Discrete structure solutions for December 14, 2001 Foundation Exam

Part A

1. Let A, B and C denote sets and suppose that $A - B \subseteq C$ and $A \not\subseteq C$ (A is NOT a subset of C). Prove or disprove that $A \cap B \neq \emptyset$.

Proof by contradiction. Assume that $A - B \subseteq C$ and $A \not\subseteq C$, but $A \cap B = \emptyset$. If A and B are disjoint, then any element of A does not belong to B, which means that $A \subseteq A - B$. Then $A \subseteq A - B$ and $A - B \subseteq C$ imply $A \subseteq C$, in contradiction to $A \not\subseteq C$.
Since the assumption that $A - B \subseteq C$, and $A \not\subseteq C$, but $A \cap B = \emptyset$ results in a contradiction, we can conclude that if $A - B \subseteq C$ and $A \not\subseteq C$, then $A \cap B \neq \emptyset$.

Direct proof. Assume that $A - B \subseteq C$ and $A \not\subseteq C$. To prove that $A \cap B \neq \emptyset$ it is sufficient to show that there exists at least one element that belongs to both sets, A and B.
$A \not\subseteq C$ means that there exists some element $x \in A$ and $x \notin C$. Since $A - B \subseteq C$, we can deduce that $x \in B$, otherwise it would belong to C. So, we found an element $x \in A$ and $x \in B$. This completes the proof that $A \cap B \neq \emptyset$.

2. a) How many distinguishable ways are there to rearrange the letters in the word COMBINATORICS?

There are 13 letters in the word COMBINATORICS, including three duplicates, two C's, two O's and two I's. So, the total number of arrangements is $13!/(2!)^3$.

b) How many distinguishable arrangements are possible with the restriction that all vowels ("A", "I", "O") are always grouped together to form a contiguous block?

If all five vowels are consecutive, they form a single block. Then first we need to count permutations of the consonants and one block of vowels. Given eight consonants with one duplicate (two C's), we have $9!/2!$. But every arrangement of consonants and the block of vowels can be combined with any permutation of vowels inside the block. For five vowels including two duplicates we have $5!/(2!)^2$ possible permutations inside the block. Then by the product rule we get the answer: $(9! \times 5!)/(2!)^3$.

c) How many distinguishable arrangements are possible with the restriction that all vowels are alphabetically ordered and all consonants are alphabetically ordered? For example: BACICINOONRST is one such arrangement.

Any arrangement is completely defined by specifying which 5 of 13 positions should be occupied by vowels (or equivalently which 8 out of 13 should be occupied by consonants). So we just need to count the number of ways to select 5 positions out of 13 (or equivalently 8 positions out of 13), that is $13!/(8!5!)$. Given any such selection, both consonants and vowels are distributed alphabetically into assigned slots.

Part B

1. Use mathematical induction to prove that

$$\frac{1}{2}+\frac{2}{3}+\ldots+\frac{\boldsymbol{n}}{\boldsymbol{n}+1}<\frac{\boldsymbol{n}^2}{\boldsymbol{n}+1}, \text{ for all } n \geq 2.$$

Use induction on n^{3} 2.

<u>Base Case</u>: n=2. LHS = 1/2 + 2/3 = 7/6
RHS = 2^2/3 = 4/3
Since LHS < RHS, the base case is true.

<u>Inductive Hypothesis</u>: Assume for an arbitrary value of n=k, k ^{3} 2 that

$$\sum_{i=1}^{k}\frac{i}{i+1}<\frac{k^2}{k+1}$$

<u>Inductive Step</u>: Under this assumption, we must show for n=k+1 that

$$\sum_{i=1}^{k+1}\frac{i}{i+1}<\frac{(k+1)^2}{(k+1)+1}$$

$$\sum_{i=1}^{k+1}\frac{i}{i+1}=\sum_{i=1}^{k}\frac{i}{i+1}+\frac{k+1}{k+2}$$

$$<\frac{k^2}{k+1}+\frac{k+1}{k+2}$$

$$=\frac{k^2(k+2)}{(k+1)(k+2)}+\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k^3+2k^2}{(k+1)(k+2)}+\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{k^3+3k^2+2k+1}{(k+1)(k+2)}$$

$$<\frac{k^3+3k^2+3k+1}{(k+1)(k+2)}$$ **, since k is positive**

$$= \frac{(k+1)^3}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)+1}$$ **, completing the induction.**

