Computer Science Foundation Exam

May 3, 2002

Section II A

DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,
and you must work entirely on your own.**

Name: ______________________________

SSN: ______________________________

In this section of the exam, there are two (2) problems.

You must do both of them.

Each counts for 25% of the total exam grade.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone

Credit cannot be given when your results are unreadable.

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: Work both of the following problems (1 and 2).

1) Prove by using induction on n that for all positive integers $n > 0$,

$$\sum_{i=1}^{n} \frac{(i-1)2^i}{i(i+1)} = \frac{2^{n+1}}{n+1} - 2$$

2) For both parts of this question, let A, B and C be finite sets. (Note that $\neg$ denotes the complement of a set.)

a) Prove or disprove: if $A \subseteq B$ and $A \subseteq C$, then $\neg(B \cap C) \subseteq \neg A$.
b) Prove or disprove: if $A - B = B - C$, then $A = \varnothing$.

Solution to Problem 1:

2) Prove by using induction on *n* that for all positive integers *n* >0,

$$\sum_{i=1}^{n} \frac{(i-1)2^i}{i(i+1)} = \frac{2^{n+1}}{n+1} - 2$$

Use induction on n.

Base case. n=1 LHS = (1-1)2^1/[(1)(2)] = 0
RHS = 2^{1+1}/(1+1) - 2 = 0
Thus the equation given is true for n=1.

Inductive hypothesis. Assume for an arbitrary value of n=k that

$$\sum_{i=1}^{k} \frac{(i-1)2^i}{i(i+1)} = \frac{2^{k+1}}{k+1} - 2$$

Inductive step. Under the assumption above, prove for n=k+1 that

$$\sum_{i=1}^{k+1} \frac{(i-1)2^i}{i(i+1)} = \frac{2^{(k+1)+1}}{(k+1)+1} - 2$$

We will prove this by simplifying the left hand side.

$$\sum_{i=1}^{k+1} \frac{(i-1)2^i}{i(i+1)} = \sum_{i=1}^{k} \frac{(i-1)2^i}{i(i+1)} + \frac{k2^{k+1}}{(k+1)(k+2)}$$

$= \frac{2^{k+1}}{k+1} - 2 + \frac{k2^{k+1}}{(k+1)(k+2)}$, **using the inductive hypothesis**

$= \frac{2^{k+1}}{k+1}\left(1 + \frac{k}{k+2}\right) - 2$, **factoring out** $\frac{2^{k+1}}{k+1}$

$= \frac{2^{k+1}}{k+1}\left(\frac{(k+2)+k}{k+2}\right) - 2$, **getting a common denominator**

$= \frac{2^{k+1}}{k+1}\left(\frac{2(k+1)}{k+2}\right) - 2$, **factoring out a 2 from 2k+2 in the numerator**

$= \frac{2^{k+2}}{k+2} - 2$, **cancelling the (k+1)'s and multiplying 2^{k+1} by 2.**

Solution to Problem 2:

For both parts of this question, let A, B and C be finite sets. (Note that $\neg$ denotes the complement of a set.)

a) Prove or disprove: if $A \subseteq B$ and $A \subseteq C$, then $\neg(B \cap C) \subseteq \neg A$.
b) Prove or disprove: if $A - B = B - C$, then $A = \varnothing$.

a) Proof: We must show that if an arbitrary element x $\in \neg(B \cap C)$, then x$\in \neg A$.
This is equivalent to showing if x $\notin (B \cap C)$, then x$\notin$A.
Since any statement is equivalent to its contrapositive, we can prove the above by proving its contrapositive:

If x$\in$A, then x$\in$ B $\cap$ C.

To prove this, assume that x$\in$A. Under this assumption, we can deduce that x$\in$B since A $\subseteq$ B and that x$\in$C since A $\subseteq$ C. (Defn of subset.) But if these two things are true, we can conclude that x$\in$ B $\cap$ C, as desired.

b) Disproof: Let A = B = C = {1}. In this situation we have A - B = $\varnothing$ and B - C = $\varnothing$, but A $\neq \varnothing$.

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Section II B

DISCRETE STRUCTURES

NO books, notes, or calculators may be used, and you must work entirely on your own.

Name: ________________________________

SSN: ________________________________

In this section of the exam, there are two (4) problems.

You must do two (2) of them.

Each counts for 25% of the total exam grade.

You must clearly identify the problems you are solving.

Show the steps of your work carefully.

Problems will be graded based on the completeness of the solution steps and not graded based on the answer alone

Credit cannot be given when your results are

unreadable.

PART B: Work any two of the following problems (3 through 6).

3) Suppose $R, S \subseteq A \times A$ are two symmetric relations on a set A. Prove or disprove each of the following propositions.

a) the relation $R \circ S$ is symmetric.
b) the relation $R \circ S \cup S \circ R$ is symmetric

4) Suppose $f : A \rightarrow B$ and $g : B \rightarrow A$ are two functions. Assume that for any element $y \in B$, $(f \circ g)(y) = y$, but there is at least one element $x \in A$ such that $(g \circ f)(x) \neq x$. Prove that f is onto, but not one to one.

5) Consider the following relation on a set of positive integers $n > 0$.

$R = \{(a, b) \mid \gcd(a, b) = 15 \text{ and } \operatorname{lcm}(a, b) = 180\}$.

Explicitly write out all pairs in R and show that no other pairs are members of R.

6) Consider six-digit numbers with all distinct digits that do NOT start with 0. Answer the following questions about these numbers. Leave the answer in factorial form.

a) How many such numbers are there?
b) How many of these numbers contain a 3 but not 6?
c) How many of these numbers contain either 3 or 6 (or both)?

