Computer Science Foundation Exam

May 5, 2006

Section II A

KEY

DISCRETE STRUCTURES

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: **Work both of the following problems (1 and 2).**

1) (25 pts) (PRF) Using induction on n, prove for all positive integers n that

$$\sum_{i=1}^{n} \frac{i(i+1)(i+2)}{6} = \frac{n(n+1)(n+2)(n+3)}{24}.$$

2) (PRF) Let A, B and C be any three sets. Prove or disprove the following propositions:

a) (5 pts) If $A \subseteq B \cup C$, then either $A \subseteq B$ or $A \subseteq C$.

b) (10 pts) $(A - C) \cap (C - B) = \emptyset$

c) (10 pts) $\text{Power}(A) - \text{Power}(B) \subseteq \text{Power}(A - B)$

Solution to Problem 1:

(PRF) (25 pts) Using induction on n, prove for all positive integers n that

$$\sum_{i=1}^{n} \frac{i(i+1)(i+2)}{6} = \frac{n(n+1)(n+2)(n+3)}{24}.$$

Base case: n=1. LHS = $\sum_{i=1}^{1} \frac{i(i+1)(i+2)}{6} = \frac{1(2)(3)}{6} = 1$ **(2 pts)**

RHS = $\frac{1(2)(3)(4)}{24} = 1$, thus, the statement holds for n=1.

Inductive hypothesis: Assume for an arbitrary positive integer value n=k that

$$\sum_{i=1}^{k} \frac{i(i+1)(i+2)}{6} = \frac{k(k+1)(k+2)(k+3)}{24}$$ **(3 pts)**

Inductive step: Prove for n=k+1 that $\sum_{i=1}^{k+1} \frac{i(i+1)(i+2)}{6} = \frac{(k+1)(k+2)(k+3)(k+4)}{24}$.
(3 pts)

$$\sum_{i=1}^{k+1} \frac{i(i+1)(i+2)}{6} = (\sum_{i=1}^{k} \frac{i(i+1)(i+2}{6}) + \frac{(k+1)(k+2)(k+3)}{6}$$ **(4 pts)**

$$= \frac{k(k+1)(k+2)(k+3)}{24} + \frac{(k+1)(k+2)(k+3)}{6}$$, using the I.H. **(4 pts)**

$$= \frac{k(k+1)(k+2)(k+3)}{24} + \frac{4(k+1)(k+2)(k+3)}{24}$$ **(4 pts)**

$$= \frac{(k+1)(k+2)(k+3)[k+4]}{24}$$, factoring out (k+1)(k+2)(k+3) **(5 pts)**

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{24}$$, as desired.

Solution to Problem 2:

(PRF) Let A, B and C be any three sets. Prove or disprove the following propositions:

a) (5 pts) If $A \subseteq B \cup C$, then either $A \subseteq B$ or $A \subseteq C$.
b) (10 pts) $(A - C) \cap (C - B) = \emptyset$
c) (10 pts) Power(A)–Power(B) $\subseteq$ Power ($A - B$)

a) If $A \subseteq B \cup C$, then either $A \subseteq B$ or $A \subseteq C$.

It can be disproved by the following counter example. Take $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{2, 4\}$. Then $A \subseteq B \cup C = \{1, 2, 3, 4\}$, but A is neither a subset of B , nor a subset of C.

Grading: 2 pts for explicitly listing A, B and C. 3 pts if it works and is explained.

b) $(A - C) \cap (C - B) = \emptyset$

Proof by contradiction. Assume that $(A - C) \cap (C - B) \neq \emptyset$ to show that it results to contradiction. $(A - C) \cap (C - B) \neq \emptyset$ means that there exists some $x \in (A - C) \cap (C - B)$. By the definition of intersection we can imply, that there exists x for which the following proposition is true: $p = (x \in A - C) \wedge (x \in C - B)$. Using the definition of set difference we can rewrite p as

: $p = (x \in A) \wedge (x \notin C) \wedge (x \in C) \wedge (x \notin B)$. But $(x \notin C) \wedge (x \in C)$ = False, so

$p = (x \in A) \wedge [(x \notin C) \wedge (x \in C)] \wedge (x \notin B) = (x \in A) \wedge \text{False} \wedge (x \notin B)$ = False.

So, the assumption that intersection $(A - C) \cap (C - B) \neq \emptyset$ is not empty results to contradiction which proves that this assumption is false, i.e. intersection is empty.

Other solution is to use the membership table.

Grading: Other proofs can work than this one. 3 points for the basic set up of the proof whether it be direct, by contradiction or membership table. 5 points for the actual "guts" of the proof, and 2 points for the conclusion (e.g. the contradiction)

c) Power(A)–Power(B) $\subseteq$ Power ($A - B$)

This is false. To disprove take a counterexample: $A = \{1, 2\}$, $B = \{2, 3\}$, Power(A) = $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, Power($B$) = $\{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$, Power($A$) − Power($B$) = $\{\{1\}, \{1, 2\}\}$, $A - B = \{1\}$, Power ($A - B$) = $\{\emptyset, \{1\}\}$. Thus, $\{1, 2\} \in$ Power(A) − Power(B), but $\{1, 2\} \notin$ Power ($A - B$), so the proposition Power(A)−Power(B) $\subseteq$ Power ($A - B$) is disproved.

Grading: 5 pts to explicitly list out A and B, and calculate Power(A) and Power(B) correctly, 5 pts if the example is a counterexample and is shown to be so.