1) Induction

The nth Harmonic number, H_n is defined as follows: $H_n = \sum_{i=1}^{n} \frac{1}{i}$. Using induction on n, prove that the following summation formula is true for all positive integers n:

$$\sum_{i=1}^{n} \frac{i}{i+2} = (n+3) - 2H_{n+2}$$

(Hint: k+1 = (k+3) - 2.)

Base case: n=1 LHS = $\sum_{i=1}^{1} \frac{i}{i+2} = \frac{1}{3}$ RHS = $(1+3) - 2H_3 = 4 - 2(\frac{11}{6}) = \frac{1}{3}$

Thus, the given equation holds for n=1. **(4 pts)**

Inductive Hypothesis: Assume for an arbitrary value of n=k that

$$\sum_{i=1}^{k} \frac{i}{i+2} = (k+3) - 2H_{k+2} \qquad \textbf{(2 pts)}$$

Inductive Step: Prove for n=k+1 that

$$\sum_{i=1}^{k+1} \frac{i}{i+2} = (k+1+3) - 2H_{k+2+1} = (k+4) - 2H_{k+3} \textbf{ (2 pts)}$$

$$\sum_{i=1}^{k+1} \frac{i}{i+2} = (\sum_{i=1}^{k} \frac{i}{i+2}) + \frac{k+1}{k+3} \qquad \textbf{(3 pts)}$$

$$= (k+3) - 2H_{k+2} + \frac{(k+3)-2}{k+3} \qquad \textbf{(IH-3 pts, Algebra-3 pts)}$$

$$= (k+3) - 2H_{k+2} + 1 - \frac{2}{k+3} \qquad \textbf{(2 pts)}$$

$$= (k+4) - 2(H_{k+2} + \frac{1}{k+3}) \qquad \textbf{(2 pts)}$$

$$= (k+4) - 2H_{k+3} \text{, since } H_{k+3} = \sum_{i=1}^{k+3} \frac{1}{i} = (\sum_{i=1}^{k+2} \frac{1}{i}) + \frac{1}{k+3} = H_{k+2} + \frac{1}{k+3}$$

(4 pts)

Proving the inductive step and showing that the formula given is true for all n.

2) Sets

Let A, B, and C be arbitrary sets. Prove or disprove the following:

a) $(A - B) \cap (C - B) = (A \cap B) \cap \sim(B \cup \sim C)$
(Note: ~A denotes the complement of set A.)

b) If $A \cap B \subseteq A \cap C$ and $A \cap C \subseteq B \cap C$, then show that $A \cap B = A \cap C$, but it is possible that $A \cap C \neq B \cap C$.

a) Disproof by counter example: B={} A=C={1}
(6 pts)

LHS=$(\{1\}-\{\}) \cap (\{1\}-\{\}) = \{1\} \cap \{1\} = \{1\}$
RHS= $\{1\} \cap \{\} \cap \sim(B \cup \sim C) = \{\} \neq$ LHS
(4 pts)

b) If $A \cap B \subseteq A \cap C$ and $A \cap C \subseteq B \cap C$, then show that $A \cap B = A \cap C$, but it is possible that $A \cap C \neq B \cap C$.

First part:
Show $A \cap B = A \cap C$
But we are already given $A \cap B \subseteq A \cap C$
Therefore, need to show $A \cap C \subseteq A \cap B$ (2 pts)
Let $x \in A \cap C$
$\Rightarrow x \in A$ and $x \in C \Rightarrow x \in A$ and $x \in A$ and $x \in C$ by Idempotent Law (2 pts)
$\Rightarrow x \in A$ and $x \in A \cap C$
$\Rightarrow x \in A$ and $x \in B \cap C$ by premise stating "$A \cap C \subseteq B \cap C$" (4 pts)
$\Rightarrow x \in A$ and $x \in B$ and $x \in C$ (2 pts)
$\Rightarrow x \in A$ and $x \in B$ by Conjunctive Simplification
$\Rightarrow x \in A \cap B$