3) Suppose $R, S \subseteq A \times A$ are two symmetric relations on a set A. Prove or disprove each of the following propositions.

a) the relation $R \circ S$ is symmetric.
b) the relation $R \circ S \cup S \circ R$ is symmetric

a) This is false. Consider the following counter example:

A = {1,2,3}
R = {(1,2), (2,1)}
S = {(2,3), (3,2)}
R°S = {(1,3)}
Here we have two relations R and S that are both symmetric, but R°S is not.

b) Proof.

We must show that if (x,y)∈ R°S ∪ S°R, then (y,x)∈ R°S ∪ S°R.

If (x,y)∈ R°S ∪ S°R, then there are two cases to consider:

1) (x,y)∈ R°S or
2) (x,y)∈ S°R

In the first case, there must exist an element z∈A such that (x,z)∈R and (z,y)∈S, by the definition of relation composition.

Since R and S are symmetric, (z,x)∈R and (y,z)∈S. It follows that (y,x)∈ S°R, by the definition of relation composition.Now we can deduce that (y,x)∈ R°S ∪ S°R.

In the second case, there must exist an element z∈A such that (x,z)∈S and (z,y)∈R, by the definition of relation composition.

Since R and S are symmetric, (z,x)∈S and (y,z)∈R. It follows that (y,x)∈ R°S, by the definition of relation composition. Now we can deduce that (y,x)∈ R°S ∪ S°R.

Thus, we have shown that if if (x,y)∈ R°S ∪ S°R, then (y,x)∈ R°S ∪ S°R, proving that if R and S are symmetric then R°S ∪ S°R is as well.

4) Suppose $f : A \to B$ and $g : B \to A$ are two functions. Assume that for any element $y \in B$, $(f \circ g)(y) = y$, but there is at least one element $x \in A$ such that $(g \circ f)(x) \neq x$. Prove that f is onto, but not one to one.

To show that f is onto, we must show that there exists some element x∈A such that f(x) = y, for all elements y∈B.

Let y be an arbitrary element of the set B, we must show that there is some x for which f(x) = y.

We know that f°g(y) = y. Thus, we have f(g(y)) = y. But we know that the function g maps from B to A. Thus, g(y)∈A. Let x = g(y). Then we have that f(x) = y, as desired.

Now, we must show that f is NOT one-to-one.

In essence, we must show that for two distinct elements x∈A and y∈A, f(x)=f(y).

We know that for at least one element x∈A that g°f(x) ≠x. Let g°f(x)=z for some x∈A, and z∈A such that x≠z.

g(f(x)) = z.
Let w = f(x), where w∈B. Thus, g(w) = z.

We know that f(g(w)) = w.
But, g(w) =z from above, so we have that
f(z) = w.
Thus, we have deduced that f(x) = f(z) = w, and that x≠w proving that f is NOT injective.

5) Consider the following relation on a set of positive integers $n > 0$.

$$R = \{(a, b) \mid \gcd(a, b) = 15 \text{ and } \mathrm{lcm}(a, b) = 180\}.$$

Explicitly write out all pairs in R and show that no other pairs are members of R.

For all pairs of positive integers a and b, we have that

ab = (gcd(a,b))(lcm(a,b))

Since gcd(a,b) = 15, let a=15a' and b=15b', where a' and b' are integers, with gcd(a',b') =1. (If this weren't so, then gcd(a,b) > 15.)

ab = 15a'15b' = 15(180)

a'b' = 12
Since a' and b' are relatively prime positive integers, the only possible solutions for ordered pairs (a',b') are (1,12), (3,4), (4,3) and (12,1). We can use these to construct R.

R = {(15, 180), (45, 60), (60,45), (180,15)}

We ruled out all other possibilities by using the ab=(gcd(a,b))(lcm(a,b)) restriction and the gcd(a',b') =1 restriction.

6) Consider six-digit numbers with all distinct digits that do NOT start with 0. Answer the following questions about these numbers. Leave the answer in factorial form.

a) How many such numbers are there?
b) How many of these numbers contain a 3 but not 6?
c) How many of these numbers contain either 3 or 6 (or both)?

a) There are 9 choices for the first digit, and then 9 choices for the second digit (0 has been added as a choice), 8 for the third, 7 for the fourth, 6 for the fifth, and 5 for the sixth. Total = (9)(9)(8)(7)(6)(5) = 9(9!)/4! = 136080.

b) We need to separate the counting into two categories

1) 3 is the first digit
2) 3 is NOT the first digit

For the first category, we have one choice for the first digit, followed by 8 choices for the second digit (not 3 or 6), 7 choices for the third digit, 6 choices for the fourth digit, 5 choices for the fifth digit and 4 choices for hte sixth digit.
Total = (8)(7)(6)(5)(4) = 8!/3!

For the second category, we have 7 choices for the first digit (not 0, 3, or 6), now we must guarantee that a 3 is picked. There are five PLACES to place the 3. For the remaining 4 digits, we have 7 choices, 6 choices, 5 choices and 4 choices, respectively for each of these. (To see this, imagine the 3 was placed 2nd. Then for the third digit you could choose any number except for the first digit, 3 and 6. Similarly, no matter where the 3 is placed, you always have 7 choices for the next placed digit, then 6, etc.)
Total = (7)(5)(7)(6)(5)(4) = 35(7!)/3!

The total of both of these categories is 8!/3! + 35(7!)/3! = 36120

c) Count the number of numbers that contain neither:

There are 7 choices for the first digit(not 0,3 or 6), 7 choices for the second digit, 6 choices for the third digit, 5 choices for the fourth digit, 4 choices for the fifth digit and 3 choices for the sixth digit. Total = (7)(7)(6)(5)(4)(3) = 7(7!)/2!

Now, the answer to the question given is the value above subtracted from the answer in part a. Thus, this answer is 9(9!)/4! - 7(7!)/2! = 118440.