Computer Science Foundation Exam

December 17, 2004

Section II A

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: __________________________________

SSN: ___________________________________

In this section of the exam, there are two (2) problems.

You must do both of them.

Each counts for 25% of the Discrete Structures exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and <u>not</u> graded based on the answer alone.

Credit cannot be given when your results are unreadable.

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: **Work both of the following problems (1 and 2).**

1) Induction

You are a scientist working with a strain of bacteria that seems to never die. In particular, you started with 16 bacterial cells on day 1, and by day 2, there were 28 bacterial cells. Following those two days, you noticed the following pattern: all the cells that were alive the previous day continue to be alive the following day and all the cells that were alive two days ago spawn six new copies of themselves. Thus, on day 3, there are 28+6(16) = 124 bacterial cells. (The 28 were the ones alive on day 2, and of those, 16 were alive on day 1, and each of these created six new cells.) Similarly, on day 4, there are 124+6(28) = 292 bacterial cells.

a) Let f(n) represent the number of bacterial cells alive on day n. We are given that f(1) = 16 and f(2) = 28. For all integers n>2, write a recurrence relation that f(n) satisfies. (Note: a recurrence relation is simply an equation that defines the value of a function in terms of the function evaluated at other values. An example of a recurrence relation is t(n) = t(n-1)-t(n-2).)

b) Using strong induction (with two base cases) prove that $f(n) = 4(3)^n + (-2)^{n+1}$.

2) Sets

a) Let A be the set {2, 3, 5, 7} and B be the set {1, 3, 5}. Explicitly list each element in the following sets:

i) AxB
ii) $A \cup B$
iii) $A \cap B$
iv) A - B

b) You are given sets A, B and C from the universe of positive integers such that $A \cap B \subseteq C$ and $C \subseteq A \cup B$.

i) Prove that $C - A \subseteq B - A$.
ii) Show by means of a counter-example, that $B - A \subseteq C - A$, is not always true.

Solution to Problem 1:

You are a scientist working with a strain of bacteria that seems to never die. In particular, you started with 16 bacterial cells on day 1, and by day 2, there were 28 bacterial cells. Following those two days, you noticed the following pattern: all the cells that were alive the previous day continue to be alive the following day and all the cells that were alive two days ago spawn six new copies of themselves. Thus, on day 3, there are 28+6(16) = 124 bacterial cells. (The 28 were the ones alive on day 2, and of those, 16 were alive on day 1, and each of these created six new cells.) Similarly, on day 4, there are 124+6(28) = 292 bacterial cells.

a) Let f(n) represent the number of bacterial cells alive on day n. We are given that f(1) = 16 and f(2) = 28. For all integers n>2, write a recurrence relation that f(n) satisfies. (Note: a recurrence relation is simply an equation that defines the value of a function in terms of the function evaluated at other values. An example of a recurrence relation is t(n) = t(n-1)-t(n-2).)

b) Using strong induction (with two base cases) prove that $f(n) = 4(3)^n + (-2)^{n+1}$.

a) f(n) = f(n-1) + 6f(n-2), since f(n-1) represents the number of cells alive on the previous day, and f(n-2) represents all of those cells alive two days ago that spawned 6 new cells each. (5 pts)

b) Use induction on n.

Base case(s): **n=1 LHS=f(1) = 16, RHS=$4(3)^1+(-2)^{1+1}$ = 12+4 = 16**
n=2 LHS=f(2) = 28, RHS=$4(3)^2+(-2)^{2+1}$ = 36-8 = 28(2 pts)

Inductive hypothesis: Assume for all positive integer values of n ≤ k, where k is an arbitrary positive integer greater than or equal to 3 that $f(n) = 4(3)^n+(-2)^{n+1}$. (2 pts)

Inductive step: For all integers k>1, prove that $f(k+1)= 4(3)^{k+1}+(-2)^{k+2}$. (2 pts)

f(k+1) = f(k) + 6f(k-1), using the recurrence relation from part a. (3 pts)
$= 4(3)^k+(-2)^{k+1} + 6(4(3)^{k-1}+(-2)^k)$, using the inductive hypothesis (3 pts)
$= 4(3)^k+(-2)^{k+1} + 24(3)^{k-1}+6(-2)^k$, (1 pt)
$= 4(3)^k+(-2)^{k+1} + 8(3)^k-3(-2)^{k+1}$, (3 pts)
$= 12(3)^k+(1-3)(-2)^{k+1}$, (2 pts)
$= 4(3)^{k+1}+(-2)(-2)^{k+1}$, (1 pt)
$= 4(3)^{k+1}+(-2)^{k+2}$, completing the proof. (1 pt)

Thus, $f(n) = 4(3)^n + (-2)^{n+1}$, for all positive integer values of n.

