Satisfiability problems on sums of Kripke frames

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Abstract

We consider the operation of sum on Kripke frames, where a family of frames-summands is indexed by elements of another frame. In many cases, the modal logic of sums inherits the finite model property and decidability from the modal logic of summands [BR10], [Sha18]. In this paper we show that, under a general condition, the satisfiability problem on sums is polynomial space Turing reducible to the satisfiability problem on summands. In particular, for many modal logics decidability in PSpace is an immediate corollary from the semantic characterization of the logic.

Keywords: *sum of Kripke frames, finite model property, decidability, Turing reduction, PSPACE, Japaridze's polymodal logic, lexicographic product of modal logics, lexicographic sum of modal logics*

1 Introduction

In classical model theory, there is a number of results ("composition theorems") that reduce the theory (first-order, MSO) of a compound structure (e.g., sum or product) to the theories of its components, see, e.g., [Mos52, FV59, She75, Gur79] or [Gur85]. In this paper we use the composition method in the context of modal logic.

Given a family $(\mathsf{F}_i \mid i \text{ in } \mathsf{I})$ of frames (structures with binary relations) indexed by elements of another frame I, the *sum of the frames* F_i*'s over* I is obtained from their disjoint union by connecting elements of i-th and j-th distinct components according to the relations in I. Given a class $\mathcal{F}$ of frames-summands and a class $\mathcal{I}$ of frames-indices, $\sum_{\mathcal{I}} \mathcal{F}$ denotes the class of all sums of F_i's in $\mathcal{F}$ over I in $\mathcal{I}$.

In many cases, this operation preserves the finite model property and decidability of the logic of summands [BR10], [Sha18]. In this paper we show that transferring results also hold for the complexity of the modal satisfiability problems on sums. In particular, it follows that for many logics PSpace-completeness is an immediate corollary of semantic characterization.

It is a classical result by R. Ladner that the decision problem for the logic of (finite) preorders S4 is in PSpace [Lad77]. In [Sha08], it was shown that the polymodal provability logic GLP is also decidable in PSpace. In spite of the significant difference between these two logics, there is a uniform proof for both these two particular systems, as well as for many other important modal logics: the general phenomenon is that the modal satisfiability problem on sums over Noetherian (in particular, finite) orders is polynomial space Turing reducible to the modal satisfiability problem on summands. In the case of S4 these summands are frames of form $(W, W \times W)$ with the satisfiability problem being in NP, so in PSpace. And hence, S4 is in PSpace. In the case of GLP, the class of summands is even simpler: the only summand required is an irreflexive singleton [Bek10]. (We will discuss these and other examples in Sections 4.5, 5.3, 5.4.)

The paper has the following structure. Sections 2 contains preliminary material. Section 3 is about truth-preserving operations on sums of frames; it contains necessary tools for the complexity results, and also quotes recent results on the finite model property. This section is based on [Sha18]. The central Section 4 is about complexity. The reduction between sums and summands is described in Theorem 9 and in its more technical (but more tunable) version Theorem 15; these are the main results of the paper. This section elaborates earlier works [Sha08, Sha05] (in particular, our new results significantly generalize and simplify Theorems 22 and 35 in [Sha08]). In Section 5 we discuss modifications of the sum operation: (iterated) lexicographic sums of frames, which are important in the context of provability logics [Bek10]; lexicographic products of frames, earlier studied in [Bal09, Bal10, BM13, BFD16]; the operation of refinement of modal logics introduced in [BR10]. Further results and directions are discussed in Section 6.

2 Preparatory syntactical and semantical definitions

Let $\mathrm{A} \leq \omega$. The set $\mathrm{ML}(\mathrm{A})$ of *modal formulas over* A (or A-*formulas*, for short) is built from a countable set of *variables* $\mathrm{PV} = \{p_0, p_1, \ldots\}$ using Boolean connectives $\bot, \to$ and unary connectives $\Diamond_a$, $a < \mathrm{A}$ (*modalities*). The connectives $\vee, \wedge, \neg, \top, \Box_a$ are defined as abbreviations in the standard way, in particular $\Box_a\varphi$ is $\neg\Diamond_a\neg\varphi$.

An (A-)frame is a structure $\mathsf{F} = (W, (R_a)_{a<\mathrm{A}})$, where $W \neq \varnothing$ and $R_a \subseteq W \times W$ for $a < \mathrm{A}$. A *(Kripke) model on* F is a pair $\mathsf{M} = (\mathsf{F}, \theta)$, where $\theta : \mathrm{PV} \to 2^W$. We write $\mathrm{dom}(\mathsf{F})$ for W, which is called the *domain* of F. We write $u \in \mathsf{F}$ for $u \in \mathrm{dom}(\mathsf{F})$. For $u \in W$, $V \subseteq W$, we put $R_a(u) = \{v \mid uR_av\}$, $R_a[V] = \cup_{v\in V} R_a(v)$.

The *truth relation* in a model is defined in the usual way, in particular $\mathsf{M}, w \models \Diamond_a\varphi$ means that $\mathsf{M}, v \models \varphi$ for some v in $R_a(w)$. A formula φ is *satisfiable in a model* M if $\mathsf{M}, w \models \varphi$ for some w in M. A formula is *satisfiable in a frame* F (*in a class* $\mathcal{F}$ *of frames*) if it is satisfiable in some model on F (in some model on a frame in $\mathcal{F}$). φ is valid in a frame F (in a class $\mathcal{F}$ of frames) if $\neg\varphi$ is not satisfiable in F (in $\mathcal{F}$). Validity of a set of formulas means validity of every formula in this set.

The notions of p-morphism, generated subframe and submodel are defined in the standard way, see e.g. [GKWZ03, Section 1.4]. The notation $\mathsf{F} \twoheadrightarrow \mathsf{G}$ means that G is a p-morphic image of F; $\mathsf{F} \cong \mathsf{G}$ means that F and G are isomorphic.

A (*propositional normal modal*) *logic* is a set L of formulas that contains all classical tautologies, the axioms $\neg\Diamond_a\bot$ and $\Diamond_a(p_0 \vee p_1) \to \Diamond_a p_0 \vee \Diamond_a p_1$ for each a in A, and is closed under the rules of modus ponens, substitution and monotonicity (if $\varphi \to \psi \in L$, then $\Diamond_a\varphi \to \Diamond_a\psi \in L$, for each a in A).

The set $\mathrm{Log}\,\mathcal{F}$ of all formulas that are valid in a class $\mathcal{F}$ of A-frames is a logic (see, e.g., [CZ97]); it is called the *logic of* $\mathcal{F}$; such logics are called *Kripke complete.* A logic has the *finite model property* if it is the logic of a class of finite frames (a frame is finite, if its domain is). $\mathrm{Fr}\,L$ denotes the class of all frames validating L.

Remark that for a Kripke complete logic, its decision problem is the validity problem on the class of all its frames (or on any other class $\mathcal{F}$ such that $L = \mathrm{Log}(\mathcal{F})$). The dual problem is the satisfiability: $\mathrm{Sat}(\mathcal{F})$ is the set of all satisfiable in $\mathcal{F}$ formulas (in the signature of $\mathcal{F}$). Remark that for the class PSpace (as well as for any other deterministic complexity class), $\mathrm{Sat}\,\mathcal{F} \in \mathrm{PSpace}$ iff $\mathrm{Log}\,\mathcal{F} \in \mathrm{PSpace}$.

Natural numbers are considered as finite ordinals. Given a sequence $\boldsymbol{v} = (v_0, v_1, \ldots)$, we write $\boldsymbol{v}(i)$ for v_i.

3 Sums

We fix $\mathrm{A} \leq \omega$ for the alphabet and consider the language $\mathrm{ML}(\mathrm{A})$.

Consider a non-empty family $(\mathsf{F}_i)_{i\in I}$ of A-frames $\mathsf{F}_i = (W_i, (R_{i,a})_{a\in\mathrm{A}})$. The *disjoint union* of these frames is the A-frame $\bigsqcup_{i\in I}\mathsf{F}_i = (\bigsqcup_{i\in I} W_i, (R_a)_{a\in\mathrm{A}})$, where $\bigsqcup_{i\in I} W_i = \bigcup_{i\in I}(\{i\}\times W_i)$ is the disjoint union of sets W_i, and

$$(i,w)R_a(j,v) \quad \text{iff} \quad i = j \,\&\, wR_{i,a}v.$$

Suppose that I is the domain of another A-frame $\mathsf{I} = (I, (S_a)_{a\in\mathrm{A}})$.

Definition 1. The *sum of the family* $(\mathsf{F}_i)_{i\in I}$ *of* A*-frames over the* A*-frame* $\mathsf{I} = (I, (S_a)_{a\in\mathrm{A}})$ is the A-frame $\sum_{i\in\mathsf{I}}\mathsf{F}_i = (\bigsqcup_{i\in I} W_i, (R^{\Sigma}_a)_{a\in\mathrm{A}})$, where

$$(i,w)R^{\Sigma}_a(j,v) \quad \text{iff} \quad i = j \,\&\, wR_{i,a}v \text{ or } i \neq j \,\&\, iS_aj.$$

The sum of models $\sum_{i\in\mathsf{I}}(\mathsf{F}_i,\theta_i)$ is the model $(\sum_{i\in\mathsf{I}}\mathsf{F}_i,\theta)$, where $(i,w)\in\theta(p)$ iff $w\in\theta_i(p)$.

For classes $\mathcal{I}$, $\mathcal{F}$ of A-frames, let $\sum_{\mathcal{I}}\mathcal{F}$ be the class of all sums $\sum_{i\in\mathsf{I}}\mathsf{F}_i$ such that $\mathsf{I}\in\mathcal{I}$ and $\mathsf{F}_i\in\mathcal{F}$ for every i in I.

Remark 1. We do not require that S_a's are partial orders or even transitive relations. Also, we note that the relations R^{Σ}_a are independent of reflexivity of the relations S_a: if $\mathsf{I}' = (I, (S'_a)_{a\in N})$, where S'_a is the reflexive closure of S_a for each $a\in\mathrm{A}$, then $\sum_{i\in\mathsf{I}}\mathsf{F}_i = \sum_{i\in\mathsf{I}'}\mathsf{F}_i$.

Example 1 (Skeleton and clusters). A *cluster* is a frame of the form $(W, W\times W)$.

Let $\mathsf{F} = (W,R)$ be a preorder. The *skeleton* of F is the partial order $\mathrm{sk}\mathsf{F} = (\overline{W}, \leq_R)$, where $\overline{W}$ is the quotient set of W by the equivalence $R\cap R^{-1}$, and for $C, D\in\overline{W}$, $C\leq_R D$ iff $\exists w\in C\,\exists v\in D\; wRv$. The restriction of F on an element of $\overline{W}$ is called a *cluster in* F.

