

Computer Science Foundation Exam Solutions

May 2, 2003

Section II B

DISCRETE STRUCTURES

**NO books, notes, or calculators may be used,
and you must work entirely on your own.**

Name: _____

SSN: _____

In this section of the exam, there are two (4) problems.

You must do two (2) of them.

Each counts for 25% of the total exam grade.

You must clearly identify the problems you are solving.

Show the steps of your work carefully.

**Problems will be graded based on the completeness of
the solution steps and not graded based on the answer alone**

Credit cannot be given when your results are

unreadable.

PART B: Work any two of the following problems (3 through 6).

3) Students A, B, C, D, E, F, G, H, I, and J must sit in ten chairs lined up in a row. Answer the following questions based on the restrictions given below. (Note that each part is independent of the others, thus no restriction given in part a applies to the rest of the parts, etc.)

a) How many ways can the students sit if the two students on the ends of the row have to be vowel-named students?

There are three choices for the student on the left, and then 2 choices for the student on the right. Following those two choices, we can arrange the rest of the 8 students left in $8!$ ways. Thus, the total number of ways the students can sit is $(3)(2)(8!)$.

Grading: 5 pts, 2 pts for the 3 and 2, 3 pts for the $8!$

b) How many ways can the students sit if no two students with vowel names can sit adjacent to each other?

Place all seven consonants like so (C designates an arbitrary consonant):

___ C ___ C ___ C ___ C ___ C ___ C ___ C ___

Now, the empty slots (___) represent possible locations for the vowels. There are $P(8,3) = (8)(7)(6)$ ways to place the vowels. The 7 consonants can be ordered in $7!$ ways. Thus, there are $(8)(7)(6)(7!)$ ways the students can sit without any vowel-named students sitting next to each other.

Grading: 10 pts, 4 pts for the idea, 3 pts for the $P(8,3)$ and 3 pts for the $7!$

c) Given that students A, B, C, and D are male, and that the rest of the students are female, how many ways can the students be arranged such that the average number of females adjacent to each male is 0.25? (Note: to determine the average number of females each male is adjacent to, sum up the total number of females adjacent to each male and then divide by the total number of males. For example, in the arrangement AEBFCDDGHIJ, each male is adjacent to 1.25 females, on average.)

Notice that the only ways in which the average number of females adjacent to males is 0.25 is when all four males are at the left or right end of the row of chairs. If this isn't the case, then more than one female will be adjacent to a male. If this occurs, then the average will be at least 0.5. Since the males and female can sit in any arrangement amongst themselves, for both cases, they can sit in $(4!)(6!)$ ways. Totalling both possibilities (males to the left, males to the right), the total number of arrangements desired is $(2)(4!)(6!)$.

Grading: 10 pts, 4 pts for deducing where males sit, 3 pts for $4!$ and 3 pts for $6!$

4)

a) Let $n = 2^5 3^2 5^4 7^3$ and $m = 2^3 3^4 5^3 7^4$. Find $\gcd(n,m)$ without using the Euclidean Algorithm. Leave your answer in prime factorized form.

$2^3 3^2 5^3 7^3$, 5 pts all or nothing

b) Let n be a positive integer. Which of the following expressions could be a perfect square? (For each expression that cannot be a perfect square, prove that this is the case. For each expression that could be a perfect square, provide the value of n for which the expression is a perfect square.)

i) $n(n+1)$ ii) $n(n+2)$ iii) $n(n+3)$

i and ii) Notice that $(n+1)^2 = n^2 + 2n + 1$. Also, $n(n+1) = n^2 + n$, and $n(n+2) = n^2 + 2n$

Thus, for all integers $n > 0$:

$$n^2 < n(n+1) < n(n+2) < (n+1)^2.$$

Since there can be no perfect squares between n^2 and $(n+1)^2$, it is impossible for either $n(n+1)$ or $n(n+2)$ to be a perfect square.

iii) Note that the argument used for the first two parts doesn't hold here. The easiest way to find a value of n that satisfies this problem is to set the expression $n(n+3)$ to a perfect square. It makes sense to set this to $(n+1)^2$ and solve for n :

$$\begin{aligned}n(n+3) &= (n+1)^2, \\n^2 + 3n &= n^2 + 2n + 1 \\3n &= 2n + 1 \\n &= 1\end{aligned}$$

Plugging into the expression verifies that $(1)(1+3) = 4 = 2^2$.

No other positive solution exists because we can show that $n(n+3) < (n+2)^2$ for all positive integers n .

Grading: 20 pts, 8 pts for cases i and ii, 4 pts for case iii.

5) Let A , B , and C be finite sets and f and g be functions such that $f:A \rightarrow B$, $g:B \rightarrow C$ for each of the following questions.

a) Give a small example with f is surjective, g is injective, but $g \circ f$ is neither. (Please explicitly list the sets A , B , and C that you choose, as well as the functions f and g .)

Let $A = \{1, 2\}$, $B = \{10\}$, $C = \{50, 55\}$

Let $f = \{(1, 10), (2, 10)\}$ and $g = \{(10, 50)\}$.

Then we have $g \circ f = \{(1, 50), (2, 50)\}$. This function is not surjective since there is no value a in the set A such that $g \circ f(a) = 55$. This function is not injective because $g \circ f(1) = g \circ f(2)$.

Grading: 10 pts, 3 pts for labeling the sets, 2 pts for a surjective f , 2 pts for a injective g , 3 pts for the composition being neither.

b) Prove that if $g \circ f$ is injective, then f must be injective as well.

We must prove that if $f(x) = f(y)$, then $x = y$, for all values of x and y such that $x \in A$ and $y \in A$.

Let x and y be arbitrary elements of A such that $f(x) = f(y)$.

Now, let both of these be inputs to the function g . Thus,

$$g(f(x)) = g(f(y))$$

But, since $g \circ f$ is injective, it follows that $x = y$ as desired.

Grading: 15 pts, 4 pts for stating what needs to be proved, 2 pts for starting with $f(x)=f(y)$, 5 pts for taking g of both sides of this equality, 4 pts for invoking the given information.

6) Let S be a relation over the positive real numbers (so $S \subseteq \mathbb{R}^+ \times \mathbb{R}^+$), defined as follows:

$$S = \{ (a,b) \mid a=b^n, \text{ for some positive rational number } n. \}$$

Is S reflexive? symmetric? transitive? Give proof for each of your answers.

The relation is reflexive, since for all positive rational values a , $a^1 = a$, so $(a,a) \in R$.

The relation is symmetric as well. To show this, we must prove that if $(a,b) \in R$, then $(b,a) \in R$.

Given that $(a,b) \in R$, there exists a positive rational number n such that $a = b^n$. Now, take both sides of this equation to the power $1/n$, yielding:

$a^{1/n} = b$. Since n is a positive rational number, so is $1/n$. Thus, it follows that $(b,a) \in R$, as desired.

Finally we must show that R is also transitive. We must prove the following:

if $(a,b) \in R$, and $(b,c) \in R$, then $(a,c) \in R$, for all positive rational values a , b , and c .

Given that $(a,b) \in R$, and $(b,c) \in R$, we have

$a = b^n$ and $b = c^m$, for some positive rational values m and n . Substituting for b in the first equation, we have:

$a = b^n = (c^m)^n = c^{mn}$. Since m and n are both positive rational numbers, so is their product. Hence, $(a,c) \in R$ as desired.

Grading: reflexive: 5 pts, symmetric: 10 pts, transitive: 10 pts.