Computer Science Foundation Exam

May 3, 2001

Section II A

DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,
and you must work entirely on your own.**

Name: _____

SSN: _____

In this section of the exam, there are two (2) problems.

You must do both of them.

Each counts for 25% of the total exam grade.

Show the steps of your work carefully.

**Problems will be graded based on the completeness of
the solution steps and <u>not</u> graded based on the answer alone**

Credit cannot be given when your results are unreadable.

FOUNDATION EXAM (DISCRETE STRUCTURES)

Answer two problems of Part A and two problems of Part B. Be sure to show the steps of your work including the justification. The problem will be graded based on the completeness of the solution steps (including the justification) and **not** graded based on the answer alone. NO books, notes, or calculators may be used, and you must work entirely on your own.

PART A: **Work both of the following problems (1 and 2).**

1) Use induction on n to prove the following inequality for all positive integers n:

$$\sum_{i=1}^{n^2} \sqrt{i} \le \frac{n(n+1)(4n-1)}{6}$$

(Hint: Remember that $\sqrt{(x^2 - y)} \le x$ for all positive integers x and non-negative integers $y \le x^2$.)

2) Suppose A, B and C are sets. Prove that $A - (B - C) \subseteq (A - B) \cup C$.
Note: You may **<u>not</u>** use the "truth table" or "membership" method for the proof.

Solution to Problem 1:
Use induction on n to prove the following inequality for all positive integers n:

$$\sum_{i=1}^{n^2} \sqrt{i} \le \frac{n(n+1)(4n-1)}{6}$$

Base Case: n=1. LHS = √(1) = 1
$$\text{RHS} = 1(1+1)(4(1) - 1)/6 = 1$$

Assume for an arbitrary value of n=k that

$$\sum_{i=1}^{k^2} \sqrt{i} \le \frac{k(k+1)(4k-1)}{6}$$

Under this assumption, prove for n=k+1 that

$$\sum_{i=1}^{(k+1)^2} \sqrt{i} \le \frac{(k+1)((k+1)+1)(4(k+1)-1)}{6} = \frac{(k+1)(k+2)(4k+3)}{6}$$

$$\sum_{i=1}^{(k+1)^2} \sqrt{i} = \sum_{i=1}^{k^2} \sqrt{i} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{i}$$

$$\le \frac{k(k+1)(4k-1)}{6} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{i} \text{ , using the inductive hypothesis.}$$

$$\le \frac{k(k+1)(4k-1)}{6} + \sum_{i=k^2+1}^{(k+1)^2} \sqrt{(k+1)^2} \text{ , because each term in the summation is}$$

$$\text{less than or equal to that last term when } i = (k+1)^2.$$

$$= \frac{k(k+1)(4k-1)}{6} + \sum_{i=k^2+1}^{(k+1)^2} (k+1)$$

$$= \frac{k(k+1)(4k-1)}{6} + [(k+1)^2 - k^2](k+1) \text{ , because the first summation from}$$

$$\text{the IH contains } k^2 \text{ terms while the summation from the IS contains } (k+1)^2\text{, leaving the difference for this summation.}$$

$$= \frac{k(k+1)(4k-1)}{6} + (2k+1)(k+1)$$

$$= \frac{(k+1)}{6} \big(k(4k-1) + 6(2k+1) \big)$$

$$= \frac{(k+1)}{6} \left(4k^2 - k + 12k + 6 \right)$$

$$= \frac{(k+1)}{6} \left(4k^2 + 11k + 6 \right)$$

$$= \frac{(k+1)(k+2)(4k+3)}{6}$$

Solution to Problem 2:
Suppose *A*, *B* and *C* are sets. Prove that $A - (B - C) \subseteq (A - B) \cup C$.

We must prove the following:

Let x be an arbitrary element from the set A – (B – C).
We need to prove that x ∈ $(A - B) \cup C$.