It is easy to see that every preorder F is isomorphic to the sum $\sum_{C\in\mathrm{sk}\mathsf{F}}(C, C\times C)$ of its clusters over its skeleton.

Example 2. Suppose that $\mathsf{F} = (W,R)$ satisfies the property of *weak transitivity* $xRzRy \Rightarrow xRy \vee x = y$. Let I be the skeleton of the preorder (W,R^*), where R^* is the transitive reflexive closure of R. Then F is isomorphic to a sum $\sum_{i\in\mathsf{I}}\mathsf{F}_i$ such that every $\mathsf{F}_i = (W_i, R_i)$ satisfies the property $x\neq y \Rightarrow xR_iy$.

3.1 Basic truth-preserving operations

The theorem below is a collection of facts illustrating how sums interact with p-morphisms, generated subframes, and disjoint unions. Their proofs are straightforward from definitions, see [Sha18] for the details.

Theorem 1 ([Sha18])**.**

1. *Let* I *be an* A*-frame,* $(\mathsf{F}_i)_{i\in\mathsf{I}}$ *is a sequence of* A*-frames. If* J *is a generated subframe of* I*, then* $\sum_{i\in\mathsf{J}}\mathsf{F}_i$ *is a generated subframe of* $\sum_{i\in\mathsf{I}}\mathsf{F}_i$*.*

2. *Let* I*,* J *be* A*-frames,* $(\mathsf{F}_i)_{i\in\mathsf{I}}$*,* $(\mathsf{G}_j)_{j\in\mathsf{J}}$ *families of* A*-frames. Suppose that all the relations in* J *are irreflexive.*

   (a) If $f : \mathsf{I}\twoheadrightarrow\mathsf{J}$ *and* $\mathsf{F}_i\twoheadrightarrow\mathsf{G}_{f(i)}$ *for all* i *in* I*, then* $\sum_{i\in\mathsf{I}}\mathsf{F}_i\twoheadrightarrow\sum_{j\in\mathsf{J}}\mathsf{G}_j$*.*

(b) *If* $\mathsf{I} = \mathsf{J}$ *and* $\mathsf{F}_i \twoheadrightarrow \mathsf{G}_i$ *for all* i *in* I, *then* $\sum_{i\in\mathsf{I}} \mathsf{F}_i \twoheadrightarrow \sum_{i\in\mathsf{I}} \mathsf{G}_i$.

(c) *If* $f : \mathsf{I} \twoheadrightarrow \mathsf{J}$, *then* $\sum_{i\in\mathsf{I}} \mathsf{G}_{f(i)} \twoheadrightarrow \sum_{j\in\mathsf{J}} \mathsf{G}_j$.

3. (a) *Let* I *be an* A*-frame,* $(\mathsf{J}_i)_{i\in\mathsf{I}}$ *a family of* A*-frames, and* $(\mathsf{F}_{ij})_{i\in\mathsf{I},j\in\mathsf{J}_i}$ *a family of* A*-frames. Then*

$$\sum_{i\in\mathsf{I}}\sum_{j\in\mathsf{J}_i}\mathsf{F}_{ij} \quad\cong\quad \sum_{(i,j)\in\sum_{k\in\mathsf{I}}\mathsf{J}_k}\mathsf{F}_{ij}.$$

(b) *Let* I *be a non-empty set,* $(\mathsf{J}_i)_{i\in\mathsf{I}}$ *a family of* A*-frames, and* $(\mathsf{F}_{ij})_{i\in\mathsf{I},j\in\mathsf{J}_i}$ *a family of* A*-frames. Then*

$$\bigsqcup\nolimits_{i\in I}\sum_{j\in\mathsf{J}_i}\mathsf{F}_{ij} \quad\cong\quad \sum_{(i,j)\in\bigsqcup_{k\in I}\mathsf{J}_k}\mathsf{F}_{ij}.$$

(c) *Let* I *be an* A*-frame,* $(J_i)_{i\in\mathsf{I}}$ *a family of non-empty sets, and* $(\mathsf{F}_{ij})_{i\in\mathsf{I},j\in J_i}$ *a family of* A*-frames. Then*

$$\sum_{i\in\mathsf{I}}\bigsqcup\nolimits_{j\in J_i}\mathsf{F}_{ij} \quad\cong\quad \sum_{(i,j)\in\sum_{k\in\mathsf{I}}(J_k,(\varnothing)_{\mathrm{A}})}\mathsf{F}_{ij},$$

where $(\varnothing)_{\mathrm{A}}$ *denotes the sequence of length* A *in which every element is the empty set.*

3.2 Interchangeable summands

Classes of A-frames $\mathcal{F}$ and $\mathcal{G}$ are said to be *interchangeable*, in symbols $\mathcal{F} \equiv \mathcal{G}$, if $\mathcal{F}$ and $\mathcal{G}$ have the same modal logic in the language enriched with the universal modality. Formally, for an A-frame $\mathsf{F} = (W, (R_0, R_1, \ldots))$, let $\mathsf{F}^{(\forall)}$ be the $(1 + \mathrm{A})$-frame $(W, (W \times W, R_0, R_1, \ldots))$. For a class $\mathcal{F}$ of A-frames, $\mathcal{F}^{[\forall]} = \{\mathsf{F}^{[\forall]} \mid \mathsf{F} \in \mathcal{F}\}$. We put

$$\mathcal{F} \equiv \mathcal{G} \text{ if } \operatorname{Log}\mathcal{F}^{(\forall)} = \operatorname{Log}\mathcal{G}^{(\forall)}.$$

In [Sha18], it was shown that if $\mathcal{F}$ and $\mathcal{G}$ are interchangeable, then for every class $\mathcal{I}$ of A-frames, the logics of sums $\sum_{\mathcal{I}}\mathcal{F}$ and $\sum_{\mathcal{I}}\mathcal{G}$ are equal; moreover, these classes of sums are interchangeable again, thus we have $\operatorname{Log}\sum_{\mathcal{J}}(\sum_{\mathcal{I}}\mathcal{F}) = \operatorname{Log}\sum_{\mathcal{J}}(\sum_{\mathcal{I}}\mathcal{G})$ for any other class of frames-indices $\mathcal{J}$, and so on.

Theorem 2 ([Sha18])**.** *Let* $\mathcal{I}$, $\mathcal{F}$, $\mathcal{G}$ *be classes of* A*-frames. If* $\mathcal{F} \equiv \mathcal{G}$, *then* $\sum_{\mathcal{I}}\mathcal{F} \equiv \sum_{\mathcal{I}}\mathcal{G}$.

In particular, it follows that if the logic of the class $\mathcal{F}^{(\forall)}$ has the finite model property, then the logic of the class of sums $\sum_{\mathcal{I}}\mathcal{F}$ is equal to the logic of the class of sums $\sum_{\mathcal{I}}\mathcal{G}$, where $\mathcal{G}$ is a class of finite frames.

3.3 Decomposition of sums

To reduce satisfiability in sums to the satisfiability in summands, we will use an auxiliary notion: *satisfiability under conditions*.

Definition 2. A sequence $\mathbf{\Gamma} = (\Gamma_a)_{a\in\mathrm{A}}$, where Γ_a are sets of A-formulas, is called a *condition* (in the language $\mathrm{ML}(\mathrm{A})$).

Consider a model $\mathsf{M} = (W, (R_a)_{a\in\mathrm{A}}, \theta)$, w in M. By induction on the length of an A-formula φ, we define the relation $\mathsf{M}, w \models_{\mathbf{\Gamma}} \varphi$ ("*under the condition* $\mathbf{\Gamma}$, φ *is true at* w *in* M"): as usual, $\mathsf{M}, w \not\models_{\mathbf{\Gamma}} \bot$, $\mathsf{M}, w \models_{\mathbf{\Gamma}} p$ iff $\mathsf{M}, w \models p$ for a variable p, $\mathsf{M}, w \models_{\mathbf{\Gamma}} \varphi \to \psi$ iff $\mathsf{M}, w \not\models_{\mathbf{\Gamma}} \varphi$ or $\mathsf{M}, w \models_{\mathbf{\Gamma}} \psi$; for $a \in \mathrm{A}$,

$$\mathsf{M}, w \models_{\mathbf{\Gamma}} \Diamond_a\varphi \quad\text{iff}\quad \varphi \in \Gamma_a \text{ or } \exists v \in R_a(w)\ \ \mathsf{M}, v \models_{\mathbf{\Gamma}} \varphi.$$

In particular, if all Γ_a are empty, then we have the standard notion of truth in a Kripke model:

$$\mathsf{M}, w \models_{(\varnothing)_{\mathrm{A}}} \varphi \quad \text{iff} \quad \mathsf{M}, w \models \varphi,$$

where $(\varnothing)_{\mathrm{A}}$ denotes the condition consisting of empty sets. The truth under conditions is respected by the standard operations on Kripke models:

Proposition 3. *Let $\mathbf{\Gamma}$ be a condition, φ a formula.*

1. *If M' is a generated submodel of M, then $\mathsf{M}', w \models_{\mathbf{\Gamma}} \varphi$ iff $\mathsf{M}, w \models_{\mathbf{\Gamma}} \varphi$ for every every w in M'.*

2. *If $\mathsf{M} = \bigsqcup_{i \in I} \mathsf{M}_i$, then $\mathsf{M}, (i, w) \models_{\mathbf{\Gamma}} \varphi$ iff $\mathsf{M}_i, w \models_{\mathbf{\Gamma}} \varphi$ for every i in I and every w in M_i.*

3. *If $f : \mathsf{M} \twoheadrightarrow \mathsf{M}'$, then $\mathsf{M}, w \models_{\mathbf{\Gamma}} \varphi$ iff $\mathsf{M}', f(w) \models_{\mathbf{\Gamma}} \varphi$ for every w in M.*

Proof. This proof completely reflects the proof of these facts for the standard truth relation in Kripke models and can be obtained by a straightforward induction of the length of φ. □

Let $\mathrm{sub}(\varphi)$ be the set of all subformulas of φ. We put

$$\varphi[\mathsf{M}, \mathbf{\Gamma}] = \{\psi \in \mathrm{sub}(\varphi) \mid \mathsf{M}, v \models_{\mathbf{\Gamma}} \psi \text{ for some } v\}.$$

In particular, $\varphi[\mathsf{M}, (\varnothing)_{\mathrm{A}}]$ is the set of all subformulas of φ that are satisfiable in M.

