Computer Science Foundation Exam

August 1, 2003

Section II A Solutions

DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,
and you must work entirely on your own.**

SOLUTION KEY

In this section of the exam, there are two (2) problems.

You must do both of them.

Each counts for 25% of the total exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone

Credit cannot be given when your results are

unreadable.

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: **Work both of the following problems (1 and 2).**

1) The n^{th} Harmonic number, denoted H_n is defined as follows:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

Prove that the following equation is true for all positive integers n, using induction on n:

$$\sum_{i=1}^{n} \frac{i}{i+1} = (n+1) - H_{n+1}$$

2) Given arbitrary sets A, B, and C chosen from the universe of {1, 2, 3, 4, 5}, prove or disprove the following two assertions. (Note: In order to disprove an assertion, please give a single counter-example to the assertion.)

a) If $B \subseteq C$, then $(B - A) \subseteq (C - A)$.

b) If $(B - A) \subseteq (C - A)$, then $B \subseteq C$.

Solution to Problem 1:

The nth Harmonic number, denoted H_n is defined as follows:

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

Prove that the following equation is true for all positive integers n, using induction on n:

$$\sum_{i=1}^{n} \frac{i}{i+1} = (n+1) - H_{n+1}$$

Base case: n=1 LHS = $\sum_{i=1}^{1} \frac{i}{i+1} = \frac{1}{1+1} = \frac{1}{2}$

RHS = $(1+1) - H_{1=1} = 2 - (1 + \frac{1}{2}) = 2 - \frac{3}{2} = \frac{1}{2}$

Thus, the equation holds for n=1.

Inductive hypothesis: Assume for some k > 0, that

$$\sum_{i=1}^{k} \frac{i}{i+1} = (k+1) - H_{k+1}$$

Inductive step: Prove for n=k+1 that

$$\sum_{i=1}^{k+1} \frac{i}{i+1} = ((k+1)+1) - H_{(k+1)+1} = (k+2) - H_{k+2}$$

$$\sum_{i=1}^{k+1} \frac{i}{i+1} = (\sum_{i=1}^{k} \frac{i}{i+1}) + \frac{k+1}{k+2}$$

$$= (k+1) - H_{k+1} + \frac{k+1}{k+2}$$

$$= (k+1) - H_{k+1} + \frac{k+2-1}{k+2}$$

$$= (k+1) - H_{k+1} + \frac{k+2}{k+2} - \frac{1}{k+2}$$

$$= (k+1) - H_{k+1} + 1 - \frac{1}{k+2}$$

$$= (k+2) - (H_{k+1} + \frac{1}{k+2})$$

$= (k+2) - H_{k+2}$, **proving the formula true for all positive integers n.**

Solution to Problem 2:

Given arbitrary sets A, B, and C chosen from the universe of {1, 2, 3, 4, 5}, prove or disprove the following two assertions. (Note: In order to disprove an assertion, please give a single counter-example to the assertion.)

a) If B $\subseteq$ C, then (B - A) $\subseteq$ (C - A).

b) If (B - A) $\subseteq$ (C - A), then B $\subseteq$ C.

a) This statement is true. We must prove that for an arbitrary element x , that if x Î (B - A), then x Î (C-A).

We assume that x Î (B - A), thus, we have x Î B and x Ï A.

Furthermore, we are given that B Í C. By the definition of subset, we can deduce that x Î C.

But, if x Î C and x Ï A, by the definition of set difference, we have x Î C - A as desired.

b) This statement is false. Consider the following counter-example:

A = {1, 2}, B = {1}, and C = {2}.

In this situation, we have B - A = Æ, C - A = Æ so (B - A) Í (C - A), but clearly, B Ë C.

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Section II B Solutions

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: ______________________________

SSN: _______________________________

In this section of the exam, there are four (4) problems.

You must do two (2) of them.

Each counts for 25% of the total exam grade.

You must clearly identify the problems you are solving.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone

Credit cannot be given when your results are

unreadable.

PART B: Work any two of the following problems (3 through 6).

3) Prove that (q ∧ (p → ¬q)) → ¬p is a tautology using the laws of logic. (Note: You may not use truth tables.)

(q Ù **(p → Øq))** **→ Øp**	Û	**(q** Ù **(Ø p** Ú **Øq))** **→ Øp,**	**Defn of implication**
	Û	**(q** Ù **Ø(p** Ù **q))** **→ Øp,**	**DeMorgan's Law**
	Û	**Ø(q** Ù **Ø(p** Ù **q))** Ú **Øp,**	**Defn of implication**
	Û	**(Ø q** Ú **ØØ(p** Ù **q))** Ú **Øp,**	**DeMorgan's Law**
	Û	**(Ø q** Ú **(p** Ù **q))** Ú **Øp,**	**Double Negation**
	Û	**(Øp** Ú **Ø q)** Ú **(p** Ù **q),**	**Commutative (or)**
	Û	**Ø(p** Ù **q)** Ú **(p** Ù **q),**	**DeMorgan's Law**
	Û	**T**	**Inverse Law**

4) The set K contains all three-digit integers from 000 to 999.