Solution to Problem 2:

a) Let A be the set {2, 3, 5, 7} and B be the set {1, 3, 5}. Explicitly list each element in the following sets:

i) AxB
ii) $A \cup B$
iii) $A \cap B$
iv) A - B

b) You are given sets A, B and C from the universe of positive integers such that $A \cap B \subseteq C$ and $C \subseteq A \cup B$.

i) Prove that $C - A \subseteq B - A$.
ii) Show by means of a counter-example, that $B - A \subseteq C - A$, is not always true.

a) i) { (2,1), (2, 3), (2,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5), (7,1), (7,3), (7,5) } (2 pts)
ii) {1, 2, 3, 5, 7} (2 pt)
iii) {3, 5} (2 pts)
iv) {2, 7} (2 pts)

b) i) We must show that if an arbitrary element $x \in C-A$, then $x \in B-A$. (2 pts)

Since $C \subseteq A \cup B$, it follows that $x \in (A \cup B) - A$. (2 pts) But,

$(A \cup B) - A = (A \cup B) \cap \neg A$, by definition of set difference (2 pts)
$= (A \cap \neg A) \cup (B \cap \neg A)$, by the distributive property (2 pts)
$= \varnothing \cup (B - A)$, using the inverse law and defn of set difference (2 pts)
$= B - A$, using the identity law (2 pts)

Thus, we can conclude that $x \in B-A$, as desired.

ii) Let A = {1}, B = {1,2} and C = {1}. In this example, we have $A \cap B=\{1\}$, $A \cup B = \{1,2\}$, so the given conditions hold, but B-A = {2) while $C-A=\varnothing$, so that the former is NOT a subset of the latter. (5 pts)

Note: There could easily be quite different valid solutions to both of these questions, adjust grading for different solutions accordingly.

Computer Science Foundation Exam

December 17, 2004

Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: ______________________________

SSN: _______________________________

In this section of the exam, there are four (4) problems.

You must do two (2) of them.

Each counts for 25% of the Discrete Structures exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone.

Credit cannot be given when your results are unreadable.

PART B: Work any two of the following problems (3 through 6).

3) Relations

a) Let $A = \{1, 2, 3, 4, 5\}$. R is a relation defined on A, (so $R \subseteq A \text{x} A$.) In particular, $R = \{(1, 3), (1, 5), (1, 1), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (2, 2), (4, 4)\}$
Is R (i) reflexive? (ii) symmetric? (iii) transitive? Prove your answers.

b) For natural numbers a and b, define $a\text{R}b$ iff a^2+b is even. Prove that R defines an equivalence relation on N.

a) **(i) No, because $(5,5) \notin R$. (2 pts)**
(ii) Yes, because (3 pts)
$(1, 3) \in R$ and $(3, 1) \in R$.
$(1, 5) \in R$ and $(5, 1) \in R$.
…. etc

(iii) No, because $(5,1) \in R$ and $(1,5) \in R$ but $(5,5) \notin R$. (3 pts)

b)
Reflexive: for any $a \in N$, aRa since $a^2+a=a(a+1)$ is even, as the product of consecutive natural numbers. (4 pts)
Symmetric: If aRb, then a^2+b is even. It follows that either a and b are both even or both odd. If they are both even, b^2+a is the sum of even numbers, hence, even. If they both odd, b^2+a is the sum of odd numbers and, hence again, even. In both cases b^2+a is even, so bRa. (5 pts)
Transitive: If aRb and bRc, then a^2+b and b^2+c are even, so $(a^2+b)+(b^2+c)$ is even; in other words, $(a^2+c)+(b^2+b)$ is even. Since (b^2+b) is even, (a^2+c) is even too; therefore, aRc. (4 pts).

R is equivalence relation since it is Reflexive, Symmetric, and Transitive. (4 pts)

4) Functions

Consider the subset N $\subseteq$ Z and the function f: Z $\rightarrow$ N defined by

$$f(n)=\begin{cases} 1, & if \quad n=0; \\ 2n, & if \quad n>0; \\ 1-2n, & if \quad n<0; \end{cases}$$

(a) Prove that f is injective.
(b) Prove that f is surjective.