A triple $(\varphi, \Phi, \mathbf{\Gamma})$, where $\Phi \subseteq \mathrm{sub}(\varphi)$, is called a *tie*. A tie $(\varphi, \Phi, \mathbf{\Gamma})$ is *satisfiable* in a frame F (in a class $\mathcal{F}$ of frames) if there exists a model M on F (on a frame in $\mathcal{F}$) such that $\Phi = \varphi[\mathsf{M}, \mathbf{\Gamma}]$. Hence, we have:

Proposition 4. *A formula φ is satisfiable in a class $\mathcal{F}$ of frames iff there exists $\Phi \subseteq \mathrm{sub}(\varphi)$ such that $\varphi \in \Phi$ and the tie $(\varphi, \Phi, (\varnothing)_{\mathrm{A}})$ is satisfiable in $\mathcal{F}$.*

In other words, if the same ties are satisfiable in classes $\mathcal{F}$ and $\mathcal{G}$, then their logics coincide. Moreover, the following holds:

Theorem 5 ([Sha18])**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be classes of* A*-frames. The following are equivalent:*

- *A tie is satisfiable in $\mathcal{F}$ iff it is satisfiable in $\mathcal{G}$.*
- *A formula is satisfiable $\mathcal{F}^{(\forall)}$ iff it is satisfiable in $\mathcal{G}^{(\forall)}$.*

The latter condition means that the logics of $\mathcal{F}^{(\forall)}$ and $\mathcal{G}^{(\forall)}$ are equal, i.e., $\mathcal{F} \equiv \mathcal{G}$.

Definition 3. Let V be a set of elements of a model $\mathsf{M} = (W, (R_a)_{a \in \mathrm{A}}, \theta)$. Given a formula φ and a condition $\mathbf{\Gamma}$, let $\mathbf{\Delta}$ be the condition defined as follows: for $a \in \mathrm{A}$,

$$\mathbf{\Delta}(a) = \mathbf{\Gamma}(a) \cup \{\chi \in \mathrm{sub}(\varphi) \mid \exists w \in R_a[V] \backslash V \ \mathsf{M}, w \models_{\mathbf{\Gamma}} \chi\}.$$

$\mathbf{\Delta}$ is called the *external condition of V in M with respect to φ and $\mathbf{\Gamma}$*.

Lemma 6 ([Sha18])**.** *Consider a sum of models $\mathsf{M} = \sum_{\mathsf{I}} \mathsf{M}_i$, i in I, and the set $V = \{i\} \times \mathrm{dom}(\mathsf{M}_i)$. If $\mathbf{\Delta}$ is the external condition of V in M with respect to some given φ, $\mathbf{\Gamma}$, then for all v in M_i, χ in $\mathrm{sub}(\varphi)$,*

$$\mathsf{M}, (i, v) \models_{\mathbf{\Gamma}} \chi \quad \textit{iff} \quad \mathsf{M}_i, v \models_{\mathbf{\Delta}} \chi. \tag{1}$$

Lemma 7 below is a particular corollary of Lemma 6. It will be important for the proofs of our complexity results.

Consider $a \in \mathrm{A}$ and models M_0, M_1. The model $\mathsf{M}_0 +^a \mathsf{M}_1$ is obtained from the disjoint union of M_0 and M_1 by adding all the pairs of form $((0,w),(1,v))$ to the a-th relation; that is, in our general notation, $\mathsf{M}_0 +^a \mathsf{M}_1 = \sum^a_{(2,<)} \mathsf{M}_i$.

For an A-condition $\mathbf{\Gamma} = (\Gamma_0, \ldots, \Gamma_{\mathrm{A}-1})$ and a set of A-formulas Ψ, we put $\mathbf{\Gamma} \cup^a \Psi = (\Gamma'_0, \ldots, \Gamma'_{\mathrm{A}-1})$, where $\Gamma'_a = \Gamma_a \cup \Psi$, and $\Gamma'_b = \Gamma_b$ for $b \neq a$.

Lemma 7. *For* A*-models* $\mathsf{M}_0, \mathsf{M}_1$*, an* A*-formula* φ*, an* A*-condition* $\mathbf{\Gamma}$*, and* $a \in \mathrm{A}$*, we have*

$$\varphi[\mathsf{M}_0 +^a \mathsf{M}_1, \mathbf{\Gamma}] = \varphi[\mathsf{M}_0, \mathbf{\Gamma} \cup^a \varphi[\mathsf{M}_1, \mathbf{\Gamma}]] \cup \varphi[\mathsf{M}_1, \mathbf{\Gamma}]. \tag{2}$$

Proof. Put $\mathbf{\Delta} = \mathbf{\Gamma} \cup^a \varphi[\mathsf{M}_1, \mathbf{\Gamma}]$. By Lemma 6, we have for every w in M_0, every v in M_1, and every $\chi \in \mathrm{sub}(\varphi)$:

$$\begin{array}{rcl} \mathsf{M}_0 +^a \mathsf{M}_1, (0,w) \models_{\mathbf{\Gamma}} \chi & \text{iff} & \mathsf{M}_0, w \models_{\mathbf{\Delta}} \chi, \\ \mathsf{M}_0 +^a \mathsf{M}_1, (1,v) \models_{\mathbf{\Gamma}} \chi & \text{iff} & \mathsf{M}_1, v \models_{\mathbf{\Gamma}} \chi. \end{array}$$

Now (2) follows. ∎

3.4 Sums over Noetherian orders

Definition 4. Consider a unimodal frame $\mathsf{I} = (I, S)$ and a family $(\mathsf{F}_i)_{i \in I}$ of A-frames (or A-models). For $a \in N$, the *a-sum* $\sum^a_{\mathsf{I}} \mathsf{F}_i$ is the sum $\sum_{\mathsf{I}'} \mathsf{F}_i$, where I' is the A-frame whose domain is I, the a-th relation is S and other relations are empty. If $\mathcal{F}$ is a class of A-frames, $\mathcal{I}$ is a class of 1-frames, then $\sum^a_{\mathcal{I}} \mathcal{F}$ is the class of all sums $\sum^a_{\mathsf{I}} \mathsf{F}_i$, where $\mathsf{I} \in \mathcal{I}$ and all F_i are in $\mathcal{F}$.

A strict partial order $(I, <)$ is *Noetherian* (or *converse well-founded*) if it has no infinite ascending chains. Let NPO be the classes of all non-empty Noetherian partial orders (we say that a partial order is non-empty, if its domain is).

A strict partial order $(I, <)$ is called a (*transitive irreflexive*) *tree* if it has a least element (the *root*) and for all $i \in I$ the set $\{j \mid j < i\}$ is a finite chain. Let Tr_f be the class of all finite trees.

Consider a finite tree $\mathsf{I} = (I, <)$. The *branching of* i *in* I, denoted by $\underline{(}i, \mathsf{I})$, is the number of immediate successors of i (j *is an immediate successor of* i, if $i < j$ and there is no k such that $i < k < j$); the *branching of* I, denoted by $\underline{(}\mathsf{I})$, is $\max\{\underline{(}i, \mathsf{I}) \mid i \text{ in } \mathsf{I}\}$. The *height of* I, denoted by $ht(\mathsf{I})$, is $\max\{|V| \mid V \text{ is a chain in } \mathsf{I}\}$. For $h, b \in \omega$, let $\mathrm{Tr}(h, b)$ be the class of all finite trees with height $\leq h$ and branching $\leq n$:

$$\mathrm{Tr}(h, b) = \{\mathsf{I} \in \mathrm{Tr}_f \mid ht(\mathsf{I}) \leq h \,\&\, \underline{(}\mathsf{I}) \leq b\}.$$

Let $\bigsqcup \mathcal{F}$ be the class of frames of form $\bigsqcup_I \mathsf{F}_i$, where I is a non-empty set, $\mathsf{F}_i \in \mathcal{F}$ for all $i \in I$, and let $\bigsqcup_{\leq k} \mathcal{F}$ be the class of such frames with $0 < |I| \leq k$. Likewise for $\bigsqcup_{<k} \mathcal{F}$.

Let $\#\varphi$ denote the number of subformulas of a formula φ.

Theorem 8 ([Sha18])**.** *Let* $\mathcal{F}$ *be a class of* A*-frames,* $a \in \mathrm{A}$*,* $\mathcal{I}$ *a class of Noetherian orders containing all finite trees.*

1. *We have* $\mathrm{Log} \sum^a_{\mathrm{NPO}} \mathcal{F} = \mathrm{Log} \sum^a_{\mathrm{Tr}_f} \mathcal{F} = \mathrm{Log} \sum^a_{\mathcal{I}} \mathcal{F}$*. Moreover, for every* A*-formula* φ *we have:*

$$\varphi \text{ is satisfiable in } \sum\nolimits^a_{\mathrm{NPO}} \mathcal{F} \text{ iff } \varphi \text{ is satisfiable in } \sum\nolimits^a_{\mathrm{Tr}(\#\varphi, \#\varphi)} \mathcal{F}.$$

2. *Assume that $\mathcal{I}$ is closed under finite disjoint unions. Then*

$$\sum\nolimits_{\mathrm{NPO}}^{a} \mathcal{F} \equiv \sum\nolimits_{\mathrm{Tr}_f}^{a} \mathcal{F} \equiv \sum\nolimits_{\mathcal{I}}^{a} \mathcal{F}.$$

Moreover, for every A*-tie* $\tau = (\varphi, \Phi, \mathbf{\Gamma})$ *we have:*

$$\tau \text{ is satisfiable in } \sum\nolimits_{\mathrm{NPO}}^{a} \mathcal{F} \text{ iff } \tau \text{ is satisfiable in } \bigsqcup\nolimits_{\leq \#\varphi} \sum\nolimits_{\mathrm{Tr}(\#\varphi,\#\varphi)}^{a} \mathcal{F}.$$

This theorem will be the crucial semantic tool for the complexity results.

4 Complexity

The main goal of this section is to show that the modal satisfiability problem on sums over Noetherian orders is polynomial space Turing reducible to the modal satisfiability problem on summands.

For problems A and B, we put $A \leq_{\mathrm{T}}^{\mathrm{PSpace}} B$ if there exists a polynomial space bounded oracle deterministic machine M with oracle B that recognizes A [SG77] (it is assumed that every tape of M, including the oracle tape, is polynomial space bounded).

Theorem 9. *Let $a < \mathrm{A} < \omega$, $\mathcal{F}$ a class of* A*-frames, $\mathcal{I}$ a class of Noetherian orders containing all finite trees. Then:*

1. $\mathrm{Sat} \sum_{\mathcal{I}}^{a} \mathcal{F} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{Sat}\, \mathcal{F}^{(\forall)}$.
2. *If also $\mathcal{I}$ is closed under finite disjoint unions, then* $\mathrm{Sat}(\sum_{\mathcal{I}}^{a} \mathcal{F})^{(\forall)} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{Sat}\, \mathcal{F}^{(\forall)}$.

