

## FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

### PART A: Work both of the following problems (1 and 2).

1. Let  $A$ ,  $B$ , and  $C$  denote three arbitrary sets. If  $A \subset B$  (that is,  $A$  is a subset of  $B$  or  $A$  is equal to  $B$ ), prove that  $(C - B) \cap A = \emptyset$ .

**(Solution One)** We prove the result by contradiction.

That is, assume  $x \in (C - B) \cap A$  --- (1) for some  $x$ , we prove this leads to a contradiction.

From (1),  $x \in (C - B)$  --- (2) and  $x \in A$  --- (3), by the definition of  $\cap$ .

Thus, (2) implies  $x \in C$  and  $x \notin B$  --- (4), by the definition of set difference.

However, (3) implies  $x \in B$  because  $A \subset B$  by assumption, which contradicts to (4).

**(Solution Two)** We prove the result directly using laws (as much as possible).

$$(C - B) \cap A = (C \cap \neg B) \cap A, \text{ by the definition of set difference}$$

$$= C \cap ((\neg B) \cap A) \text{ --- (1), by the associative law.}$$

Since  $A \subset B$  by assumption, so

$$((\neg B) \cap A) \subset ((\neg B) \cap B), \text{ by the definition of } \cap$$

$$= \emptyset, \text{ by the law } ((\neg X) \cap X) = \emptyset$$

Thus,  $((\neg B) \cap A) = \emptyset$  --- (2), since the only subset of  $\emptyset$  is itself.

Substituting (2) into (1) yields

$$(C - B) \cap A = C \cap \emptyset$$

$$= \emptyset, \text{ by the law } X \cap \emptyset = \emptyset.$$

Thus, the proof is completed.

**(Solution Three)** This proof is very similar to that of Solution Two.

$$(C - B) \cap A = (C \cap \neg B) \cap A, \text{ by the definition of set difference}$$

$$= C \cap ((\neg B) \cap A) \text{ --- (1), by the associative law.}$$

We now claim that  $((\neg B) \cap A) = \emptyset$  --- (2), proved by the method of contradiction.

That is, suppose there exists  $x \in ((\neg B) \cap A)$  --- (3), we show this leads to a contradiction.

From (3),  $x \in (\neg B)$  --- (4) and  $x \in A$  --- (5), by the definition of  $\cap$ .

Since  $A \subset B$  by assumption, so (5) implies  $x \in B$ , which contradicts to (4). This contradiction proves (2).

Substituting (2) into (1) yields

$$(C - B) \cap A = C \cap \emptyset$$

$$= \emptyset, \text{ by the law } X \cap \emptyset = \emptyset.$$

Thus, the proof is completed.

2. Prove the induction step in an induction proof for the following summation result:

$$\sum_{i=2}^n \frac{1}{i^2} < 1 - \frac{1}{n}, \text{ for all integer } n \geq 2.$$

That is, state the induction hypothesis precisely, then prove the induction step based on the induction hypothesis.

**Note: In the following, we provide a complete induction proof even though the Basis Step is not required.**

We use induction on  $n \geq 2$ .

(Basis Step) Consider  $n = 2$ . In this case,

$$\text{The LHS} = \sum_{i=2}^2 \frac{1}{i^2} = \frac{1}{2^2} = \frac{1}{4}.$$

$$\text{The RHS} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Thus, LHS < RHS, so the Basis Step is proved.

(Induction Hypothesis) Consider  $n = k$ . Assume the following is true:

$$\sum_{i=2}^k \frac{1}{i^2} < 1 - \frac{1}{k}, \text{ for some integer } k \geq 2.$$

(Induction Step) Consider  $n = k + 1$ . We need to prove

$$\sum_{i=2}^{k+1} \frac{1}{i^2} < 1 - \frac{1}{k+1} \text{ --- (1)}.$$

Note that

$$\begin{aligned} \text{The LHS of (1)} &= \sum_{i=2}^k \frac{1}{i^2} + \frac{1}{(k+1)^2}, \text{ by the definition of summation} \\ &< \left(1 - \frac{1}{k}\right) + \frac{1}{(k+1)^2}, \text{ by the Induction Hypothesis} \\ &= 1 - \frac{(k+1)^2 - k}{k(k+1)^2} = 1 - \frac{k^2 + k + 1}{k(k+1)^2} \\ &< 1 - \frac{k^2 + k}{k(k+1)^2}, \text{ since } -(k^2 + k + 1) < -(k^2 + k) \\ &= 1 - \frac{k(k+1)}{k(k+1)^2} = 1 - \frac{1}{k+1} \\ &= \text{RHS of (1)}. \end{aligned}$$

Thus, (1) is proved, so the Induction Step is proved.

