Image De-noising Problem

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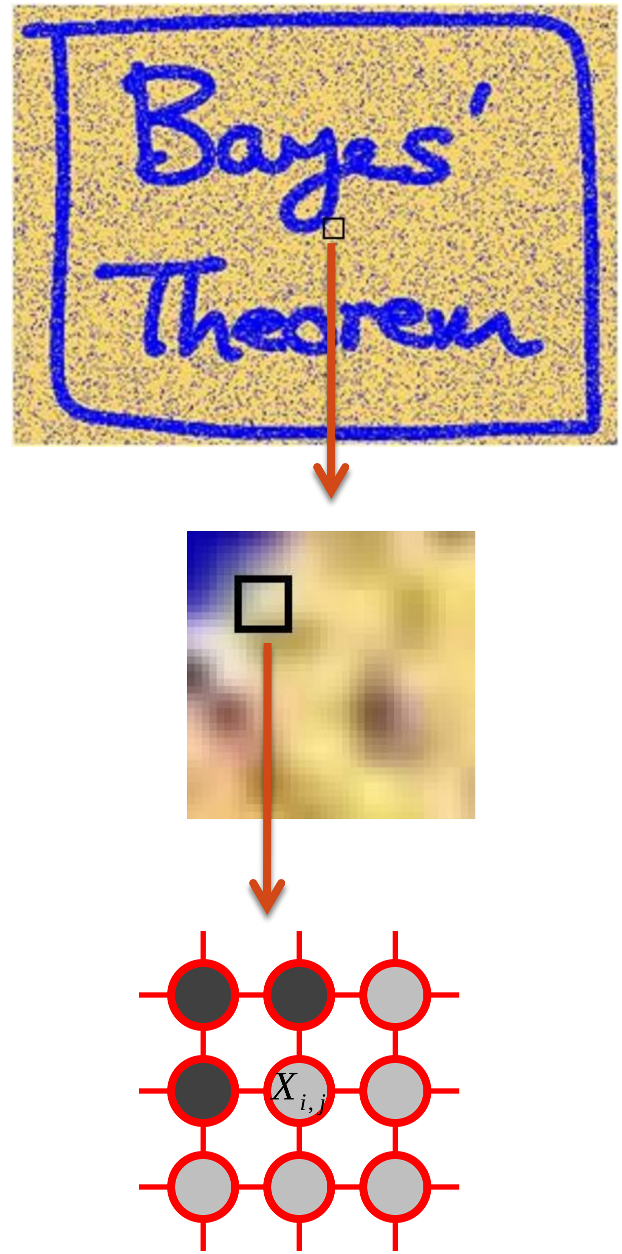
Outline

- Problem
  - Description
  - Definition
- Proof

Problem

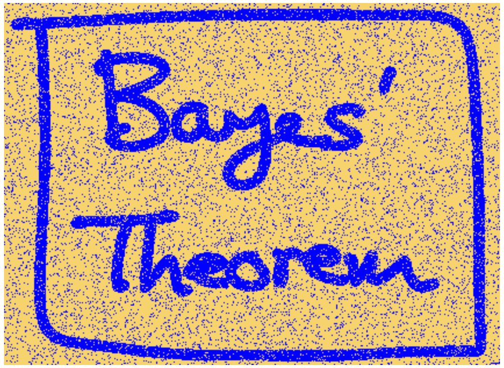
Problem

- Description
  - Each image I consists of m by n binary pixels
  - Each pixel I(i,j) corresponding to a boolean variable $X_{i,j} = \{1, 0\}$
  - Each two adjacent pixels are connected with an edge

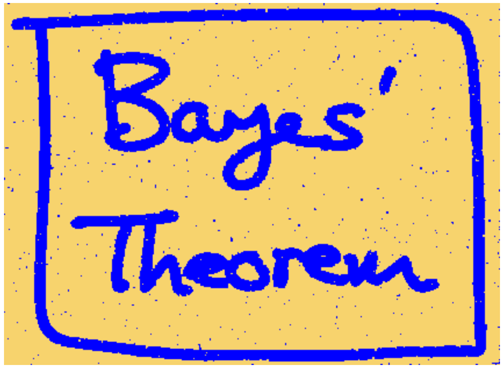


Problem

- Goal
  - Given a binary noise image I'
  - Generate a new binary image I that eliminates the noise in I'