This theorem will be proven in Section 4.3. For technical reasons, first we will address complexity of the *conditional satisfiability problem.* Let A be finite, $\mathcal{F}$ a class of A-frames. We shall be interested whether a given tie $(\varphi, \Phi, \mathbf{\Gamma})$ is satisfiable in $\mathcal{F}$. The following simple observation shows that, w.l.g., we may assume that every $\mathbf{\Gamma}(a)$, $a < \mathrm{A}$, consists of subformulas of φ, and hence that $\mathbf{\Gamma}$ is a finite sequence of finite sets:

Proposition 10. *A tie $(\varphi, \Phi, (\Gamma_a)_{a \in \mathrm{A}})$ is satisfiable in a class $\mathcal{F}$ iff $(\varphi, \Phi, (\mathrm{sub}(\varphi) \cap \Gamma_a)_{a \in \mathrm{A}})$ is.*

Proof. It is immediate from Definition 2 that if $\mathbf{\Gamma}(a) \cap \mathrm{sub}(\varphi) = \mathbf{\Delta}(a) \cap \mathrm{sub}(\varphi)$ for all $a \in \mathrm{A}$, then for every model M and every w in M we have $\mathsf{M}, w \models_{\mathbf{\Gamma}} \varphi$ iff $\mathsf{M}, w \models_{\mathbf{\Delta}} \varphi$. ☐

The *conditional satisfiability problem on $\mathcal{F}$* is to decide whether a given tie $(\varphi, \Phi, \mathbf{\Gamma})$ such that $\mathbf{\Gamma} \subseteq \mathrm{sub}(\varphi)^{\mathrm{A}}$ is satisfiable in $\mathcal{F}$. In symbols,

$$\mathrm{CSat}\, \mathcal{F} = \{(\varphi, \Phi, \mathbf{\Gamma}) \mid \varphi \text{ is an A-formula } \& \ \Phi \subseteq \mathrm{sub}(\varphi) \ \& \ \mathbf{\Gamma} \subseteq \mathrm{sub}(\varphi)^{\mathrm{A}} \ \& \ \text{ the tie } (\varphi, \Phi, \mathbf{\Gamma}) \text{ is satisfiable in } \mathcal{F}\}.$$

In the next Section 4.1 we will describe a decision procedure for the conditional satisfiability problem on sums over Noetherian orders $\sum_{\mathrm{NPO}}^{a} \mathcal{F}$ with the oracle $\mathrm{CSat}\, \mathcal{F}$. Next, in Section 4.2, we will describe reductions between $\mathrm{CSat}\, \mathcal{F}$ and $\mathrm{Sat}\, \mathcal{F}^{(\forall)}$, which will complete the proof of Theorem 9.

4.1 Arithmetic of conditional satisfiability

It will be convenient to encode subformulas of a given φ as Boolean vectors of length $\#\varphi$, considered as characteristic functions on $\mathrm{sub}(\varphi)$. For $\mathrm{A} \le \omega$, the set of modal formulas is linearly ordered by a polynomial time computable relation $\sqsupseteq$ such that if ψ is a subformula of φ, then $\varphi \sqsupseteq \psi$ (e.g., put $\varphi \sqsupseteq \psi$ if ψ is shorter than φ, and assume that formulas of the same length are ordered lexicographically). Let $(\psi_0, \ldots, \psi_{\#\varphi-1})$ be the $\sqsupseteq$-chain of all subformulas of φ (hence, $\psi_0 = \varphi$); for $\mathbf{v} \in 2^{\#\varphi}$, we write $\varphi[\mathbf{v}]$ for $\{\psi_i \mid \mathbf{v}(i) = 1\}$; similarly, a sequence $\mathbf{U} = (\mathbf{u}_a)_{a\in\mathrm{A}}$ of such vectors represents the condition $\mathbf{\Gamma} = (\varphi[\mathbf{u}_a])_{a\in\mathrm{A}}$. Hence, for a finite A, every tie $\tau = (\varphi, \Phi, \mathbf{\Gamma})$ with $\mathbf{\Gamma} \subseteq \mathrm{sub}(\varphi)^{\mathrm{A}}$ is represented by a triple $\tau' = (\varphi, \mathbf{v}, (\mathbf{u}_a)_{a\in\mathrm{A}})$, where $\mathbf{v}, \mathbf{u}_0, \ldots \mathbf{u}_{\mathrm{A}-1} \in 2^{\#\varphi}$; this triple is also called a *tie*. In this case, by the satisfiability of τ' we mean the satisfiability of τ.

Let $\mathbf{0}_\varphi$ denote the sequence of length A of zero vectors of length $\#\varphi$ (that is, $\mathbf{0}_\varphi$ represents the condition, consisting of empty sets). In view of Proposition 10, we have the following reformulation of Proposition 4:

Proposition 11. *φ is satisfiable in $\mathcal{F}$ iff there exists $\mathbf{v} \in 2^{\#\varphi}$ such that $\mathbf{v}(0) = 1$ and the tie $(\varphi, \mathbf{v}, \mathbf{0}_\varphi)$ is satisfiable in $\mathcal{F}$.*

For Boolean vectors $\mathbf{v} = (v_0, \ldots, v_{l-1})$, $\mathbf{u} = (u_0, \ldots, u_{l-1})$, let $\mathbf{v} + \mathbf{u}$ be their element-wise disjunction $(\max\{v_0, u_0\}, \ldots, \max\{v_{l-1}, u_{l-1}\})$.

Lemma 12. *Let $\mathcal{G}$ be a class of A-frames, $0 < b < \omega$. A tie $(\varphi, \mathbf{v}, \mathbf{U})$ is satisfiable in $\bigsqcup_{\le b}\mathcal{G}$ iff there exist $k \le b$, $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1} \in 2^{\#\varphi}$ such that $\mathbf{v} = \sum_{i<k} \mathbf{v}_i$ and for every $i < k$ the tie $(\varphi, \mathbf{v}_i, \mathbf{U})$ is satisfiable in $\mathcal{G}$.*

Proof. By Proposition 3, we have $\varphi[\bigsqcup_{i<k} \mathsf{M}_i, \mathbf{\Gamma}] = \bigcup_{i<k} \varphi[\mathsf{M}_i, \mathbf{\Gamma}]$ for every A-models $\mathsf{M}_0, \ldots, \mathsf{M}_{k-1}$ and every A-condition $\mathbf{\Gamma}$. □

For Boolean vectors $\mathbf{v}, \mathbf{u}_0, \ldots, \mathbf{u}_{\mathrm{A}-1}$ of the same length and $a \in \mathrm{A}$, we put $(\mathbf{u}_0, \ldots, \mathbf{u}_{\mathrm{A}-1}) +^a \mathbf{v} = (\mathbf{u}'_0, \ldots, \mathbf{u}'_{\mathrm{A}-1})$, where $\mathbf{u}'_a = \mathbf{u}_a + \mathbf{v}$, and $\mathbf{u}'_b = \mathbf{u}_b$ for $b \ne a$.

Similarly to models, for A-frames F_0 and F_1 and $a \in \mathrm{A}$ we define $\mathsf{F}_0 +^a \mathsf{F}_1$ as $\sum^a_{(2,<)} \mathsf{F}_i$; for classes $\mathcal{F}$ and $\mathcal{G}$ of A-frames, let $\mathcal{F} +^a \mathcal{G} = \{\mathsf{F} +^a \mathsf{G} \mid \mathsf{F} \in \mathcal{F} \,\&\, \mathsf{G} \in \mathcal{G}\}$

Lemma 13. *Let $\mathcal{F}$ and $\mathcal{G}$ be classes of A-frames, $a \in \mathrm{A}$. Then a tie $(\varphi, \mathbf{v}, \mathbf{U})$ is satisfiable in $\mathcal{F} +^a \mathcal{G}$ iff there exist $\mathbf{v}_0, \mathbf{v}_1 \in 2^{\#\varphi}$ such that*

1. *$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$, and*
2. *$(\varphi, \mathbf{v}_0, \mathbf{U} +^a \mathbf{v}_1)$ is satisfiable in $\mathrm{CSat}\,\mathcal{F}$, and*
3. *$(\varphi, \mathbf{v}_1, \mathbf{U})$ is satisfiable in $\mathrm{CSat}\,\mathcal{G}$.*

Proof. Let M_0 be a model on a frame in $\mathcal{F}$, M_1 a model on a frame in $\mathcal{G}$. Let $\mathbf{\Gamma}$ be the condition represented by $\mathbf{U}$, i.e., $\mathbf{\Gamma} = (\varphi[\mathbf{U}(a)])_{a\in\mathrm{A}}$, and let $\Psi = \varphi[\mathsf{M}_1, \mathbf{\Gamma}]$. By Lemma 7,

$$\varphi[\mathsf{M}_0 +^a \mathsf{M}_1, \mathbf{\Gamma}] = \varphi[\mathsf{M}_0, \mathbf{\Gamma} \cup^a \Psi] \cup \Psi. \tag{3}$$

Assume that $\varphi[\mathbf{v}] = \varphi[\mathsf{M}_0 +^a \mathsf{M}_1, \mathbf{\Gamma}]$. Consider tuples $\mathbf{v}_0, \mathbf{v}_1 \in 2^{\#\varphi}$ such that $\varphi[\mathbf{v}_0] = \varphi[\mathsf{M}_0, \mathbf{\Gamma} \cup^a \Psi]$, $\varphi[\mathbf{v}_1] = \varphi[\mathsf{M}_1, \mathbf{\Gamma}]$. By (3), $\varphi[\mathbf{v}] = \varphi[\mathbf{v}_0] \cup \varphi[\mathbf{v}_1]$, and so $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$.

Now assume that $\varphi[\mathbf{v}_0] = \varphi[\mathsf{M}_0, \mathbf{\Gamma} \cup^a \Psi]$ and $\varphi[\mathbf{v}_1] = \Psi$ for some $\mathbf{v}_0, \mathbf{v}_1 \in 2^{\#\varphi}$. For $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ we have $\varphi[\mathbf{v}] = \varphi[\mathbf{v}_0] \cup \varphi[\mathbf{v}_1]$. By (3), we have $\varphi[\mathbf{v}] = \varphi[\mathsf{M}_0 + \mathsf{M}_1, \mathbf{\Gamma}]$. □

Lemma 14. *Let $\mathcal{F}$ be a class of* A*-frame, $a \in \mathrm{A}$, $0 < h, b < \omega$. A tie $(\varphi, \mathbf{v}, \mathbf{U})$ is satisfiable in $\sum^a_{\mathrm{Tr}(h+1,b)} \mathcal{F}$ iff it is satisfiable is $\mathcal{F}$, or there exist a positive $k \le b$ and $\mathbf{u}, \mathbf{v}_0, \dots, \mathbf{v}_{k-1} \in 2^{\#\varphi}$ such that*

1. *$\mathbf{v} = \mathbf{u} + \sum_{i<k} \mathbf{v}_i$, and*
2. *the tie $(\varphi, \mathbf{u}, \mathbf{U} +^a \sum_{i<k} \mathbf{v}_i)$ is satisfiable in $\mathcal{F}$, and*
3. *for all $i < k$, the tie $(\varphi, \mathbf{v}_i, \mathbf{U})$ is satisfiable in $\sum^a_{\mathrm{Tr}(h,b)} \mathcal{F}$.*

Proof. By the definition of $\sum^a_{\mathrm{Tr}(h+1,b)} \mathcal{F}$, the tie $\tau = (\varphi, \mathbf{v}, \mathbf{U})$ is satisfiable in $\sum^a_{\mathrm{Tr}(h+1,b)} \mathcal{F}$ iff τ is satisfiable in $\mathcal{F}$ or in $\mathcal{F} +^a \mathcal{G}$, where $\mathcal{G} = \bigsqcup_{\le b} \sum^a_{\mathrm{Tr}(h,b)} \mathcal{F}$.