(a) Assume that $f(m)=f(n)=k$ for m, $n \in$Z and $k \in$ N.

a. If k=1, then m=n=0 (3 pts)

b. For $k \neq 1$. According to the definition of f, if k is even, then both m and n are positive, while if k is odd, then both m and n are negative. In the former case, we compute that

$$n=\frac{f(n)}{2}=\frac{k}{2}=\frac{f(m)}{2}=m \text{ (3 pts)}$$

and in the latter case, we similarly compute

$$n=\frac{1-f(n)}{2}=\frac{1-k}{2}=\frac{1-f(m)}{2}=m \text{ .(3 pts)}$$

Thus, in any case we find m=n, so f is injective (4 pts).

(b) To see that f is surjective, suppose that $k \in$ N. If k=1, then $k = f(0)$ (3 pts). If k is even and non-zero, then we take $n = k/2$ satisfying $f(n)=k$ (3 pts), while if k is odd, then we take n =(1-k)/2 satisfying $f(n)=k$(3 pts). Thus, in any case we find that there is some $n \in$ N so that $f(n) = k$, and f is indeed surjective (3 pts).

5) Counting

Each part of this question will involve the following triangular grid of numbers:

```
Row 0            1
Row 1           2 3
Row 2          4 5 6
Row 3         7 8 9 10
...
Row 100
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Note: Row 0 of the grid contains 1. For all integer values of k>0, Row k of the grid is formed by placing the next k+1 consecutive integers directly below the previous row, so that the first and last numbers in the row are one before and one spot after, respectively the first and last numbers of the previous row. Also, each number on each row is separated by one space.

a) What will the first and last numbers on Row 100 be?

b) An ant starts on the grid square labeled 1. On each move, the ant must move to the subsequent row and must either move directly to his left or directly to his right. (For example, from grid square 4, the ant can either move to grid square 7 or 8.) How many different paths can the ant take from grid square 1 to Row 6 of the grid?

c) Using the same rules as part b, how many paths can the ant take from grid square 1 to grid square 40?

a) Notice that there is 1 number in row 0, 2 numbers in row 1, 3 numbers in row 2, etc. In essence, the last value on row k is the sum of the positive integers from 1 to k+1. Thus, the last number on row 100 is $\sum_{k=1}^{101} k = \frac{101(102)}{2} = 5151$**.(5 pts) Since there are 101 numbers on this row, the first number on the row was 5051.(3 pts)**

b) The ant has exactly 2 choices of where to move for each step. To get to row 6, the ant must move exactly 6 steps, each of which he has two choices for. Using the multiplication principle, the total number of ways for the ant to get to row 6 is 2^6 **= 64.(5 pts)**

c) Since $\frac{(8)(9)}{2} = 36$**, 36 is the last number on row 7. That means that 40 is the fourth number on row 8.(4 pts) In order to get to 40, the ant must make exactly 8 moves.(1 pt) Furthermore, we can determine that exactly 5 of these moves must be to the left and 3 to the right, since the grid square for 40 is exactly 2 squares to the left of the grid square for 1.(2 pts) (We can see this because the grid square for 41, which is halfway in between 37 and 45 is below the grid square for 1.) Since the ant must make five moves to his left and three to his right, in any order, we must essentially**

count the number of permutations of 5 R's and 3 L's. (Alternatively, we can count the number of ways to choose the three positions out of 8 total positions to move to the right.) This number is $\frac{8!}{5!3!} = 56.$ **(5 pts) (Using the alternative explanation, by definition, we get the answer to be** $\binom{8}{3} = 56$**.)**

6) Number Theory

In each of the following cases, find the greatest common divisor of a and b using Euclidean Algorithm, and express gcd(a,b) in the form ma+nb for suitable integers m and n. (Note: You will only execute the extended Euclidean algorithm once for part a, and then use that result to quickly find solutions for m and n for parts b and c.)

(a) a=93, b =119
(b) a=-93, b=119
(c) a=-93, b=-119

(a)

119=1*93+26
93=3*26+15
26=1*15+11
15=1*11+4
11=2*4+3
4=1*3+1
3=3*1 (7 pts).

gcd(93,119)=1 (3 pts)

1=4-1*3=4-(11-2*4)= 3*4-11=3*(15-11)-11=3*15-4*11=3*15-4*(26-15)=7*15-4*26=7*(93-3*26)-4*26=7*93-25*26=7*93-25*(119-93)= 32*93-25*119 → 1=(32)*93+(-25)*119 (4 pts)
m=32, n=-25, There are infinite (3 pts)

(b)
1=(-32)*(-93)+(-25)*119
m=-32, n=-25 (4 pts)

(c)
1=(-32)*(-93)+(25)*(-119)
m=-32, n=25 (4 pts)

Solution to Problem ____ (Please write in the problem number 3,4,5 or 6 you are solving)