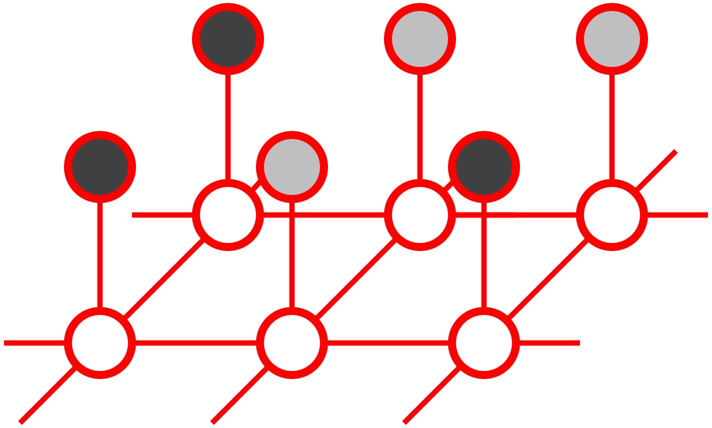
Noise image I'



New image I

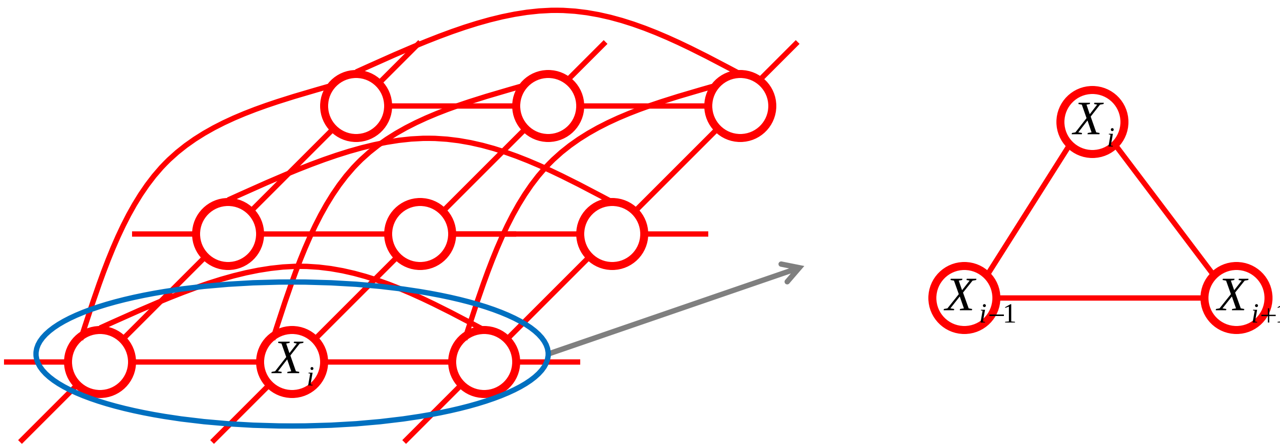
Problem

- Definition 1
  - *The* ***observation*** *is the value of node of known noise image, and a* ***instant*** *is a function which assigns a value to each variable in the new image*



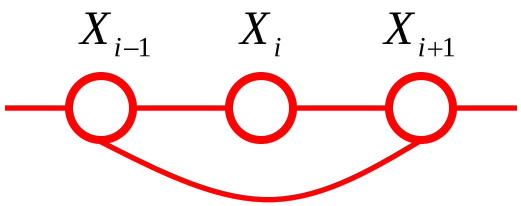
Problem

- Definition 2
  - *A* ***clique*** $Q_k(X_k)$ *is a complete sub-graph with a set of variables in clique*



Problem

- Definition 3
  - *A **clique table(CT)** T for a variable X with a set of variables in **clique** Q is a function that maps each **instant** of Q to a binary value*



Clique k

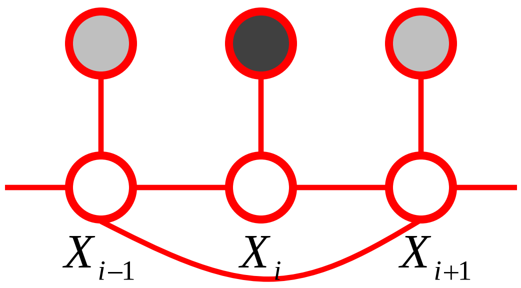
ϕ_{Q_k}	X_{i-1}	X_i	X_{i+1}
1	x_{i-1}	x_i	x_{i+1}
1	$\overline{x}_{i-1}$	x_i	x_{i+1}
1	x_{i-1}	$\overline{x}_i$	x_{i+1}
1	x_{i-1}	x_i	$\overline{x}_{i+1}$
1	x_{i-1}	$\overline{x}_i$	$\overline{x}_{i+1}$
1	$\overline{x}_{i-1}$	x_i	$\overline{x}_{i+1}$
1	$\overline{x}_{i-1}$	$\overline{x}_i$	x_{i+1}
1	$\overline{x}_{i-1}$	$\overline{x}_i$	$\overline{x}_{i+1}$

Problem

- Simple rules
  - If the observations of a clique have same value, the variables in the clique can't against all of them
  - If the observation of a clique have different value, the variables in the clique can't follow all of them