By Lemma 13, τ is satisifable in $\mathcal{F} +^a \mathcal{G}$ iff there exist $\mathbf{u}, \mathbf{u}' \in 2^{\#\varphi}$ such that $\mathbf{v} = \mathbf{u} + \mathbf{u}'$, $(\varphi, \mathbf{u}, \mathbf{U} +^a \mathbf{u}')$ is satisfiable in $\mathcal{F}$, and $(\varphi, \mathbf{u}', \mathbf{U})$ is satisfiable in $\mathcal{G}$. By Lemma 12, $(\varphi, \mathbf{u}', \mathbf{U})$ is satisfiable in $\mathcal{G}$ iff there exist $0 < k \le b$ and $\mathbf{v}_0, \dots, \mathbf{v}_{k-1} \in 2^{\#\varphi}$ such that $\mathbf{u}' = \sum_{i<k} \mathbf{v}_i$, and for all $i < k$ the tie $(\varphi, \mathbf{v}_i, \mathbf{U})$ is satisfiable in $\sum^a_{\mathrm{Tr}(h,b)} \mathcal{F}$. ☐

This lemma allows to describe the procedure CSatSum (Algorithm 1) which using an oracle for CSat $\mathcal{F}$ decides whether a given tie is satisfiable is $\sum^a_{\mathrm{Tr}(h,b)} \mathcal{F}$. Namely, we have:

Theorem 15. *Let $a < \mathrm{A} < \omega$, $\mathcal{F}$ a class of* A*-frames, $(\varphi, \mathbf{v}, \mathbf{U})$ an* A*-tie, $0 < h, b < \omega$. Then*

$$(\varphi, \mathbf{v}, \mathbf{U}) \textit{ is satisfiable in } \sum\nolimits^a_{\mathrm{Tr}(h,b)} \mathcal{F} \textit{ iff } \mathrm{CSatSum}(\varphi, \mathbf{v}, \mathbf{U}, h, b) \textit{ returns } \mathrm{true}.$$

This theorem will be our main technical tool for complexity results. For the first of its corollaries, we show how to reduce the conditional satisfiability on sums to the conditional satisfiability on summands.

Theorem 16. *Let $a < \mathrm{A} < \omega$, $\mathcal{F}$ a class of* A*-frames, $\mathcal{I}$ a class of Noetherian orders containing all finite trees. Then:*

1. Sat $\sum^a_{\mathcal{I}} \mathcal{F} \le^{\mathrm{PSpace}}_{\mathrm{T}}$ CSat $\mathcal{F}$.
2. *If also $\mathcal{I}$ is closed under finite disjoint unions, then* CSat $\sum^a_{\mathcal{I}} \mathcal{F} \le^{\mathrm{PSpace}}_{\mathrm{T}}$ CSat $\mathcal{F}$.

Proof. By Theorem 8(1), a formula φ is satisfiable in $\sum^a_{\mathcal{I}} \mathcal{F}$ iff φ is satisfiable in $\sum^a_{\mathrm{Tr}(\#\varphi,\#\varphi)} \mathcal{F}$, and by Proposition 11, this means that there exists a satisfiable in $\sum^a_{\mathrm{Tr}(\#\varphi,\#\varphi)} \mathcal{F}$ tie $(\varphi, \mathbf{v}, \mathbf{0}_\varphi)$ with $\mathbf{v}(0) = 1$. From Theorem 15 we obtain

Lemma 17. *A formula φ is satisfiable in $\sum^a_{\mathcal{I}} \mathcal{F}$ iff there exists $\mathbf{v} \in 2^{\#\varphi}$ such that $\mathbf{v}(0) = 1$ and* $\mathrm{CSatSum}(\varphi, \mathbf{v}, \mathbf{0}_\varphi, \#\varphi, \#\varphi)$ *returns* true.

Assume that $\mathcal{I}$ is closed under finite disjoint unions. By Theorem 8(2), a tie $(\varphi, \mathbf{v}, \mathbf{U})$ is satisfiable in $\sum^a_{\mathcal{I}} \mathcal{F}$ iff it is satisfiable in $\bigsqcup_{\le \#\varphi} \sum^a_{\mathrm{Tr}(\#\varphi,\#\varphi)} \mathcal{F}$. By Lemma 12, this means that there exists $k \le \#\varphi$ and tuples $\mathbf{v}^{(0)}, \dots, \mathbf{v}^{(k-1)} \in 2^{\#\varphi}$ such that $\mathbf{u} = \sum_{i<k} \mathbf{v}^{(i)}$ and $(\varphi, \mathbf{v}^{(i)}, \mathbf{U}, \#\varphi, \#\varphi)$ is satisfiable in $\sum^a_{\mathrm{Tr}(\#\varphi,\#\varphi)} \mathcal{F}$ for every $i < k$. Using Theorem 15 again, be obtain

Lemma 18. *If $\mathcal{I}$ is closed under finite disjoint unions, then a tie $(\varphi, \mathbf{v}, \mathbf{U})$ is satisfiable in $\sum^a_{\mathcal{I}} \mathcal{F}$ iff there exist $k \le \#\varphi$ and tuples $\mathbf{v}^{(0)}, \dots, \mathbf{v}^{(k-1)} \in 2^{\#\varphi}$ such that $\mathbf{u} = \sum_{i<k} \mathbf{v}^{(i)}$ and* $\mathrm{CSatSum}(\varphi, \mathbf{v}^{(i)}, \mathbf{U}, \#\varphi, \#\varphi)$ *returns* true *for every $i < k$.*

Set $n = \#\varphi$. Let us estimate the amount of space used by CSatSum for the case $n = h = b$. At each call CSatSum needs $O(n^2)$ space to store new variables. The depth of recursion is bounded by n. Thus, we can reduce Sat $\sum^a_{\mathcal{I}} \mathcal{F}$ and, for the case when $\mathcal{I}$ is closed under finite disjoint unions, CSat $\sum^a_{\mathcal{I}} \mathcal{F}$, to the conditional satisfiability problem on $\mathcal{F}$ in $O(n^3)$ space. ☐

ALGORITHM 1: Decision procedure for CSat $\sum^a_{\mathrm{Tr}(h,b)} \mathcal{F}$ with an oracle for CSat $\mathcal{F}$

CSatSum$(\varphi, \mathbf{v}, \mathbf{U}, h, b)$:*boolean*
  Input: A tie $(\varphi, \mathbf{v}, \mathbf{U})$; positive integers h, b
  if $(\varphi, \mathbf{v}, \mathbf{U})$ *is satisfiable in* $\mathcal{F}$ **then return** true;
  if $h > 1$ **then**
    for k *such that* $1 \le k \le b$ **do for** $\mathbf{u}, \mathbf{v}_0, \ldots, \mathbf{v}_{k-1} \in 2^{\#\varphi}$ *such that* $\mathbf{v} = \mathbf{u} + \sum_{i<k} \mathbf{v}_i$ **do**
      if $(\varphi, \mathbf{u}, \mathbf{U} +^a \sum_{i<k} \mathbf{v}_i)$ *is satisfiable in* $\mathcal{F}$ **then**
        if $\bigwedge_{i<k}$ CSatSum$(\varphi, \mathbf{v}_i, \mathbf{U}, h-1, b)$ **then return** true;
  return false.

4.2 Reductions between Sat $\mathcal{F}$, CSat $\mathcal{F}$, and Sat $\mathcal{F}^{(\forall)}$

By Proposition 11, the satisfiability problem on a class $\mathcal{F}$ is polynomial space Turing reducible to the conditional satisfiability problem on $\mathcal{F}$. Below we show that in many cases CSat $\mathcal{F}$ is polynomial time many-to-one reducible to Sat $\mathcal{F}$, in symbols CSat $\mathcal{F} \le^{\mathrm{PTime}}_{\mathrm{m}}$ Sat $\mathcal{F}$; hence, in these cases, the above two problems are equivalent with respect to $\le^{\mathrm{PSpace}}_{\mathrm{T}}$. This fact is close to a result obtained in [Hem96], where it was shown that in many situations there exists a (stronger than $\le^{\mathrm{PSpace}}_{\mathrm{T}}$, but weaker than $\le^{\mathrm{PTime}}_{\mathrm{m}}$) reduction of Sat $\mathcal{F}^{(\forall)}$ to Sat $\mathcal{F}$. Let us discuss reductions between the above to problems in more details.

For a binary relation R on a set W, put $R^{\le m} = \bigcup_{n \le m} R^n$, where R^0 is the identity relation on W, $R^{n+1} = R \circ R^n$ ($\circ$ denotes the composition of relations). Recall that R^* denotes the transitive reflexive closure of R: $R^* = \bigcup_{n<\omega} R^n$. R is said to be *m-transitive* if $R^{\le m}$ includes R^{m+1}, or equivalently, $R^{\le m} = R^*$ (e.g., every transitive relation is 1-transitive). For a frame $\mathsf{F} = (W, (R_a)_{a \in \mathrm{A}})$, put $R_\mathsf{F} = \bigcup_{a \in \mathrm{A}} R_a$. We say that F is *m-transitive* if the relation R_F is. In particular, if one of the relations of F is universal (i.e., is equal to $W \times W$), then F is 1-transitive.

For w in F, let $\mathsf{F}[w]$ denote the subframe of F generated by the singleton $\{w\}$ (that is, $\mathsf{F}[w]$ is the restriction of F on the set $\{v \mid w R^*_\mathsf{F} v\}$); such frames are called *cones*). A class $\mathcal{F}$ of frames is *closed under taking cones* if for every F in $\mathcal{F}$ and for every w in F, the cone $\mathsf{F}[w]$ is in $\mathcal{F}$.

A class $\mathcal{F}$ of A-frames is said to be *preconical*, if

- $\mathcal{F}$ is closed under taking cones, and
- there exists $m < \omega$ such that every frame in $\mathcal{F}$ is m-transitive, and
- for every F in $\mathcal{F}$, the relation R^*_F is downward directed (i.e., for every w, v in F there exists u such that $u R^*_\mathsf{F} w$ and $u R^*_\mathsf{F} v$).

