COT 6410 Fall 2010 Exam#1 Name: KEY

12 **1**. Choosing from among **(REC) recursive**, **(RE) re non-recursive, (coRE) co-re non-recursive**, **(NRNC) non-re/non-co-re**, categorize each of the sets in a) through d). Justify your answer by showing some minimal quantification of some known recursive predicate.

a) A = { f | f(x) ↑ for all x }

$\forall \langle x,t\rangle \sim STP(f,x,t)$ — *coRE*

b.) B = { f | domain(f) is a proper subset of ℵ ; that is f diverges at some points }

$\exists x \forall t \sim STP(f,x,t)$ — *NRNC*

c.) C = { f | f(x) > x for at least one value x }

$\exists \langle x,t\rangle [STP(f,x,t) \& VALUE(f,x,t)>x]$ — *RE*

d.) D = { $\langle f,x\rangle$ | f(x) converges in at most x steps }

$STP(f,x,x)$ — *REC*

6 **2.** Prove that the **Uniform Halting Problem** (the set **TOTAL**) is non-re within any formal model of computation. (Hint: A diagonalization proof is required here.)

Look at Notes

6 **3**. Let set **A** be recursive, **B** be re non-recursive and **C** be non-re. Choosing from among **(REC) recursive**, **(RE) re non-recursive**, **(NR) non-re**, categorize the set **D** in each of a) through d) by listing **all** possible categories. No justification is required.

a.) D = ~B *NR*

b.) D ⊆ A *REC, RE, NR*

c.) D = A ∪ B *REC, RE*

d.) D = C – A *REC, RE, NR*

10 **4**. Let set **A** be non-empty recursive, **B** be re non-recursive and **C** be non-re. Using the terminology **(REC) recursive**, (**RE) re non-recursive**, **(NR) non-re**, categorize each set by dealing with the cases I present, saying whether or not the set can be of the given category and. briefly, but convincingly, justifying each answer. You may assume, for any set **S**, the existence of comparably hard sets
S_E = **{2x|x∈S}** and S_O = **{2x+1|x∈S}**. The following is a sample of the kind of answer I require:

a.) A ∩ B = { x | x ∈ A and x ∈ B }

REC: ***Yes. Choose A = {0}, then A ∩ B = {0} or A ∩ B = { }. In either case, the set is REC***

RE: ***Yes. If A = ℵ then A ∩ B = B, which is RE***

NR: ***No. Let χ_A be a characteristic function for A and g_B be a semi-decision procedure for B, then A ∩ B is semi-decided by $g_{A \cap B}$ where $g_{A \cap B}(x) = \chi_A(x) * g_B(x)$ and so A ∩ B is always RE***

b.) A* C = { x*y | x ∈ A and y ∈ C }

REC: ***Yes. Choose A = {0}, then A* C = {0} which is REC***

NR: ***Yes. Choose A = {1}, then A* C = C which is NR***

5. Consider the set of indices **TwoOrMore = { f | |range(f)| > 1 }**.

3 **a.)** Show some minimal quantification of some known recursive predicate that provides an upper bound for the complexity of this set. (Hint: Look at **c.)** and **d.)** to get a clue as to what this must be.)

$\exists\langle x,y,t\rangle\ [x \neq y\ \&\ STP(f,x,t)\ \&\ STP(f,y,t)\ \&\ Value(f,x,t) \neq Value(f,y,t)]$

4 **b.)** Use Rice's Theorem to prove that **TwoOrMore** is undecidable.

TwoOrMore is non-trivial: C0 ∉ TwoOrMore; S ∈ TwoOrMore
TwoOrMore is an I/O Property:
Let f and g be such that $\forall x\ [f(x) == g(x)]$
Clearly the ranges of f and g are the same and hence both either are in or out of TwoOrMore
As TwoOrMore satisfies both requirements of ice's Theorem, TwoOrMore is undecidable.

3 **c.)** Show that $\mathbf{K \leq_m}$ **TwoOrMore**, where **K = { f |** $\boldsymbol{\varphi_f(f)\downarrow}$ **}**.