By induction we proved that

$$\sum_{i=2}^n \frac{1}{i^2} < 1 - \frac{1}{n}, \text{ for all integer } n \geq 2.$$

**PART B: Work any two of the following problems (3 through 6).**

3. Verify the identity:

$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$  where  $\binom{n}{k}$  is the usual binomial coefficient for  $n$  and  $k$  both positive integers with  $k < n$ .

**Hint:** Two approaches are possible and you may use either. The first is to calculate using the definition of the binomial coefficient. The second is to appeal directly (and less formally) to the interpretation of  $\binom{n}{k}$  as the number of ways of choosing  $k$  objects from  $n$  objects.

**Preferred Solution:** Think of having  $n+1$  golf balls consisting of 1 red ball and  $n$  white balls. To choose  $k+1$  of these  $n+1$  we can do either of the following:

- i. choose the red golfball and choose  $k$  of the  $n$  white golfballs
- ii. don't choose the red one and choose  $k+1$  of the white balls.

The first choice (i) can be made in  $\binom{n}{k}$  ways and the second choice (ii) can be made in  $\binom{n}{k+1}$  ways. The set of all ways to choose golfballs using (i) contains only choices that include the red golfball. The set of all ways to choose golfballs using (ii) contains only choices that exclude the red golfball. Thus, these two sets of choices are disjoint and it follows that the total number of ways to choose  $k+1$  of  $n$  is given by

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1} \text{ as required.}$$

**Alternative Solution:** By definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

To simplify the equations we divide  $(n-k)!$  into  $n!$  in the above equation and are left in the numerator with the largest  $k$  factors of  $n!$  which we will define as

$$n_k = n(n-1)\dots(n-k+1).$$

Now we write more simply,

$$\binom{n}{k} = \frac{n_k}{k!}.$$

Using this notation the required result becomes

$$\frac{(n+1)_{k+1}}{(k+1)!} = \frac{n_k}{k!} + \frac{n_{k+1}}{(k+1)!}.$$

By algebraic manipulation the Right-Hand-Side becomes

$$\begin{aligned} \frac{(k+1)n_k + n_{k+1}}{(k+1)!} &= \frac{(k+1)n_k + n_k(n-k)}{(k+1)!} = \frac{kn_k + n_k + nn_k - kn_k}{(k+1)!} \\ &= \frac{(n+1)n_k}{(k+1)!} = \frac{(n+1)_{k+1}}{(k+1)!} = \binom{n+1}{k+1}, \text{ which was to be shown.} \end{aligned}$$

4. (a) How many arrangements of the word COMPUTER contain the word MOP and do not contain the word RUT? (b) How many arrangements of the word COMPUTER contain the word MOP and do not contain the word PUT? **Justify your answers.**

**Solution:** (a) Let  $M$  = the set of all arrangements containing MOP.

Let  $R$  = the set of all arrangements containing RUT.

Because  $M = (M \cap R) \cup (M \cap \bar{R})$  is a disjoint union, we have

$$|M| = |M \cap R| + |M \cap \bar{R}|.$$

It follows that

$$|M \cap \bar{R}| = |M| - |M \cap R|. \quad (1)$$

To find  $|M|$ , think of “MOP” as a single letter. There are 6 “letters” to arrange including C, U, T, E, R, “MOP.” Thus,  $|M| = 6!$ . Similarly, to find  $|M \cap R|$  think of “MOP” as a single letter and “RUT” as a single letter. There are 4 “letters” to arrange including C, E, “MOP,” and “RUT.” Thus,  $|M \cap R| = 4!$ . The answer we seek is  $|M \cap \bar{R}| = 6! - 4! = 696$  (follows from (1)).