Proposition 19. *Let* A *be finite,* $\mathcal{F}$ *a class of* A*-frames. If* $\mathcal{F}$ *is preconical, then* CSat $\mathcal{F} \le^{\mathrm{PTime}}_{\mathrm{m}}$ Sat $\mathcal{F}$.

Proof. Given a condition $\mathbf{\Gamma}$ and a formula φ, we define the formula $[\varphi]^\mathbf{\Gamma}$ as follows: $[\bot]^\mathbf{\Gamma} = \bot$, $[p]^\mathbf{\Gamma} = p$ for variables, $[\varphi_1 \to \varphi_2]^\mathbf{\Gamma} = [\varphi_1]^\mathbf{\Gamma} \to [\varphi_2]^\mathbf{\Gamma}$, and

$$[\Diamond_a \varphi]^\mathbf{\Gamma} = \begin{cases} \top, & \text{if } \varphi \in \mathbf{\Gamma}(a), \\ \Diamond_a [\varphi]^\mathbf{\Gamma} & \text{otherwise.} \end{cases}$$

For every A-model M, we have:

$$\mathsf{M}, w \models_\mathbf{\Gamma} \varphi \text{ iff } \mathsf{M}, w \models [\varphi]^\mathbf{\Gamma} \tag{4}$$

The proof is straightforward, see [Sha18, Lemma 5] for the details.

Let $\Diamond\varphi$ abbreviate the A-formula $\bigvee_{a\in\mathrm{A}}\Diamond_a\varphi$, and let $\Diamond^0\varphi = \varphi$, $\Diamond^{m+1}\varphi = \Diamond\Diamond^m\varphi$, $\Diamond^{\le m}\varphi = \bigvee_{n\le m}\Diamond^n\varphi$.

For a tie $(\varphi, \mathbf{v}, \mathbf{U})$, we put

$$\delta_m(\varphi, \mathbf{v}, \mathbf{U}) = \bigwedge_{\psi\in\varphi[\mathbf{v}]} \Diamond^{\le m}[\psi]^{\mathbf{\Gamma}} \wedge \bigwedge_{\psi\in\mathrm{sub}(\varphi)\setminus\varphi[\mathbf{v}]} \neg\Diamond^{\le m}[\psi]^{\mathbf{\Gamma}}, \tag{5}$$

where $\mathbf{\Gamma}$ is the condition represented by $\mathbf{U}$, i.e., $\mathbf{\Gamma} = (\varphi[\mathbf{u}_a])_{a\in\mathrm{A}}$.

Since $\mathcal{F}$ is preconical, every frame in $\mathcal{F}$ is m-transitive for some finite m. We claim that

$$(\varphi, \mathbf{v}, \mathbf{U}) \in \mathrm{CSat}\,\mathcal{F} \quad \text{iff} \quad \delta_m(\varphi, \mathbf{v}, \mathbf{U}) \in \mathrm{Sat}\,\mathcal{F}. \tag{6}$$

First, assume that $(\varphi, \mathbf{v}, \mathbf{U})$ is satisfiable in a frame $\mathsf{F} \in \mathcal{F}$. This means that for a model M based on F we have $\varphi[\mathbf{v}] = \varphi[\mathsf{M}, \mathbf{\Gamma}]$, where $\mathbf{\Gamma}$ is the condition represented by $\mathbf{U}$. For every $\psi \in \varphi[\mathbf{v}]$ we choose a point w_ψ such that $\mathsf{M}, w_\psi \models_{\mathbf{\Gamma}} \varphi$, and then put $V = \{w_\psi \mid \psi \in \varphi[\mathbf{v}]\}$. The relation R^*_{F} is downward directed, hence there exists a point w in M such that $wR^*_{\mathsf{F}}v$ for all v in V; by m-transitivity, $wR^{\le m}_{\mathsf{F}}v$. It follows that if $\psi \in \varphi[\mathbf{v}]$, then $\mathsf{M}, w \models \Diamond^{\le m}[\psi]^{\mathbf{\Gamma}}$: indeed, we have $\mathsf{M}, w_\psi \models [\psi]^{\mathbf{\Gamma}}$ by (4) and $wR^{\le m}_{\mathsf{F}}w_\psi$. On the other hand, if a subformula ψ of φ is not in $\varphi[\mathbf{v}]$, then $[\psi]^{\mathbf{\Gamma}}$ is false at every point in M by (4), and so $\mathsf{M}, w \models \neg\Diamond^{\le m}[\psi]^{\mathbf{\Gamma}}$. It follows that the formula $\delta_m(\varphi, \mathbf{v}, \mathbf{U})$ is satisfiable in $\mathcal{F}$.

Now assume that $\delta_m(\varphi, \mathbf{v}, \mathbf{U})$ is true at a point w in a model M over a frame $\mathsf{F} \in \mathcal{F}$. Since $\mathcal{F}$ is closed under taking cones, we may assume that $\mathsf{F} = \mathsf{F}[w]$. Let ψ be a subformula of φ. Suppose that $\psi \in \varphi[\mathbf{v}]$. Then $\mathsf{M}, w \models \Diamond^{\le m}[\psi]^{\mathbf{\Gamma}}$. Hence, the formula $[\psi]^{\mathbf{\Gamma}}$ is true at a point u of M, which means that $\mathsf{M}, w \models \Diamond^{\le m}[\psi]^{\mathbf{\Gamma}}$ by (4). Thus, $\psi \in \mathrm{sub}(\varphi, \mathsf{M}, \mathbf{\Gamma})$, where $\mathbf{\Gamma} = (\varphi[\mathbf{u}_a])_{a\in\mathrm{A}}$. On the other hand, if $\psi \notin \varphi[\mathbf{v}]$, then $\mathsf{M}, w \models \neg\Diamond^{\le m}[\psi]^{\mathbf{\Gamma}}$; using (4) again, we obtain that $[\psi]^{\mathbf{\Gamma}}$ is false at every point of M, and so $\psi \notin \mathrm{sub}(\varphi, \mathsf{M}, \mathbf{\Gamma})$. Thus, $\varphi[\mathbf{v}] = \mathrm{sub}(\varphi, \mathsf{M}, \mathbf{\Gamma})$, that is $(\varphi, \mathbf{v}, \mathbf{U})$ is satisfiable in $\mathcal{F}$. ☐

Proposition 20. *If* A *is finite and* $\mathcal{F}$ *is a class of* A*-frames, then* $\mathrm{CSat}\,\mathcal{F} \le^{\mathrm{PTime}}_{\mathrm{m}} \mathrm{Sat}\,\mathcal{F}^{(\forall)}$.

Proof. It is trivial that $\mathrm{CSat}\,\mathcal{F} \le^{\mathrm{PTime}}_{\mathrm{m}} \mathrm{CSat}\,\mathcal{F}^{(\forall)}$ (the reduction increases by one the indexes of modalities occurring in formulas of a given tie).

The class $\mathcal{F}^{(\forall)}$ is preconical: if $\mathsf{F} \in \mathcal{F}$, then F is 1-transitive, $\mathsf{F}[w] = \mathsf{F}$ for every $w \in \mathsf{F}$, and R^*_{F} is downward-directed, since R_{F} is the universal relation on F. Hence, $\mathrm{CSat}\,\mathcal{F}^{(\forall)} \le^{\mathrm{PTime}}_{\mathrm{m}} \mathrm{Sat}\,\mathcal{F}^{(\forall)}$ by Proposition 19. ☐

Now let us describe a reduction of $\mathrm{Sat}\,\mathcal{F}^{(\forall)}$ to $\mathrm{CSat}\,\mathcal{F}$. This reduction is based on the following construction proposed in [Hem96]. Let φ be a formula in the language $\mathrm{ML}(1 + \mathrm{A})$. For subformulas $\Diamond_0\psi$ of φ starting with $\Diamond_0$, we choose distinct variables p_ψ not occurring in φ, and put for subformulas of φ: $\bot' = \bot$; $p' = p$ for variables; $(\varphi_1 \to \varphi_2)' = \varphi_1{}' \to \varphi_2{}'$; $(\Diamond_a\psi)' = \Diamond_a\psi'$ for $a > 0$; and $(\Diamond_0\psi)' = p_\psi$. We have for a class $\mathcal{F}$ of A-frames:

$$\varphi \in \mathrm{Sat}\,\mathcal{F}^{(\forall)} \text{ iff } \varphi' \wedge \bigwedge_{\Diamond_0\psi\in\mathrm{sub}(\varphi)} \big((\Diamond_0\psi' \leftrightarrow \Box_0 p_\psi) \wedge (\Box_0 p_\psi \vee \Box_0\neg p_\psi)\big) \in \mathrm{Sat}\,\mathcal{F}^{(\forall)}, \tag{7}$$

see [Hem96, Lemma 4.5] for details.

Let η denote the formula in the right-hand side of the above equivalence. Let us show how the satisfiability of η in $\mathcal{F}^{(\forall)}$ can be expressed as the satisfiability of a tie in $\mathcal{F}$. Consider the formula

$$\xi_\varphi = \bigwedge_{\psi\in\mathrm{sub}(\varphi)} (\psi' \wedge \neg p_\psi)$$

(it is only important that ξ_φ contains ψ' and $\neg p_\psi$ as subformulas for every subformula ψ of φ). Let M be a model on a frame in $\mathcal{F}^{(\forall)}$, and let Φ be the set of subformulas of ξ_φ that are satisfiable in M, that is, $\Phi = \xi_\varphi[\mathsf{M}, (\varnothing)_{1+\mathrm{A}}]$. Then η is true at a point in M iff

$$\varphi' \in \Phi \text{ and for every } \psi \in \mathrm{sub}(\varphi),\ \psi' \in \Phi \text{ iff } p_\psi \in \Phi \text{ iff } \neg p_\psi \notin \Phi. \tag{8}$$

If follows that η is satisfiable in $\mathcal{F}^{(\forall)}$ iff there exists $\Phi \subseteq \mathrm{sub}(\xi_\varphi)$ satisfying (8) and the tie $(\xi_\varphi, \Phi, (\varnothing)_{1+\mathrm{A}})$ is satisfiable in $\mathcal{F}^{(\forall)}$. The formula ξ_φ and so, the formulas in Φ, do not contain $\Diamond_0$. Hence, the satisfiability of $(\xi_\varphi, \Phi, (\varnothing)_{1+\mathrm{A}})$ in $\mathcal{F}^{(\forall)}$ is equivalent to the satisfiability of $(\dot{\xi_\varphi}, \dot{\Phi}, (\varnothing)_{\mathrm{A}})$ in $\mathcal{F}$, where $\dot{\xi_\varphi}$ and $\dot{\Phi}$ are obtained by decreasing indexes of modalities by 1. Putting everything together, we have shown that

$$\varphi \in \mathrm{Sat}\,\mathcal{F}^{(\forall)} \text{ iff there exists } \Phi \subseteq \mathrm{sub}(\xi_\varphi) \text{ satisfying (8) such that } (\dot{\xi}, \dot{\Phi}, (\varnothing)_{\mathrm{A}}) \text{ is satisfiable in } \mathcal{F}.$$

This proves

Proposition 21. *If* A *is finite and* $\mathcal{F}$ *is a class of* A*-frames, then* $\mathrm{Sat}\,\mathcal{F}^{(\forall)} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{CSat}\,\mathcal{F}$.

