Computer Science Foundation Exam

December 19, 2003

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: ________________________________

SSN: _________________________________

In this section of the exam, there are two (2) problems.

You must do both of them.

Each counts for 25% of the Discrete Structures exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: **Work both of the following problems (1 and 2).**

1) The Lucas numbers, are defined as follows:

$L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$, for all integers $n > 2$. (Thus, the third Lucas number, $L_3=4$ and the fourth Lucas number $L_4 = 7$.)

Prove that the following equation is true for all positive integers n, using induction on n:

$$\sum_{i=1}^{n} L_i^2 = L_n L_{n+1} - 2$$

2) Given arbitrary sets A, B, and C chosen from the universe of integers, prove or disprove the following two assertions. (Note: In order to disprove an assertion, please give a single counter-example to the assertion.)

a) If $A \cap C = \emptyset$, then $A - B - C = A - (B - C)$

b) If $A - B - C = A - (B - C)$ then $A \cap C = \emptyset$.

Solution to Problem 1:

The Lucas numbers, are defined as follows:

$L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$, for all integers $n > 2$. (Thus, the third Lucas number, $L_3=4$ and the fourth Lucas number $L_4 = 7$.)

Prove that the following equation is true for all positive integers n, using induction on n:

$$\sum_{i=1}^{n} L_i^2 = L_n L_{n+1} - 2$$

Use induction on n to prove the given statement.

Base case: n=1 The left-hand side of the equation evaluates to $L_1^2 = 1^2 = 1$
The right-hand side of the equation evaluates to $L_1L_2 - 2 = 1(3)-2 = 1$

Thus, the statement holds for n=1

Inductive hypothesis: Assume for an arbitrary positive integer n=k, that

$$\sum_{i=1}^{k} L_i^2 = L_k L_{k+1} - 2$$

Inductive step: Prove for n=k+1, that $\sum_{i=1}^{k+1} L_i^2 = L_{k+1} L_{k+2} - 2$

$$\sum_{i=1}^{k+1} L_i^2 = \sum_{i=1}^{k} L_i^2 + L_{k+1}^2$$

$= L_k L_{k+1} - 2 + L_{k+1}^2$, **using the inductive hypothesis**
$= L_k L_{k+1} + L_{k+1}^2 - 2$
$= L_{k+1}(L_k + L_{k+1}) - 2$
$= L_{k+1}(L_{k+2}) - 2$, **using the definition of Lucas numbers**

Grading: 3 pts for the base case, IH, and IS, each. 3 pts for splitting the sum correctly, 3 pts for invoking the IH, 4 points for factoring out L_{k+1}, and 4 points for completing the proof, that leaves 2 floating points to give or take away based on other issues.

Solution to Problem 2:

Given arbitrary sets A, B, and C chosen from the universe of integers, prove or disprove the following two assertions. (Note: In order to disprove an assertion, please give a single counter-example to the assertion.)

a) If $A \cap C = \varnothing$, then $A - B - C = A - (B - C)$

b) If $A - B - C = A - (B - C)$ then $A \cap C = \varnothing$.

a) In order to show that $A - B - C = A - (B - C)$, we must prove two things:

(1) $A - B - C \subseteq A - (B - C)$, and
(2) $A - (B - C) \subseteq A - B - C$

In order to show (1), we must prove for an arbitrarily chosen integer x, if $x \in A-B-C$, then $x \in A-(B-C)$.

By definition of set difference, if $x \in A-B-C$, then $x \in A-B$ and $x \notin C$. We can break this down further and deduce that $x \in A$, and $x \notin B$ as well. Since $x \notin B$, it follows that $x \notin B-C$, due to the definition of set difference. (All elements of B-C must be elements of B.) Since we have that $x \in A$ and $x \notin B-C$, it follows that $x \in A-(B-C)$.

To show (2), we must prove for an arbitrarily chosen integer x, if $x \in A-(B-C)$, then $x \in A-B-C$.