2. Let $f : A \rightarrow A$ be a function from some nonempty set A to itself. In the following propositions x and y are variables ranging over A and g is a variable ranging over functions from A to A. Circle all propositions that are equivalent to the proposition that f is one-to-one.

a) $(x = y) \vee [f(x) \neq f(y)]$
b) $(x = y) \rightarrow [f(x) = f(y)]$
c) $(x \neq y) \rightarrow [f(x) \neq f(y)]$
d) $[f(x) = f(y)] \rightarrow (x = y)$
e) $\neg[\exists x\, \exists y\, ((x \neq y) \wedge (f(x) = f(y))]$
f) $\exists g\, \forall x\, [g(f(x))=x]$
g) $\exists g\, \forall x\, [f(g(x))=x]$
h) $\neg[\exists y\, \forall x\, (f(x) \neq y)]$

Solution: a, c, d, e, f

b) doesn't work because the function *f*(*x*) = 1 satisfies the definition for b), but clearly isn't an injection.

g) does not work. Let *A*=Z, and let *f*(*x*) = 2*x*. Since no *x* exists such that *f*(*x*) = 1, it is impossible for *f*(*g*(1)) = 1. Thus, we have found an injective function *f*, which does NOT satisfy the definition g.

h) does not work. Use the same exact counterexample given above. For all *x*, *f*(*x*) ¹ 1. Thus, the function *f*(*x*) = 2*x* is injective over the integers, but does not satisfy the definition in h.

3. Let R denote a relation on a set A, i.e. $R \subseteq A \times A$. Assume that R is symmetric and transitive.

a) Prove or disprove that R is reflexive.
b) Prove or disprove that $R \circ R \subseteq R$.

a) DISPROVE. R is not necessarily reflexive. Let *A* = {1,2} and *R* = Æ. *R* is NOT reflexive, but is certainly symmetric and transitive.

b) True. This follows from the fact that *R* is transitive.

We must prove that if (*x*, *y*) Î *R*o*R* then (*x*, *y*) Î R.

Assume for an arbitrary *x* and *y* that (*x*, *y*) Î *R*o*R*. Then there must exist an element *z* Î *A* such that (*x*, *z*) Î *R* AND (*z*, *y*) Î *R*, by definition of relation composition.

But, we also know that R is transitive. Since this is the case and (*x*, *z*) Î *R* and (*z*, *y*) Î *R*, it follows that (*x*, *y*) Î *R*, as desired.

4. a) Use Euclid algorithm to find gcd(1028, −34).

$1028 = -30 \times (-34) + 8$
$-34 = -5 \times 8 + 6$
$8 = 1 \times 6 + 2$
$6 = 3 \times 2$

$\gcd(1028, -34) = 2$

b) Using your work from part (a) or otherwise, find a pair of integers m and n such that $1028m - 34n = \gcd(1028, -34)$.

Use the first three equations from part (a) as follows:

$8 - 6 = 2$
$-34 + 5 \times 8 = 6$
$1028 + 30 \times (-34) = 8$
Now, substitute for 8 and 6 in the first equation with the values from the second and third:

$[1028 + 30 \times (-34)] - [-34 + 5 \times (8)] = 2$
$1028 + 30 \times (-34) - 1(-34) - 5 \times (1028 + 30 \times (-34)) = 2$
$1028 + 29 \times (-34) - 5 \times (1028) - 150 \times (-34) = 2$
$1028 \times (-4) - 34 \times (-121) = 2$

A pair of integers *m* and *n* that satisfy the given equation are $m = -4$ and $n = -121$.