Remark 2. In fact, Proposition 21 provides a stronger reduction than $\leq_{\mathrm{T}}^{\mathrm{PSpace}}$.

4.3 Proof of Theorem 9

In view of the above propositions, Theorem 9 is an easy corollary of Theorem 16.

Proof of Theorem 9. We have $\mathrm{Sat}\sum_{\mathcal{I}}^{a}\mathcal{F} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{CSat}\,\mathcal{F}$ by Theorem 16(1), and $\mathrm{CSat}\,\mathcal{F} \leq_{\mathrm{m}}^{\mathrm{PTime}} \mathrm{Sat}\,\mathcal{F}^{(\forall)}$ by Proposition 19. This proves the first statement of the theorem.

To prove the second statement, we apply Theorem 16(2) and obtain $\mathrm{CSat}\sum_{\mathcal{I}}^{a}\mathcal{F} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{Sat}\,\mathcal{F}^{(\forall)}$. By Proposition 21, $\mathrm{Sat}(\sum_{\mathcal{I}}^{a}\mathcal{F})^{(\forall)} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{CSat}(\sum_{\mathcal{I}}^{a}\mathcal{F})$. □

From Theorem 9 and Proposition 19, we obtain:

Corollary 22. *Let* $a < \mathrm{A} < \omega$*,* $\mathcal{F}$ *a class of* A*-frames,* $\mathcal{I}$ *a class of Noetherian orders containing all finite trees. If* $\mathcal{F}$ *is preconical, then:*

1. $\mathrm{Sat}\sum_{\mathcal{I}}^{a}\mathcal{F} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{Sat}\,\mathcal{F}$.
2. *If also* $\mathcal{I}$ *is closed under finite disjoint unions, then* $\mathrm{Sat}(\sum_{\mathcal{I}}^{a}\mathcal{F})^{(\forall)} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{Sat}\,\mathcal{F}$.

4.4 PSPACE-hardness: some corollaries of Ladner's construction

According to Ladner's theorem, every contained in S4 unimodal logic L is PSpace-hard [Lad77]. In fact, the Ladner's proof yields PSpace-hardness for a wider class of modal logics (e.g., for logics contained in the Gödel-Löb logic GL or in the Grzegorczyk logic Grz), see [Spa93]. With minor modifications, Ladner's construction also works for logics contained in S4.1, Grz.2 etc, for the polymodal case, and in particular – for sums.

To illustrate this, let us briefly discuss the proof. Consider a quantified Boolean formula $\eta = Q_1 p_1 \ldots Q_m p_m\, \theta$, where $Q_1, \ldots, Q_m \in \{\exists, \forall\}$, and θ is a propositional Boolean formula in variables $p_1, \ldots, p_m$. Chose fresh variables $q_0, \ldots, q_m$. Let $[\eta]_{\mathrm{L}}$ be the following unimodal formula:[1]

$$q_0 \wedge \bigwedge\nolimits_{i<m} \Box^{\leq m}(q_i \to \Diamond q_{i+1}) \ \wedge\ \bigwedge\nolimits_{\substack{i \neq j\\ i,j \leq m}} \Box^{\leq m}(q_i \to \neg q_j) \ \wedge\ \Box^m(q_m \to \theta) \wedge$$
$$\bigwedge\nolimits_{\{i<m \mid Q_{i+1}=\forall\}} \Box^i(q_i \to \Diamond(q_{i+1} \wedge p_{i+1}) \wedge \Diamond(q_{i+1} \wedge \neg p_{i+1})) \wedge$$
$$\bigwedge\nolimits_{1 \leq i \leq m-1} \bigwedge\nolimits_{i \leq j \leq m-1} \Box^j((p_i \to \Box p_i) \wedge (\neg p_i \to \Box \neg p_i)),$$

[1]This variant of reduction is used in [BdRV02].

where $\Box^m = \neg\Diamond^m\neg$; likewise for $\Box^{\leq m}$.

Let $\mathsf{T}_\eta = (W_\eta, R_\eta)$ be the *quantifier tree* of η: $W_\eta = \bigcup_{k\leq m}\{\sigma \in 2^k \mid \sigma(i) = 0 \text{ iff } Q_{i+1} = \exists\}$; $\sigma_1 R_\eta \sigma_2$ iff $\sigma_1 \subset \sigma_2$ and $\mathrm{dom}(\sigma_2) = \mathrm{dom}(\sigma_1) + 1$. We remark that $\mathsf{T}(\eta)$ is antitransitive (and so is irreflexive). We have:

$$\eta \text{ is valid} \quad \Rightarrow \quad [\eta]_{\mathrm{L}} \text{ is satisfiable in } (W_\eta, R^*_\eta), \tag{9}$$

$$[\eta]_{\mathrm{L}} \text{ is satisfiable in a Kripke frame} \quad \Rightarrow \quad \eta \text{ is valid}; \tag{10}$$

see [BdRV02, Section 6.7] for the details.

Consider a unimodal logic $L \subseteq \mathrm{S4}$. Then (W_η, R^*_η) is an L-frame, hence every satisfiable in this frame formula is L-consistent. So we have:

$$\eta \text{ is valid} \quad \Rightarrow \quad [\eta]_{\mathrm{L}} \text{ is satisfiable in } (W_\eta, R^*_\eta) \quad \Rightarrow \quad [\eta]_{\mathrm{L}} \text{ is } L\text{-consistent} \quad \Rightarrow \quad \eta \text{ is valid},$$

that is, η is valid iff $[\eta]_{\mathrm{L}}$ is L-consistent. Thus, the L-consistency problem is PSpace-hard; synonymously, the (provability problem for the) logic L is PSpace-hard.

This proves Ladner's theorem, in its classical formulation. And, in fact, it proves more. Let Grz.Bin be the logic of the class of all finite transitive reflexive trees with branching ≤ 2. We have the following proper inclusions

$$\mathrm{S4} \subsetneq \mathrm{S4.1} \subsetneq \textsc{Grz} \subsetneq \textsc{Grz.Bin},$$

see, e.g., [CZ97], and [GDJ74] for the latter inclusion. Observe that (W_η, R^*_η) is a finite transitive reflexive tree. It immediately follows from the above reasonings that every logic contained in Grz is PSpace-hard. Moreover, the branching of (W_η, R^*_η) is ≤ 2; thus, the result holds for logics contained in Grz.Bin:

$$L \subseteq \textsc{Grz.Bin} \quad \Rightarrow \quad L \text{ is PSpace-hard.}$$

This formulation does not include an important logic GL: its frames are irreflexive. However, we can reformulate (9) in the following way: if $R_\eta \subseteq R \subseteq R^*_\eta$ (or, equivalently, $R^* = R^*_\eta$), then

$$\eta \text{ is valid} \quad \Rightarrow \quad [\eta]_{\mathrm{L}} \text{ is satisfiable in } (W_\eta, R); \tag{11}$$

the prove is straightforward and is an immediate analog of the proof of (9) given in [BdRV02].

Definition 5. A class $\mathcal{F}$ of unimodal frames is *thick* if for every finite transitive reflexive tree $\mathsf{T} = (T, \leq)$ with branching ≤ 2 there exist a relation R on T and a frame $\mathsf{F} \in \mathcal{F}$ such that $R^* = \leq$ and F is isomorphic to (T, R).

From (11) we obtain that if $\mathcal{F}$ is thick then

$$\eta \text{ is valid} \quad \Rightarrow \quad [\eta]_{\mathrm{L}} \text{ is satisfiable in } \mathcal{F}. \tag{12}$$

Thus, for a class $\mathcal{F}$ of unimodal frames and a unimodal logic L, from (12) and (10) we have:

$$\mathcal{F} \text{ is thick and } L \subseteq \mathrm{Log}\,\mathcal{F} \quad \Rightarrow \quad L \text{ is PSpace-hard.} \tag{13}$$

In particular, this formulation allows to apply Ladner's construction for logics below GL, since GL is the logic of a thick class (consisting of finite irreflexive transitive trees).

The logics S4.2 is not the logic of a thick class: it does not have trees of height > 1 within its frames. However, trees are contained as subframes in S4.2-frames. And this is also enough for PSpace-hardness due to the following relativization argument proposed by E. Spaan [Spa93].

Recall that *a subframe* of a frame $\mathsf{F} = (W, (R_a)_{a\in \mathrm{A}})$ is the restriction $\mathsf{F}{\upharpoonright}V = (V, (R_a \cap (V \times V))_{a\in\mathrm{A}})$, where $V \neq \varnothing$. For a class $\mathcal{F}$ of frames, let $\mathrm{Sub}\,\mathcal{F}$ be its closure under the subframe operation: $\mathrm{Sub}\,\mathcal{F} = \{\mathsf{F}{\upharpoonright}V \mid \mathsf{F} \in \mathcal{F} \text{ and } \varnothing \neq V \subseteq \mathrm{dom}(\mathsf{F})\}$. The satisfiability in $\mathrm{Sub}\,\mathcal{F}$ can be reduced to the satisfiability in $\mathcal{F}$ (see the proof of Theorem 2.2.1 in [Spa93]). Namely, for an A-formula φ and a variable q, let $[\varphi]_q$ be the A-formula inductively defined as follows: $[\bot]_q = \bot$, $[p]_q = p$ for variables, $[\psi_1 \to \psi_2]_q = [\psi_1]_q \to [\psi_2]_q$, and $[\Diamond_a\psi]_q = \Diamond_a([\psi]_q \wedge q)$ for $a \in \mathrm{A}$. We put $[\varphi]_{\mathrm{Rel}} = q \wedge [\varphi]_q$, where q is the first variable not occurring in φ. If $\mathsf{M} = (\mathsf{F}, \theta)$ is a model and $V = \theta(q)$, then for every $v \in V$ we have: $\mathsf{M}{\upharpoonright}V, v \models \varphi$ iff $\mathsf{M}, v \models [\varphi]_{\mathrm{Rel}}$; the proof is by induction on φ. This immediately yields

Proposition 23 ([Spa93])**.** *φ is satisfiable in $\mathrm{Sub}\,\mathcal{F}$ iff $[\varphi]_{\mathrm{Rel}}$ is satisfiable in $\mathcal{F}$.*

Thus, $\mathrm{Sat}\,\mathrm{Sub}\,\mathcal{F}$ is polynomial time reducible to $\mathrm{Sat}\,\mathcal{F}$.