By definition of set difference, if $x \in A-(B-C)$, then $x \in A$ and $x \notin B-C$. In order for the latter to occur, either $x \in B$ and $x \in C$ OR $x \notin B$. The first can not occur, since the given information states that $A \cap C = \varnothing$ and we know that $x \in A$. Thus if follows that $x \notin B$, (and as already stated $x \notin C$.) Using these pieces of information ($x \in A$, $x \notin B$, and $x \notin C$), we have by definition of set difference that $x \in A-B-C$.

b) We will prove the following statement using proof by contradiction. Assume to the contrary of what is being proved that $A \cap C \neq \varnothing$. Thus, there exists an integer x such that $x \in A \cap C$. In this situation, $x \notin A-B-C$. (We have shown for this to occur, $x \in A$, $x \notin B$, and $x \notin C$ must be true.) However, given that $x \in C$, we must have that $x \notin B-C$. Finally, since $x \in A$, we can conclude by definition that $x \in A-(B-C)$. Thus, we have contradicted our premise that that $A - B - C = A - (B - C)$. Thus, the only logical conclusion is that the initial assumption was faulty. Thus, $A \cap C = \varnothing$.

Grading: part a(15 pts), part b(10 pts). For part A assign 7 points for each subset relation that must be proven, and 1 floating point. Assign the points for part B as you see fit.

Computer Science Foundation Exam

December 19, 2003

Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: ______________________________

SSN: ______________________________

In this section of the exam, there are four (4) problems.

You must do two (2) of them.

Each counts for 25% of the Discrete Structures exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

PART B: Work any two of the following problems (3 through 6).

3) Relations

a) Show that a relation R over the integers ($R \subseteq Z \times Z$) defined as $R=\{(a,b) \mid a^2=b^2$ and $a,b \in Z\}$ is an equivalence relation.

In order to show that R is an equivalence relation, we must show that it is reflexive, symmetric and transitive.

First we show that R is reflexive. We must prove that for all integers a, $(a,a) \in R$. Since $a^2 = a^2$ for integers a, it does follows that $(a,a) \in R$ for all integers a and that R is reflexive.

To show that R is symmetric, we must prove that if $(a,b) \in R$, then $(b,a) \in R$. To prove this statement, start with the assumption $(a,b) \in R$. By definition of R it follows that $a^2 = b^2$ and because equality is commutative, we have that $b^2 = a^2$, which means that $(b,a) \in R$, as desired. Thus R is symmetric.

Finally, we must prove that R is transitive. We must show for any integers a, b and c, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$. To prove this statement, start with the assumptions $(a,b) \in R$ and $(b,c) \in R$, to yield that $a^2 = b^2$ and $b^2 = c^2$. Substitute the value of b^2 in the second equation into the first to yield $a^2 = c^2$. It then follows by definition of R that $(a,c) \in R$, proving that R is transitive and a equivalence relation.

Grading: 2 pts for stating the elements that must be proven, 5 points for each of the three proofs.

b) Is the relation R given in part (a) a function? Briefly, why or why not?

No. We have that $(1,1) \in R$ and $(1,-1) \in R$, but a function requires each element in the domain to map to exactly one element in the codomain. But here, the element 1 maps to two elements in the codomain, not one.

Grading: 8 pts, 1 or 2 point for having an idea of the defintion of a function but incorrectly applying it. 4 points if they give the example backwards, using (1,1) and (-1,1) for example.

4) Functions

a) Formally define the two properties of a relation $R \subseteq A \times B$ for it to be a function from A to B.

For all $a \in A$, there must exist an ordered pair in R with $(a,b) \in R$. (2 pts)

For all $a \in A$, there must exist an unique $b \in B$ such that $(a,b) \in R$. (2 pts)

b) Define a bijective function from A to B.

A bijective function is both injective (one-to-one) and surjective(onto). An injective funciton is one where no two distinct elements in the domain map to the same element in the codomain. A surjective function is one where each element b in the codomain has at least one matching element a in the domain such that $(a,b) \in R$. (6 pts)

c) Is it possible to have a bijection from the set {1,2,3,4,5} to the set {-2,-1,0,1}? Why or why not?

No. By the pigeonhold principle, we see that if each element in {1,2,3,4,5} must map to an element in {-2,-1,0,1), then there must be at least one element in the codomain that is mapped from at least two elements in the domain, since the cardinality of the domain is larger than that of the codomain. (5 pts)

d) Is it possible to have a bijection from the set of even integers to the set of integers? Why or why not?