Problem

- Example
  - Assume dark node is 0 and light node is 1, since variables can't follow the value of observations, $(x_{i-1} \wedge \overline{x}_i \wedge x_{i+1}) = 0$



Clique k

ϕ_{C_k}	X_{i-1}	X_i	X_{i+1}
1	x_{i-1}	x_i	x_{i+1}
1	$\overline{x}_{i-1}$	x_i	x_{i+1}
0	x_{i-1}	$\overline{x}_i$	x_{i+1}
1	x_{i-1}	x_i	$\overline{x}_{i+1}$
1	x_{i-1}	$\overline{x}_i$	$\overline{x}_{i+1}$
1	$\overline{x}_{i-1}$	x_i	$\overline{x}_{i+1}$
1	$\overline{x}_{i-1}$	$\overline{x}_i$	x_{i+1}
1	$\overline{x}_{i-1}$	$\overline{x}_i$	$\overline{x}_{i+1}$

Problem

- Therefore, as the example with the function of clique k could be represented as the

$$\phi_{Q_k} = (x_{i-1} \wedge x_i \wedge x_{i+1}) \vee (x_{i-1} \wedge x_i \wedge \overline{x}_{i+1}) \vee (\overline{x}_{i-1} \wedge x_i \wedge x_{i+1}) \vee$$

$$(x_{i-1} \wedge \overline{x}_i \wedge \overline{x}_{i+1}) \vee (\overline{x}_{i-1} \wedge x_i \wedge \overline{x}_{i+1}) \vee (\overline{x}_{i-1} \wedge \overline{x}_i \wedge x_{i+1}) \vee$$

$$(\overline{x}_{i-1} \wedge \overline{x}_i \wedge \overline{x}_{i+1})$$

Problem

- Definition 4
  - *A **Markov network** is a pair (G,Q) where G is a undirected graph whose nodes are variables and Q is a set which consists of the **clique** of each variable*

$$score(G,Q)=\prod_{k=1}^{K}\phi_{Q_k}(Q_k)$$

Proof

Formal definition: IDP

- Given: A Markov network (G,Q) and an instant of variables X
- Question Does there exist an assignment to X so that

$$\prod_{k=1}^{K} \phi_{Q_k}\left(Q_k\right) = 1$$

Formal definition: 3-SAT

- Given: A set U of boolean variables and a collection $C_i \subseteq U \cup \bar{U}$, and $|C_i| = 3$
- Question: Does there exist an assignment to U so that all clauses are true?

Proof

- IDP ∈ NP

```
for i=1 to K
    for j=1 to 7(the formula number of clique table)
        check if the variables in clique i is satisfying one of the
            formulas
    end
end
```

Proof

- Trick
  - Just check the inverse value of the zero formula in the clique table
- Example
  - As shown in previous table, the zero formula is $\left(x_{i-1} \wedge \overline{x}_i \wedge x_{i+1}\right)$
  - Check the variables value of inverse formula $\left(\overline{x}_{i-1} \wedge x_i \wedge \overline{x}_{i+1}\right)$
  - If one(or more) of the element of inverse formula is true, then

$$\begin{aligned}\phi_{Q_k} = &\left(x_{i-1} \wedge x_i \wedge x_{i+1}\right) \vee \left(x_{i-1} \wedge x_i \wedge \overline{x}_{i+1}\right) \vee \left(\overline{x}_{i-1} \wedge x_i \wedge x_{i+1}\right) \vee \\ &\left(x_{i-1} \wedge \overline{x}_i \wedge \overline{x}_{i+1}\right) \vee \left(\overline{x}_{i-1} \wedge x_i \wedge \overline{x}_{i+1}\right) \vee \left(\overline{x}_{i-1} \wedge \overline{x}_i \wedge x_{i+1}\right) \vee \\ &\left(\overline{x}_{i-1} \wedge \overline{x}_i \wedge \overline{x}_{i+1}\right) \\ = &1\end{aligned}$$

Proof

- Input of IDP
  - X = boolean variables
  - Q = cliques
  - T = table of each clique
- Input of 3SAT
  - $U = X \cup \overline{X}$
  - $C = \{C_1, \ldots, C_K\}$, where C_k is the inverse of zero formula

Proof

- $\text{Yes}(\text{IDP}) \rightarrow \text{Yes}(\text{3SAT})$
  - If IDP is true, which means each clique function $\phi_{Q_k} = 1$, the corresponding clause C_k is true.
  - Clearly 3SAT is true
- $\text{Yes}(\text{3SAT}) \rightarrow \text{Yes}(\text{IDP})$
  - If 3SAT is true, for each clause, one of the formulas in corresponding table T except zero formula must be true
  - IDP is true

Conclusion

$$\text{IDP} \propto \text{3SAT}$$

References

- James D. Park, Using Weighted MAX-SAT Engines to Solve MPE, in *AAAI*, 2002