A – (B – C) = A – (B ∩ ¬C) defn of – (here, ¬ denotes the complement of a set.)
$$= A \cap \neg(B \cap \neg C) \quad \textbf{defn of} -$$
$$= A \cap (\neg B \cup \neg\neg C) \quad \textbf{DeMorgan's}$$
$$= A \cap (\neg B \cup C) \quad \textbf{Law of Double Complement}$$
$$= (A \cap \neg B) \cup (A \cap C) \quad \textbf{Distributive}$$
$$= (A - B) \cup (A \cap C) \quad \textbf{defn of} -$$

This shows that iff x ∈ A – (B – C), x∈ (A – B) ∪ (A ∩ C).

We can now break our proof down into two parts:

1) Prove if x∈ (A – B), then x∈ $(A - B) \cup C$.
2) Prove if x∈ (A ∩ C), then x∈ $(A - B) \cup C$.

To show 1, notice, that by the definition of union, we MUST have x∈ $(A - B) \cup C$ if x∈ $(A - B)$.

To show 2, notice, that by the definition of intersection, we MUST have x∈ C if x∈ (A ∩ C). If this is the case, then by definition of union, clearly x∈ $(A - B) \cup C$.

Thus, we have shown that for any arbitrary x such that x∈ A – (B – C), that x ∈ $(A - B) \cup C$. It follows that A – (B – C) $\subseteq (A - B) \cup C$.

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Section II B

DISCRETE STRUCTURES

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Name: _____

SSN: _____

In this section of the exam, there are two (4) problems.

You must do <u>two</u> (2) of them.

Each counts for 25% of the total exam grade.

You must clearly identify the problems you are solving.

Show the steps of your work carefully.

Problems will be graded based on the completeness of
the solution steps and <u>not</u> graded based on the answer alone

Credit cannot be given when your results are unreadable.

3) Let ? be an unknown Boolean logical operator. The logical statement
$[(p \wedge q) \vee r] \Rightarrow (q ? r)$ is equivalent to $[\neg(p \wedge q) \vee r]$.
Given this information, there are 2 possible truth tables for the Boolean logical
operator ?. List both of these truth tables.

p	q	r	$(p \wedge q) \vee r$	$(q ? r)$	$[(p \wedge q) \vee r] \Rightarrow (q ? r)$	$[\neg(p \wedge q) \vee r]$.
0	0	0	0	XXX	1	1
0	0	1	1	1	1	1
0	1	0	0	XXX	1	1
0	1	1	1	1	1	1
1	0	0	0	XXX	1	1
1	0	1	1	1	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

**Here we see that based on the values of $(p \wedge q) \vee r$ and $[(p \wedge q) \vee r] \Rightarrow (q ? r)$, we
can definitively fill out five slots for our unknown column. These 5 slots specify the
operation $(q ? r)$ for all possibilities where at least one of the two values is 1. (Notice
that we can deduce that $1?0 = 0$ from row 7 and fill in that value in row 3, so that we
really only have 2 unknown values, both of which must be 1 or 0.) Thus, the only
ambiguity possible for the operator is that $0?0=0$ OR $0?0=1$. Thus, we have the two
following truth tables for ?:**

?	r=0	r=1
q=0	0	1
q=1	0	1

?	r=0	r=1
q=0	1	1
q=1	0	1

4) a) Prove or disprove: If $R_1 \subset R_2$ and $S_1 \subset S_2$, then $R_1 \circ S_1 \subset R_2 \circ S_2$.

This statement is true. We must prove the following statement:

If $(x,y) \in R_1 \circ S_1$, then $(x,y) \in R_2 \circ S_2$, for any arbitrary element (x,y).

**Using our given information, we know there must exist an element z such
that $(x,z) \in R_1$ and $(z,y) \in S_1$, by definition of relation composition.**

Then, by definition of subsets, we can deduce that $(x,z) \in R_2$ and $(z,y) \in S_2$,

since we are given that $R_1 \subset R_2$ and $S_1 \subset S_2$.

Now, by definition of relation composition, we can conclude that $(x,y) \in R_2 \circ S_2$ as desired.

b) Prove or disprove: If $R_1 \neq R_2$ and $S_1 \neq S_2$, then $R_1 \circ S_1 \neq R_2 \circ S_2$.

Let each relation be a subset of Z x Z.
This claim is false. Let $R_1 = \{(1,2)\}$, $R_2 = \{(1,3)\}$, $S_1 = \{(2,1)\}$, and $S_2 = \{(3,1)\}$.