Second part:

Existential proof of $A \cap C \neq B \cap C$ can be accomplished by a single example:
A={} B={1} C={1}

Here, $A \cap C = \{\}$ while $B \cap C = \{1\}$
(5 pts)

3) Relations

a) Let $A = \{1, 2, 3, 4, 5\}$. R is a relation defined on A, (so $R \subseteq A \times A$.) In particular, $R = \{ (1,1), (1, 2), (1, 4), (1, 5), (2, 1), (2, 2), (2, 4), (3, 3), (4, 1), (5, 1) \}$. Is R (i)reflexive? (ii)symmetric? (iii)transitive? Prove your answers.

b) Define R to be a relation over the positive integers as follows:

$$R = \{ (a,b) \mid a = cb, \text{ for some positive integer c}\}$$

Prove that R is a partial-ordering relation. (Show that it is reflexive, anti-symmetric and transitive.)

a)　　(i) No, because $(4,4) \notin R$. **(2 pts)**
　　　　(ii) No, because $(2, 4) \in R$ but $(4, 2) \notin R$. **(2 pts)**
　　　　(iii) No, because $(5,1) \in R$ and $(1,5) \in R$ but $(5,5) \notin R$. **(2 pts)**

b) First we show that R is reflexive. Consider any ordered pair (a,a). We have that $(a,a) \in R$ because $a = 1(a)$, thus we can let $c = 1$. **(3 pts)**

Now, we must show that R is antisymmetric. In order to do this we must show the following: **(9 pts - breakdown is below)**

if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$. **(2 points for stating this.)**

The given information yields the following two equations:

$a = c_1 b$
$b = c_2 a$ **(2 pts for getting these equations)**

where c_1 and c_2 are positive integers.

Substitute the left-hand side of the second equation in for b in the first equation:

$a = c_1(c_2 a)$ **(2 pts for this substitution)**

Simplifying, we find that

$c_1 c_2 = 1$

Since both c_1 and c_2 are positive integers it follows that both are equal to 1. **(2 pts for this deduction)**

(No other product of positive numbers is 1.) If this is the case, then $b = 1(a)$ so, $b = a$, as desired. **(1 pt for the conclusion)**

To show that R is transitive, we must prove the following: **(7 pts -breakdown below)**

if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$. **(2 pts)**

In order to prove this, assume that $(a,b) \in R$ and $(b,c) \in R$. Thus, we have

$a = c_1 b$
$b = c_2 c$ **(2 pts)**

Substitute for b into the first equation to yield:

$a = c_1(c_2 c) = (c_1 c_2)c$ **(2 pts)**

Since c_1 and c_2 are positive integers it follows that their product is as well. Thus, by the definition of R, we have that $(a,c) \in R$ as desired, showing that R is a partial-ordering relation. **(1 pt)**

4) Functions

      a) Let A and B be sets such that $|A| = 5$ and $|B| = 3$. How many functions f:A→B can there be? How many of those functions are injective?

      b) For arbitrary sets, A, B and C, let f and g be functions with f: A→B and g: B→C. Prove that if g°f: A→C is surjective then g must be surjective as well.

a) For a function f, each of the five values in the domain can map to one of three values. In essence, we make five choices for function values, and for each choice we have 3 choices. Since the five choices are independent of one another, we can have a total of 3^5 = 243 total functions. **(5 pts)**

Of those, none can be injective since the cardinality of the domain is larger than the cardinality of the codomain. (Thus, two values in the domain MUST map to the same value in the codomain.) **(5 pts)**

b) We will prove the contrapositive of the given statement:

if g is NOT surjective then g°f is not either.