(b) As in (a), if  $P$  = set of all arrangements containing PUT, then

$$|M \cap \bar{P}| = |M| - |M \cap P|.$$

We still have  $|M| = 6!$ ; however, to find  $|M \cap P|$  we think of “MOPUT” as a single letter occurring along with C, E, R in the arrangements. Again we have 4 “letters” so that  $|M \cap P| = 4!$ . The required answer is

$$|M \cap \bar{P}| = |M| - |M \cap P| = 6! - 4! = 696.$$

**NOTE: The only difference between (a) and (b) is in the explanation of how the answer is obtained. No explanation means no full credit.**

5. Let  $A$  denote an arbitrary non-empty set, and let  $L$  denote the relation defined over  $A$  as follows:

$$L = \{(a, a) \mid a \in A\}$$

Suppose  $R$  is a **transitive** relation over  $A$ , that is,  $R \subset A \times A$ , and for all  $x, y, z \in A$ , if  $(x, y) \in R$  and  $(y, z) \in R$  then  $(x, z) \in R$ . Prove that the relation  $R \cup L$  is transitive.

To prove  $R \cup L$  is transitive, let  $(a, b) \in R \cup L$  --- (1) and let  $(b, c) \in R \cup L$  --- (2), we need to prove  $(a, c) \in R \cup L$  --- (3).

From (1),  $(a, b) \in R$  or  $(a, b) \in L$ ; that is  $(a, b) \in R$  or  $a = b$  --- (4), by the definition of relation  $L$ . Similarly, (2) implies  $(b, c) \in R$  or  $(b, c) \in L$ ; that is  $(b, c) \in R$  or  $b = c$  --- (5). Combining (4) and (5), there are the following 4 cases to consider:

(Case 1)  $(a, b) \in R$  and  $(b, c) \in R$ . In this case,

$(a, c) \in R$ , because  $R$  is transitive by assumption

$$\subset R \cup L.$$

(Case 2)  $(a, b) \in R$  and  $b = c$ . In this case,

$$(a, c) = (a, b) \in R \subset R \cup L.$$

(Case 3)  $a = b$  and  $(b, c) \in R$ . In this case,

$$(a, c) = (b, c) \in R \subset R \cup L.$$

(Case 4)  $a = b$  and  $b = c$ . In this case,

$$(a, c) = (c, c) \in L \subset R \cup L.$$

Thus, we proved  $(a, c) \in R \cup L$  in all cases, so (3) is proved.

6. Let  $p, q, r$  denote three propositions (i.e. statements). Prove that the logical expression  $(p \text{ and } q) \Rightarrow r$  is equivalent to the expression  $p \Rightarrow (q \Rightarrow r)$ .

**(Solution One)** Proof using the laws of logic:

$$\begin{aligned} (p \text{ and } q) \Rightarrow r &\equiv \neg(p \text{ and } q) \text{ or } r, \text{ using the law } (x \Rightarrow y) \equiv (\neg x) \text{ or } y \\ &\equiv ((\neg p) \text{ or } (\neg q)) \text{ or } r, \text{ using De Morgan's law} \\ &\equiv (\neg p) \text{ or } ((\neg q) \text{ or } r), \text{ the associative law} \\ &\equiv (\neg p) \text{ or } (q \Rightarrow r), \text{ using the law } (x \Rightarrow y) \equiv (\neg x) \text{ or } y \\ &\equiv p \Rightarrow (q \Rightarrow r), \text{ using the law } (x \Rightarrow y) \equiv (\neg x) \text{ or } y. \end{aligned}$$

**(Solution Two)** Proof using the truth table (we use T for True, F for False):

$p$	$q$	$r$	$p \text{ and } q$	$(p \text{ and } q) \Rightarrow r$	$q \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$
F	F	F	F	T	T	T
F	F	T	F	T	T	T
F	T	F	F	T	F	T
F	T	T	F	T	T	T
T	F	F	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

Note that the column for the expression  $(p \text{ and } q) \Rightarrow r$ , and the column for the expression  $p \Rightarrow (q \Rightarrow r)$ , have identical truth values in all combinations of the truth values for propositions  $p, q$ , and  $r$ . Thus, these two logical expressions are equivalent.