In this counter example, all four relations are different but $R_1 \circ S_1 = R_2 \circ S_2$.

5) a) How many permutations of the word **FOUNDATION** are there?

10!/(2!2!), since there are 10 letters total with 2Os and 2Ns.

b) A valid password is 5 letters long and uses a selection of the letters in the word **FOUNDATION**. (Thus, a password may have at most 2 **N**'s, at most 2 **O**'s, and at most 1 copy of each of the other letters {**A**, **D**, **F**, **I**, **U**, **T**} in **FOUNDATION**.) How many valid passwords are there?

Split up the counting into three separate categories:

1) Passwords with 5 distinct letters
2) Passwords with 4 distinct letters
3) Passwords with 3 distinct letters

1) We have 8 distinct letters to choose from and we are choosing 5.
   There are P(8,5) = 8!/3! ways to do this.

2) We first choose either two Ns or two Os. This can be done in 2 ways.
   Then we choose three distinct letters out of the 7 remaining. We can
   choose the three letters in C(7,3) ways. Thus, we have
   C(7,3)x2 = 70 ways to choose our letters. Each of these choices gives
   rise to 5!/2! = 60 permutations.

3) We have to choose all Ns and Os, which leaves us one choice out of the
   remaining 6 letters. We can choose this letter is 6 ways. For each of these
   choices, we can make 5!/(2!2!) = 30 permutations. So there is a total of
   180 permutations of this kind to count.

Adding up, we get the total number of valid passwords to be

8!/3! + 70x60 + 180 = 6720 + 4200 + 180 = 11100.

c) An ascending word is one where the letters of the word appear in alphabetical order. Thus, for example, **AFNNT** is a valid password that is ascending. How many of the valid passwords are ascending?

Using the logic from above, but just counting combinations, we get the answer to be C(8,3) + 70 + 6 = 56 + 70 + 6 = 132.
Note: The reason we only count combinations here is that each distinct combination of letters gives rise to exactly one valid password. (For each set of letters, there is one and only one way to alphabetize them.)

6) Consider the relation R on the set of integers $\mathbb{Z}=\{\dots-2, -1, 0, 1, 2, \dots\}$:
$R = \{(x, y) \mid x, y \in A$ and $x \equiv y \pmod 5)\}$.
i) Prove that this relation is an equivalence relation.

We need to prove that this relation is reflexive, symmetric and transitive.

The relation is reflexive because for all integers a, (a,a) ∈ R. To see this, note that 5 | (a – a), for all values of a since a – a = 0.

To show the relation in symmetric, we must prove the following:

If (x,y)∈R, then (y,x)∈R.

Using our given information, we have that x ≡ y (mod 5). Using mod rules we have the following:

x ≡ y (mod 5)
x – y ≡ y – y (mod 5)
x – y ≡ 0 (mod 5)
–x + x – y ≡ –x (mod 5)
– y ≡ –x (mod 5)
(–1)(–y) ≡ (–1)(–x) (mod 5)
y ≡ x (mod 5)
(y,x) ∈ R.

To show that R is transitive, we must show the following:

If (x,y)∈R and (y,z)∈R, then (x,z)∈R.

Using our given information, we have the following:

x ≡ y (mod 5)

y ≡ z (mod 5)

Adding these two equations, we get
(x + y) ≡ (y + z) (mod 5)

Subtracting y from both sides, we get
x ≡ z (mod 5)
Thus, we have shown that (x,z)∈ R.

Since R is reflexive, symmetric and transitive, we conclude that it is an equivalence relation.

ii) How many equivalence classes are induced by this relation on Z?
(Recall that: $x \equiv y$ (mod 5) means that $5 \mid (x-y)$, i. e. there exists an integer k, such that $x-y = 5 \cdot k$ For instance, $1 \equiv 6$ (mod 5) or $8 \equiv 3$ (mod 5)).

There are 5 of them. They are [0], [1], [2], [3], [4], corresponding to the different remainders you get when you divide by 5.

Solution to Problem _____ (Please write in the problem number 3,4,5 or 6 you are solving)

Solution to Problem _____ (Please write in the problem number 3,4,5 or 6 you are solving)