Let f be arbitrary and let $g_f(x) = \varphi_f(f) - \varphi_f(f) + x$
Clearly, if f ∈ K, $g_f(x) = x$***, for all x, in which case*** $g_f \in$ ***TwoOrMore;***
If f ∉ K, $g_f(x)\uparrow$***, for all x, in which case*** $g_f \notin$ ***TwoOrMore.***
Thus, f ∈ K iff $g_f \in$ ***TwoOrMore, proving that*** $K \leq_m$ ***TwoOrMore***

3 **d.)** Show that **TwoOrMore** $\mathbf{\leq_m K}$, where **K = { f |** $\boldsymbol{\varphi_f(f)\downarrow}$ **}**.

Let f be arbitrary and
let $g_f(z) = \mu\langle x,y,t\rangle\ [x \neq y\ \&\ STP(f,x,t)\ \&\ STP(f,y,t)\ \&\ Value(f,x,t) \neq Value(f,y,t)]$
If f ∈ TwoOrMore then is g_f ***is defined everywhere and so*** $g_f \in K$
If f ∉ TwoOrMore then is g_f ***is diverges everywhere and so*** $g_f \notin K$
Thus, f ∈ TwoOrMore iff $g_f \in K$***, proving that TwoOrMore*** $\leq_m K$

3 **e.)** From **a.)** through **d.)** what can you conclude about the computable complexity of **TwoOrMore** (choose from **REC, RE, RE-COMPLETE, CO-RE, CO-RE-COMPLETE, NON_RE/NON-CO_RE**)? Briefly justify your conclusion.

RE-COMPLETE. (a) says at worst RE; (b) says non-recursive; (c) shows that all re sets are reducible to TwoOrMore since K is known to be RE-COMPLETE; (d) just provides redundant confirmation of the set's RE-COMPLETE status.Thus, you can just use (d) as a proof since TwoOrMore $\equiv_m$ ***K and K is known to be RE-COMPLETE***

3 **6.** Why does Rice's Theorem have nothing to say about the following? Explain by showing some condition of Rice's Theorem that is not met by the stated property.
NOT_CONSTANT_TIME = { f | for any fixed C, ∃y $\boldsymbol{\varphi_f(y)}$ **fails to converge in C steps }**.
Consider C_0 ***and*** K_0***, where C0 is the constant 0 base function.***

(1) $C_0 \in$ ***NOT_CONSTANT_TIME***
(2) $K_0 \notin$ ***NOT_CONSTANT_TIME, where***
$K_0(0) = C_0(0)$
$K_0(y+1) = K_0(y)$

Here $C_0(x) = K_0(x)$***, for all x, yet one belongs to NOT_CONSTANT_TIME and the other does not. Thus, NOT_CONSTANT_TIME is not an I/O property and so Rice's Theorem has nothing to say about its possible undecidability.***

6 **7.** Let **S** be an arbitrary infinite re set. This means that **S** is the range of some total recursive function $\mathbf{f_s}$. It also means **S** is the domain of some partial recursive function $\mathbf{g_S}$. Additionally, the range of $\mathbf{f_S}$ is infinite and the domain of $\mathbf{g_S}$ is similarly infinite. Using either $\mathbf{f_S}$ or $\mathbf{g_S}$, show that **S** has an infinite recursive subset, call it **R**. To be complete you will need to create a characteristic function for **R**, $\chi_\mathbf{R}$, and argue that the set **R** you defined is infinite.

Define $f_R(0) = f_S(0); f_R(y+1) = f_S(\mu z [f_S(z) > f_R(y)])$

First, we need to argue that $f_R(y)$ ***is defined everywhere, but this is clear since*** f_S ***is defined everywhere;*** $f_R(0)$ ***is directly defined from*** $f_S(0)$***; and the infiniteness of S guarantees there is always a larger value enumerated by*** f_S ***than we have found as the value enumerated at*** $f_R(y)$***, for any y.***

Second, by definition $f_R(y+1) > f_R(y)$***, for all y, so the range of*** f_R ***is monotonically increasing and its range is infinite.***

Third, since $f_R(y)$***, for any y, is defined as some element enumerated by*** f_S***, its range is an infinite subset of S.***

Now, define $\chi_R(x) = \exists z (z \leq x) [f_R(z) = x]$

χ_R ***uses the bounded existential quantifier, so it always returns a value (0 or 1). If x is in the range of*** f_R***, then it must appear by the time we enumerate the first x values (0-based counting) since*** f_R ***is monotonically increasing. Thus,*** χ_R ***is a characteristic function for R, as required, and so R is recursive.***