Discrete structure solutions for August, 6, 2001 Foundation Exam

Part A.

1. a) Use Euclid's Algorithm to find the greatest common divisor of 1029 and 602.

1029 = 1x602 + 427
602 = 1x427 + 175
427 = 2x175 + 77
175 = 2x77 + 21
77 = 3x21 + 14
21 = 1x14 + 7
14 = 2x7, so GCD(1029, 602) = 7.

b) Two integers a and b are called relatively prime if gcd(a, b)=1. For example, 35 and 12 are relatively prime because they share no common factors. Show that the number of integers in the set {1, 2, 3, ..., 2^n–1} that are relatively prime to 2^n is 2^{n-1}. (*Hint*: In order for gcd(x, y) ≠ 1, x and y HAVE TO SHARE at least one prime number factor. Similarly, if x and y share no common prime factor gcd(x, y)=1. Before you start this problem, list all the distinct prime factors of 2^n.)

The only prime factor of 2^n is 2 because the number is already written in its prime factorization. Thus, if a number m is not divisible by 2, gcd(2^n, m) = 1, because the two numbers do not share any common factors. (m has a factor of 2 if and only if it is divisible by 2.) By definition, each odd number does not have a factor of 2. So, for all odd numbers m, gcd(2^n, m) = 1. In the set listed above, the odd values are 1, 3, ..., 2^n-1. If we add 1 to all these values, we get the list 2, 4, ..., 2^n. The number of values in this list is the same as the previous list. The number of values in the latter list is $2^n/2 = 2^{n-1}$. This is because we have listed every other value from the list 1, 2, ..., 2^n.

Now, we must also show that all of the other numbers in the set {1, 2, ... 2^n-1} are not relatively prime to 2^n. But, each of these numbers must be even, since we already counted all of the odd values in the original set. The gcd of two even numbers can not be 1, because they share a common factor of 2. Thus, we should not count any of these numbers.

It follows that the number of values from the set {1, 2, ... 2^n-1} that are relatively prime to 2^n is 2^{n-1}. In particular, these values are 1, 3, ..., 2^n-1, all of the odd values in the set.

2. Harmonic numbers H_k , k =1, 2, 3, … are defined by

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}.$$

Use mathematical induction to prove that $H_1 + H_2 + \ldots + H_n = (n+1)H_n - n$ for all $n \geq 1$.

Use induction on n to prove the assertion.

Base case: n=1: LHS = H_1 = 1
RHS = $(1+1)H_1$ - 1 = 2(1) - 1 = 1
Thus, the formula is true for n=1.

Inductive Hypothesis: Assume for an aribtrary value of n=k that

$$H_1 + H_2 + ... + H_k = (k+1)H_k - k$$

Inductive Step: Prove for n=k+1 that

$$H_1 + H_2 + ... + H_{k+1} = ((k+1)+1)H_{k+1} - (k+1) = (k+2)H_{k+1} - (k+1)$$

$$\sum_{i=1}^{k+1} H_i = \sum_{i=1}^{k} H_i + H_{k+1}$$

$= (k+1)H_k - k + H_{k+1}$, using the inductive hypothesis.

$= (k+1)(H_{k+1} - \frac{1}{k+1}) - k + H_{k+1}$, using the definition of harmonic #s.

$= (k+1)H_{k+1} - (k+1) * \frac{1}{k+1} - k + H_{k+1}$

$= (k+2)H_{k+1} - 1 - k$, combining H_{k+1} terms

$= (k+2)H_{k+1} - (k+1)$

This proves the inductive step using the inductive hypothesis. Thus, we can conclude that for all integers n ³ 1, $H_1 + H_2 + \ldots + H_n = (n+1)H_n - n$.

Part B

3. Let R and S be two relations on set A, i.e. $R, S \subseteq A \times A$. Prove or disprove each of the following propositions.

a) If both R and S are symmetric, then $(R - S) \subseteq A \times A$ is symmetric.

The proposition is true, so it should be proved.

__Proof.__ Let *R* and *S* be symmetric relations on *A*. To prove that *R* - *S* is symmetric we must show for any ordered pair (x,y), if (*x*, *y*) Î *R* - *S*, where *x* and *y* are distinct elements of *A*, then (*y*, *x*) Î *R* - *S*. The assumption (*x*, *y*) Î *R* - *S* implies that (*x*, *y*) Î *R* and (*x*, *y*) Ï *S* by the definition of set difference. Then (*y*, *x*) Î *R* by the symmetric property of *R*. Since we have that (x,y) Ï *S*, and S is symmetric, it is impossible for (y,x) Î S, since if (y,x) Î S, the symmetry of S would imply (x,y) Î S. Thus, we must have that (y,x) Ï *S*. (*y*, *x*) Î *R* and (*y*, *x*) Ï *S* imply that (*y*, *x*) Î *R* - *S*. Thus we showed that if (*x*, *y*) Î *R* - *S* then (*y*, *x*) Î *R* - *S*, i.e. *R* - *S* is symmetric.

b) If both R and S are transitive, then $(R - S) \subseteq A \times A$ is transitive.

The proposition is false and can be disproved by the following counterexample.

Let *A* = {1, 2, 3}, *R* ={(1, 2), (2, 3), (1, 3)} and *S* = {(1, 3)}, so *R* and *S* are transitive relations on *A*. But *R* - *S* ={ (1, 2), (2, 3)} is not transitive.

4. Suppose $A \subseteq B - C$ and $A \neq \varnothing$.

a) Prove or disprove that B can not be a subset of C.

The proposition is true, i.e. if *A* Í *B* - *C* and *A* ¹ Æ then *B* cannot be a subset of *C*.

