}{x}} \quad \textbf{(1 pt)}$$

$$1+\frac{6}{x} = \frac{3}{f(x)} \qquad \textbf{(2 pts)}$$

$$\frac{6}{x} = \frac{3}{f(x)} - 1 = \frac{3-f(x)}{f(x)} \quad \textbf{(3 pts)}$$

$$x = \frac{6f(x)}{3-f(x)} \qquad \textbf{(3 pts)}$$

Thus, $f^{-1}(x) = \dfrac{6x}{3-x}$. **(2 pts)**

Domain of $f^{-1}(x)$ is all real *x* in the interval $\dfrac{3}{7} \leq x < 3$. **(2 pts)**

Range of $f^{-1}(x)$ is all real $x \geq 1$. **(2 pts)**

7) (NTH) Number Theory (15 pts)

Find gcd(546, 495) and show that gcd(546, 495) can be represented as a linear combination of 546 and 495. (In other words, find integers *x* and *y* such that 546*x* + 495*y* = gcd(546, 495).)

Solution.

Method 1:

a	b	r	q
1	0	546	
0	1	495	1
1	-1	51	9
-9	10	36	1
10	-11	15	2
-29	32	6	2
68	-75	3	done

(1 pt for each of the first two rows of the table and 2 pts for each of the last five rows, for a total of 12 pts).

Therefore $\gcd(546,495) = 3$. **(1 pt)**
We can write $\gcd(546,495) = 546(68) + 495(-75)$. **(2 pts)**

Method 2:
546 = 495 * 1 + 51
495 = 51 * 9 + 36
51 = 36 * 1 + 15
36 = 15 * 2 + 6
15 = 6 * 2 + 3
6 = 3 * 2 + 0

Thus, gcd(546,495) = 3. **(7 pts – please assign partial credit, 1 pt off for each arithmetic error)**

3 = 15 - 6 * 2
 = 15 - (36 - 15 * 2) * 2
 = 15 * 5 - 36 * 2
 = (51 - 36 * 1) * 5 - 36 * 2
 = 51 * 5 - 36 * 7
 = 51 * 5 - (495 - 51 * 9) * 7
 = 51 * 68 - 495 * 7
 = (546 - 495 * 1) * 68 - 495 * 7
 = 546 * 68 – 495 * 75

(8 pts – same here, please assign partial credit, 1 pt off per error)