Thus, K = { 000, 001, 002, … , 998, 999 } . Let n and m be arbitrary members of K. We will represent the individual digits of n and m as $n = n_1 n_2 n_3$ and $m = m_1 m_2 m_3$.

Let R be a relation on K x K such that:
$R = \{ (n, m) \mid n \in K, m \in K, \wedge\ n_1 + n_2 + n_3 = m_1 + m_2 + m_3 \}$

a) Is R an equivalence relation? Why or why not?

Yes, it is an equivalence relation.

First, we will show that R is reflexive. Given an arbitrary element n of K, we have that $n_1+n_2+n_3 = n_1+n_2+n_3$. It follows for any element n of K, (n,n) Î R, proving R to be reflexive.

Next, we will show that R is symmetric. In order to show this, we must show that if (n,m) Î R, then (m,n) Î R, where m and n are arbitrary elements of K. Assuming that (n,m) Î R, we know that $n_1+n_2+n_3 = m_1+m_2+m_3$. But, clearly, by the definition of equality, we also have that $m_1+m_2+m_3 = n_1+n_2+n_3$. Thus, it follows that (m,n) Î R, and R is reflexive.

Finally, we must show that R is transitive. In order to do this, we must show that if (m,n) Î R and (n,p) Î R, then (m,p) Î R, where m,n and p are arbitrary elements of R. Using the assumed information we have the equations:

$m_1+m_2+m_3 = n_1+n_2+n_3$ and
$n_1+n_2+n_3 = p_1+p_2\ p_3$.

Substitute $p_1+p_2\ p_3$ for $n_1+n_2+n_3$ in the first equation to yield:

$m_1+m_2+m_3 = p_1+p_2+p_3$

From this, we have (m,p) Î R, proving R to be transitive.

Since R is reflexive, symmetric, and transitive, R is an equivalence relation.

b)If R is an equivalence relation, how many equivalence classes are there?

Two elements of K are in the same equivalence class or R if and only if the sum of their digits is equal. The minimum sum of digits of an element of K is 0 while the maximum is 27. All other integer values in between are the sum of some element of K. (This can quickly be seen by inspection.) Thus, there are 27-0+1 = 28 equivalence classes of R.

5) Consider four receptacles (R1, R2, R3, and R4) containing marbles. The marbles are either red, white, or blue but are otherwise indistinguishable.

R1: Has 10 red, 10 white, and 10 blue marbles.
R2: Has 10 red marbles.
R3: Has 10 white marbles.
R4: Has 10 blue marbles.

Marbles are selected from the jars and laid out in a row. (Thus, the order in which the marbles are chosen makes a difference. For example, RWWWBRR is a different order than RRWWWB.) How many linear arrangements can be created under the following circumstances?

a) Seven marbles are chosen, all from R1.

There are 3 choices for each of 7 marbles. Using the multiplication principle, that is 3^7 possible orders.

b) Ten marbles are chosen. The first marble chosen is from R1. Then zero or more marbles are chosen from R2, followed by zero or more marbles form R3, followed by zero or more marbles from R4. The total number of marbles chosen from these last three receptacles must be nine. (For example, WRRRBBBBBB is permissible, while, WRRWRBBBBB is not.)

There are three choices for the first marble.

The following 9 choices are chosen out of three bins, in that order.
Let r be the number of marbles chosen from R2.
Let w be the number of marbles chosen from R3.
Let b be the number of marbles chosen from R4.

We must find the total number of solutions to the equation

r+w+b = 9, where r, w, and b are all non-negative integers.

We are essentially distributing 9 marbles amongst 3 bins. This can be done in $\binom{9+3-1}{3-1} = \binom{11}{2} = 55$ **ways.**

Using the product rule, we find a total of 3(55) = 165 permissible orders.

6) a) Let gcd(x,y) denote the greatest common divisor of integers x and y. Find gcd(481, 592).

592 = 1x481 + 111
481 = 4x111 + 37
111 = 3x37 + 0

gcd(581, 592) = 37.

b) Using work from part a, find integer values of x and y such that 481x + 592y = gcd(481, 592).

Using the equations above, we have:

481 - 4x111 = 37 and
592 - 481 = 111

Substitute for 111 in the first equation to yield

481 - 4x(592 - 481) = 37
5x481 - 4x592 = 37

Thus, integer values of x and y that satisfy the equation above are x=5,y=-4.