We assume that g is NOT surjective. **(3 pt)**

 If this is the case, there exists a value $z \in C$ such that $g(y) \neq z$ for all $y \in B$. **(3 pts)**

Now, let's consider the value $g(f(x))$ for all $x \in A$. By the definition of f, we know that $f(x) \in B$. **(3 pts)**

BUT, if this is the case, it ALSO follows that $g(f(x) \neq z$, because for ALL values $y \in B$, $g(y) \neq z$. **(3 pts)**

Thus, since there is no value of x for which $g(f(x)) = z$, it follows that g°f is not surjective. **(3 pts)**

5) Counting

   a) How many four digit numbers do NOT contain any repeating digits? (Note: All four digits numbers are in between 1000 and 9999, inclusive.)

   b) A number is defined as *ascending* if each of its digits are in increasing numerical order. For example, 256 and 1278 are *ascending* numbers, but 1344 and 2687 are not. How many four digit numbers are *ascending*?

   c) A number is defined as *descending* if each of its digits are in decreasing numerical order. For example, 652 and 8721 are *descending* numbers, but 4431 and 7862 are not. How many four digit numbers are *descending*?

a) You can choose 9 values for the first digit, since 0 is not permissible, 9 values for the second digit since 0 is now permissible but the first number chosen is not, 8 values for the third digit and 7 values for the last digit using similar reasoning. Thus, the final answer is $9(9)(8)(7) = 4536$. **(5 pts)**

b) Since the first digit can not be 0, none of the digits can be 0. For each combination of four digits from the set {1, 2, 3, 4, 5, 6, 7, 8, 9} we can create exactly one ascending number. Thus, the total number of ascending numbers is $\binom{9}{4} = 126$. **(10 pts - assign partial as you see fit.)**

c) A descending number can contain 0. Thus, for each combination of four digits from the set {0, 1, 2, 3, 4, 5, 6, 7 ,8 9} we can create exactly one descending number. Thus, the total number of descending numbers is $\binom{10}{4} = 210$. **(10 pts - assign partial as you see fit.)**

6) Recursive Definitions and Number Theory

      a) Let $a_0 = 1$ and $a_n = a_{n-1}^2 - 3a_{n-1} + 2$ for all positive integers n. For how many distinct integers k does there exists a non-negative value of n for which $a_n = k$? What are these possible values of k?

      b) Find two integers x and y that satisfy the following equation: $86x + 25y = 1$.

a) $a_1 = a_0^2 - 3a_0 + 2 = 1 - 3 + 2 = 0$
    $a_2 = a_1^2 - 3a_1 + 2 = 0 - 0 + 2 = 2$
    $a_3 = a_2^2 - 3a_2 + 2 = 4 - 6 + 2 = 0$ **(5 pts for writing these out)**

At this point, since we know that the subsequent sequence value only depends on the previous one, we can ascertain that the sequence will alternate between 0 and 2 indefinitely. **(3 pts for this conclusion)**

Thus, there are three possible values of k. These are 0, 1 and 2. **(2 pts for this)**

b) Use the Extended Euclidean Algorithm to find gcd(86, 25):

$86 = 3 \times 25 + 11$
$25 = 2 \times 11 + 3$
$11 = 3 \times 3 + 2$
$3 = 1 \times 2 + 1$
$2 = 2 \times 1$ **(5 pts here, 1 pt per line)**

So, gcd(86,25) = 1.

Now, rewrite the equations as follows:

$3 - 1 \times 2 = 1$
$3 - 1 \times (11 - 3 \times 3) = 1$
$4 \times 3 - 1 \times 11 = 1$
$4 \times (25 - 2 \times 11) = 1 \times 11 = 1$
$4 \times 25 - 9 \times 11 = 1$
$4 \times 25 - 9 \times (86 - 3 \times 25) = 1$
$31 \times 25 - 9 \times 86 = 1$
So one set of solutions is x = -9 and y = 31. **(1 pt for each step and 2 points for the answer.)**

All solutions are of the form
x = -9+25n y=31 - 86n, where n is any integer.