Since A is not empty, it must contain at least one element. Let this be x. Using the given information that A Í B - C , it follows that xÎ B - C, by the definition of subset. Thus, xÎ B and xÏ C, by the definition of set difference. But the existence of such an element x that is in B but not C shows that B Í C is impossible as desired.

b) Prove or disprove that $|B| > |C|$.

It is not always the case that | *B* | > | *C* |, so the general proposition is false. The following counterexample is sufficient to disprove it. Let *B* = {1, 2, 3}, *C* = {3, 4, 5, 6}, and *A* = {1}. Then *B* - *C* ={1, 2}, *A* Í *B* - *C* and *A* ¹ Æ, but | *B* | < | *C* |.

5. How many 6-letter words can be formed by ordering the letters *ABCDEF* if *A* appears before *C* and *E* appears before *C*?

Under given restrictions there are two possible arrangements for letters *A*, *C* and *E* between themselves: either *A* appears before *E* , or *E* before *A*, i.e. *AEC* or *EAC*, so we have two choices for this task. After that we can choose 3 slots to place letters *A*, *C* and *E* out of 6 possible slots in a 6-letter word. If the order of *A*, *C* and *E* is fixed, we count *C* (6, 3) selections. After we fill 3 slots with the letters *A*, *C* and *E*, we can make 3! permutations of the letters *B*, *D* and *F* using remaining 3 slots. By the product rule the total number of orderings will be $2 \times C(6, 3) \times 3! = 2 \times 6 \times 5 \times 4 = 240$.

6. Let $x_1, x_2, \ldots, x_{2n}$ be boolean variables, where n is a positive integer. Consider the following boolean expression $Y_n = (x_1 \wedge x_2) \vee (x_3 \wedge x_4) \vee (x_5 \wedge x_6) \vee \ldots \vee (x_{2n-1} \wedge x_{2n})$. An assignment of variables that makes the boolean expression true is known as a satisfying assignment. For example, x_1 = True, x_2 = True, ... x_{2n} = True, is a satisfying assignment of the boolean expression Y_n. For any positive integer n, there are $4^n - 3^n$ satisfying assignments of Y_n. Prove this result, either through a counting argument or induction on n. (Hint: There are a total of 2^{2n} total possible assignments because there are 2n variables, each of which can take two different values. Let S_n equal the total number of satisfying assignments of Y_n, and T_n equal the total number of assignments that do NOT satisfy Y_n. Thus, $S_n + T_n = 2^{2n}$. If you are using induction it will be useful to show that $S_{n+1} = 4S_n + T_n$.)

Counting argument:

Let us count the non-satisfying assignments of the given expression. For an assignment to be non-satisfying, each clause must be false. Out of 4 possible truth assignments to variables x_1 and x_2, three make x_1 Ù x_2 false. (The ordered pairs (x_1, x_2) are (F,F), (F,T) and (T,F).) Similarly, for each of the n indepedent clauses, three possible truth assignments make the clause false. Since these are each independent clauses, (the value of one does not affect the value of another), use the product rule to find the total number of non-satisfying assignments is 3^n, (3x3...x3, n times for the n clauses.)

The total number of assignments for the clauses stated in the hint is 2^{2n}. This is equal to $(2^2)^n = 4^n$. To get the total number of satisfying assignments, we can subtract the value above from the total number of assignments.Thus the total number of satisfying assignments is $4^n - 3^n$.

Induction:

Using the hint, let S_n equal the total number of satisfying assignments of Y_n, and T_n equal the total number of assignments that do NOT satisfy Y_n.

First we will show that $S_{n+1} = 4S_n + T_n$.

Consider counting the satisfying assignments of Y_{n+1}. Clearly, any satisfying assignment of Y_n will lead to a corresponding satisfying assignment of Y_{n+1},

regardless of what x_{2n+1} and x_{2n+2} are assigned. (This is because only one clause has to be true to make all of Y_{n+1} true.) Thus, for each satisfying assignment of Y_n, there are four corresponding satisfying assignments of Y_{n+1}, or $4S_n$ of these. Now, we must try to count other satisfying assignments. The only ones we haven't considered are assignments that correspond to assignments that do not satisfy Y_n. For each of these, we must set both x_{2n+1} and x_{2n+2} to true in order to satisfy Y_{n+1}. Thus, for each non-satisfying assignment of Y_n, there is exactly one satisfying assignment of Y_{n+1}. There are T_n of these. Since we have counted all satisfying assignments of Y_{n+1} exactly once, it follows that S_{n+1} is the sum of the two calculated values or $4S_n + T_n$.

Now, our goal is to prove that $S_n = 4^n - 3^n$ using induction on n.

Base case: n=1,
LHS = S_1 = 1, since there is only one satisfying assignment of x_1 Ù x_2.
RHS = $4^1 - 3^1 = 1$
So, the formula is true for n=1.

Inductive hypothesis: Assume that for an arbitrary positive integer value of n=k that $S_k = 4^k - 3^k$.

Inductive step: We need to prove for n=k+1 that $S_{k+1} = 4^{k+1} - 3^{k+1}$.

$S_{k+1} = 4S_k + T_k$, using the result proved above.
$= 4S_k + (2^{2k} - S_k)$, since $T_k + S_k = 2^{2k}$.
$= 2^{2k} + 3S_k$
$= 4^k + 3(4^k - 3^k)$, using the inductive hypothesis
$= 4^k + 3(4^k) - 3(3^k)$
$= (1+3)(4^k) - 3^{k+1}$
$= 4(4^k) - 3^{k+1}$
$= 4^{k+1} - 3^{k+1}$, completing the inductive step.

Thus, for all positive integers n, $S_n = 4^n - 3^n$, proving that the number of satisfying assignments of Y_n is $4^n - 3^n$.