Yes. Consider the function f defined as follows: $\{(2x,x) \mid x \in Z\}$. Clearly the domain here is even integers while the codomain is just the set of integers. Furthermore, no two distinct elements in the domain map to the same element in the codomain. (To see this, note that if $f(x)=f(y)$, then $2x=2y$, which infers that $x=y$, proving that the function f above is injective.) Also, for each element in the codomain, there exists an element in the domain that maps to it. For an arbitrary element a in the codomain, we know that $f(2a) = a$. (10 pts)

5) Counting

a) A class has 8 girls and 4 boys. If the class contains 6 sets of identical twins, where each child is indistinguishable from their twin, how many different ways can the class line up to go to recess? (Do not count two configurations as distinct if the only difference between the two is twins swapping spots in line.)

All that is important here is that there are 6 pairs of twins, we are arranging 12 people in line, where 6 pairs are indistinguishable. This the exact same question as computing the number of permutations of a 12 letter word comprised of 6 pairs of letters. Using the formula for permuations with repetitions, we find the answer to be $12!/(2!)^6$. (8 pts - 3 for the 12!, 5 for dividing for repeats)

b) Unfortunately, each day when the class (the same class with 6 pairs of twins described in part A) lines up to go to recess (this is done once a day), if two boys are adjacent to each other in line, they always cause problems. But, the kids also cause problems if they are ever lined up the same way on two separate days. How many possible orders can the class line up in without having any problems?

First we will consider the possible orders of boys and girls, and then we will consider the different valid permutations while only interchanging boys with boys and girls with girls.

Consider laying out the girls with gaps in between as follows:

___ G ___ G ___ G ___ G ___ G ___ G ___ G ___ G ___

We can choose any 4 of the 9 gap(___) locations for the boys. This can be done in ${}_9C_4$ ways.

Now, lets consider the total number of orders for the boys for each of these in ${}_9C_4$ arrangements. There are 4 boys, but 2 pairs of twins. Using the formula for permutations with repetition, we get $4!/(2!)^2$ orders. Now, consider the number of ways the girls can be permuted for each of the ${}_9C_4$ arrangements discussed above. Here have 4 pairs of twins. Applying the same formula, we get $8!/(2!)^4$ permutations.

Multiply these three terms to get the final answer $({}_9C_4)4!8!/2^6$.

Grading: 17 pts, 5 points for each of the three componets of the answer, 2 pts for multiplying them

6) Number Theory

a) Find the quotient and remainder when 139 is divided by 41.

Since 139 = 3(41) + 16, the quotient is 3 and the remainder is 16. (5 pts)

b) Prove that if a and b are integers with $b > 0$, that there exists an unique ordered pair of integers (q, r) such that $a = qb+r$ with $0 \leq r < b$.

First we show that a solution exists without the restriction on r.

To see this, note that $a = 0(b) + a$. Thus, by setting q=0 and r=a, we have a solution (0,a) that satisfies the given equation. (Also note that if a is negative, then (a, a(1-b)) is a solution to the equation such that $r=a(1-b)$ is nonnegative.) (5 pts)

Now, we must show that at least one such solution exists with $0 \leq r < b$. From the side note above, we see that there always exists at least one solution with a positive value of r. Now, we will use proof by contradiction to show that the minimum positive solution for r must be less than b.

Assume to the contrary, that for some integer values of a and b, the minimal non-negative solution for r is greater than b. Let this solution be (q, r) so we have that

$a = bq + r$

Now, consider the following manipulation:

$a = bq + b + r - b$
$a = (q+1)b + (r-b)$

Let $q' = q+1$ and r'r-b. Clearly, (q', r') is another solution to the given equation. Furthermore, since $r \geq b$, it follows that $r' \geq 0$, contradicting the assumption that r was the minimal non-negative solution. Thus it follows that there is at least one solution (q,r) to the given equation for all integers a and b with $b > 0$. Now we must show that this solution is unique. (10 pts)

Assume to the contrary that two distinct solutions (q,r) and (q',r') exist, with $q \neq q'$ or $r \neq r'$. Then we have:

$a = bq + r = bq' + r'$, with $0 \leq r < b$ and $0 \leq r' < b$.
$bq - bq' = r' - r$
$b(q - q') = r' - r$

r' - r must be a multiple of b, but since both are in between 0 and b-1 inclusive, it follows that the left hand side of the equation must equal 0 (the only multiple of b that lies in between -(b-1) and b-1.) But if this is the case, then we have q-q' = 0. This implies q=q' and r=r', contradicting the assumption of the two pairs being distinct. Thus, there is always a single unique solution (q,r) to the equation a = bq + r, where a and b are integers and q and r are integers with $0 \leq r < b$. (5 pts)