It immediately follows from (13) and Proposition 23 that if $\mathrm{Sub}\,\mathcal{F}$ is thick, then the logic of $\mathcal{F}$ is PSpace-hard. Moreover, this holds for every $L \subseteq \mathrm{Log}\,\mathcal{F}$: a quantified Boolean formula η is valid iff $[[\eta]_{\mathrm{L}}]_{\mathrm{Rel}}$ is L-consistent. Indeed, if η is valid, then $[\eta]_{\mathrm{L}}$ is satisfiable in $\mathrm{Sub}\,\mathcal{F}$ by (12), so $[[\eta]_{\mathrm{L}}]_{\mathrm{Rel}}$ is satisfiable in $\mathcal{F}$ by Proposition 23, hence $[[\eta]_{\mathrm{L}}]_{\mathrm{Rel}}$ is L-consistent. The converse implication follows from (10). Thus, we have

Theorem 24 (A corollary of [Lad77] and [Spa93])**.** *Let $\mathcal{F}$ be a class of unimodal frames. If $\mathrm{Sub}\,\mathcal{F}$ is thick, then every unimodal $L \subseteq \mathrm{Log}\,\mathcal{F}$ is* PSpace*-hard.*

For an A-frame $\mathsf{F} = (W, (R_a)_{a\in\mathrm{A}})$ and $a \in \mathrm{A}$, let $\mathsf{F}^{\upharpoonright a}$ be its reduct (W, R_a). A class $\mathcal{F}$ of A-frames is said to be *thick* if for some $a \in \mathrm{A}$ the class $\{\mathsf{F}^{\upharpoonright a} \mid \mathsf{F} \in \mathcal{F}\}$ is thick.

Proposition 25. *Let $\mathcal{F}$ and $\mathcal{I}$ be classes of* A*-frames. If $\mathcal{F}$ is non-empty and $\mathrm{Sub}\,\mathcal{I}$ is thick, then $\mathrm{Sub}\sum_{\mathcal{I}}\mathcal{F}$ is thick.*

Proof. Follows from Definitions 1 and 5. □

Corollary 26. *Let $\mathcal{F}$ and $\mathcal{I}$ be classes of* A*-frames. If $\mathcal{F}$ is non-empty and $\mathrm{Sub}\,\mathcal{I}$ is thick, then $\mathrm{Sat}\sum_{\mathcal{I}}\mathcal{F}$ is* PSpace*-hard.*

4.5 Examples

For many logics, Theorem 9 gives a uniform proof of decidability in PSpace (PSpace-completeness in view of Corollary 26). Let us illustrate it with certain examples.

Example 3. It is well-known that the logic S4 as well as its expansion with the universal modality, are PSpace-complete [Lad77],[Hem96].

Let us prove it via sums. Recall that clusters are frames of form $(C, C \times C)$. Every preorder F is isomorphic to the sum $\sum_{C\in \mathrm{sk}\mathsf{F}}(C, C\times C)$ of its clusters over its skeleton $\mathrm{sk}\mathsf{F}$. The logic S4 has the finite model property, hence S4 is the logic of the class

$$\sum_{\text{finite partial orders}} \text{clusters}.$$

Thus, we have:

$$\mathrm{Sat}(\text{preorders}) \ \leq_{\mathrm{T}}^{\mathrm{PSpace}} \ \mathrm{Sat}(\text{clusters})$$

or dually

$$\mathrm{S4} \ \leq_{\mathrm{T}}^{\mathrm{PSpace}} \ \mathrm{S5}.$$

Since the satisfiability on clusters is in NP, it is in PSpace, which gives PSpace-upper bound for S4. In fact, we obtained the PSpace-completeness of the logic $\mathrm{S4}^{[\forall]} = \mathrm{Log}\{(W, W\times W, R) \mid (W,R) \text{ is a preorder}\}$ (Corollary 22).

Changing the class of summands, we obtain PSpace-completeness for other logics. If we add an irreflexive singleton S_0 to the class of summands-clusters, then we obtain PSpace-completeness of K4 and $\mathrm{K4}^{[\forall]}$.

Letting the class of summands be $\{\mathsf{S}_0\}$, we obtain PSpace-completeness for the logics GL. If the class of summands consists of a reflexive singleton $\{\mathsf{S}_1\}$, the above reasonings give PSpace-completeness of the Grzegorczyk logic, and if the class of summands is $\{\mathsf{S}_0, \mathsf{S}_1\}$ — for the weak Grzegorczyk logic (recall that the latter logic is characterized by frames whose reflexive closures are non-strict Noetherian orders [Lit07]; this class can be represented as $\sum_{\mathrm{NPO}} \{\mathsf{S}_0, \mathsf{S}_1\}$).

The results discussed in Example 3 are well-known [Lad77],[Spa93],[Hem96]. To the best of our knowledge, the following result has never been published before.

Example 4. The *weak transitivity logic* wK4 is the logic of frames satisfying the condition $xRzRy \Rightarrow xRy \vee x = y$. The *difference logic* DL is the logic of the frames such that xRy whenever $x \neq y$; let $\mathcal{F}_{\neq}$ be the class of such frames. The logic wK4 has the finite model property [Esa01], hence it is the logic of the class

$$\sum_{\text{finite partial orders}} \mathcal{F}_{\neq},$$

see Example 2. It is not difficult to check that $\mathrm{Sat}\,\mathcal{F}$ is in NP (like in the case of clusters, the size of a countermodel is linear in the length of a formula), and we obtain that

$$\mathrm{wK4} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{DL} \in \mathrm{PSpace},$$

so wK4 is in PSpace (PSpace-complete). Moreover, since the class $\mathcal{F}_{\neq}$ is preconical, $\mathrm{wK4}^{[\forall]}$ is in PSpace.

Example 5. The logic wK4.2 is the logic of weakly transitive frames (considered in the above example) satisfying the Church-Rosser property $xRy_1 \,\&\, xRy_2 \;\Rightarrow\; \exists z\,(y_1Rz \,\&\, y_2Rz)$. Observe that every frame validating wK4.2 is either in $\mathcal{F}_{\neq}$, or is isomorphic to the sum of a frame validating wK4 and a frame in $\mathcal{F}_{\neq}$.

$$\mathrm{Fr}\,(\mathrm{wK4.2}) \quad = \quad \mathcal{F}_{\neq} \cup (\mathrm{Fr}\,(wK4) \;+^0\; \mathcal{F}_{\neq}).$$

Now that wK4.2 is decidable in PSpace follows from the previous example and the following theorem.

Theorem 27. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be classes of* A*-frames,* $a \in \mathrm{A}$*. If* $\mathrm{Sat}\,\mathcal{G}^{(\forall)} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{Sat}\,\mathcal{F}^{(\forall)}$ *(in particular, if* $\mathrm{Sat}\,\mathcal{G}^{(\forall)}$ *is in* PSpace*), then* $\mathrm{Sat}(\mathcal{F} +^a \mathcal{G}) \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{Sat}\,\mathcal{F}^{(\forall)}$*.*

Proof. From Lemma 13, it follows that if $\mathrm{CSat}\,\mathcal{G} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{CSat}\,\mathcal{F}$, then $\mathrm{CSat}(\mathcal{F} +^a \mathcal{G}) \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{CSat}\,\mathcal{F}$. The rest of the proof follows from Propositions 20 and 21. □

Remark 3. This theorem can can be strengthened in two aspects. First, it can be formulated for reductions stronger than $\leq_{\mathrm{T}}^{\mathrm{PSpace}}$. Another observation is that instead of the class $\mathcal{F} +^a \mathcal{G}$, i.e., a class of sums over $\mathsf{I} = (2, <)$, one can consider sums over an arbitrary finite indexing frame I; in this case, a reduction from sums to summands would follow from Lemma 6.

We conclude this section with the following example. In topological semantics ($\Diamond$ is for the derivation), wK4 is the logic of all topological spaces [Esa01]. In [BEG11] it was shown that the logic $\mathrm{wK4T}_0$ of all T_0-spaces is equal to the logic of all finite weakly transitive frames where clusters contain at most one irreflexive point. The satisfiability for such clusters is in NP, and we obtain

Corollary 28. $\mathrm{wK4T}_0$ *is* PSpace*-complete.*

5 Variations

In this section we are interested in polymodal logics which can be characterized by models obtained via multiple application of the sum operation. An important example of such logic is the *Japaridze's polymodal logic* [Jap86, Bek10]; we also consider *lexicographic products of modal logics* introduced in [Bal09] and *refinements of logics* introduced in [BR10]; see Sections 5.3 and 5.4 below. We anticipate them with some general observations.

5.1 Iterated sums over unimodal indexes

Let $\mathcal{F}$ be a class of A-frames, $\mathcal{I}$ a class of 1-frames. Let $\boldsymbol{a} = (a_0, \ldots, a_{s-1})$ be a finite sequence of elements of A. If $\boldsymbol{a}$ is the empty sequence, let $\boldsymbol{a}$*-iterated sums* be elements of $\mathcal{F}$. For $0 < s < \omega$, let $\boldsymbol{a}$*-iterated sums* be a_0-sums of $(a_1, \ldots, a_{s-1})$-iterated sums: that is, an $\boldsymbol{a}$*-iterated sum of frames in* $\mathcal{F}$ *over frames in* $\mathcal{I}$ is a frame of form $\sum_{\mathsf{I}}^{a_0} \mathsf{H}_i$, where $\mathsf{I} \in \mathcal{I}$ and every H_i is an $(a_1, \ldots, a_{s-1})$-iterated sum of frames in $\mathcal{F}$ over frames in $\mathcal{I}$. The class of all such sums is denoted by $\sum_{\mathcal{I}}^{\boldsymbol{a}} \mathcal{F}$.

Since $\sum_{\mathcal{I}}^{\boldsymbol{a}} \mathcal{F}$ is $\sum_{\mathcal{I}}^{a_0} \sum_{\mathcal{I}}^{(a_1,\ldots,a_{s-1})} \mathcal{F}$, by Theorem 9 we obtain

Corollary 29. *Let* $\mathrm{A} < \omega$*,* $\mathcal{F}$ *a class of* A*-frames,* $\boldsymbol{a}$ *a finite sequence of elements of* A*. If* $\mathcal{I}$ *is a class of Noetherian orders that contains all finite trees and is closed under finite disjoint unions, then* $\mathrm{Sat}(\sum_{\mathcal{I}}^{\boldsymbol{a}} \mathcal{F})^{(\forall)} \leq_{\mathrm{T}}^{\mathrm{PSpace}} \mathrm{Sat}\, \mathcal{F}^{(\forall)}$*.*

Theorem 8 reduces satisfiability on sums over Noetherian orders to sums over finite trees. This theorem can also be extended for the case of iterated sums [Sha18]. Namely, let $\boldsymbol{a} = (a_0, \ldots a_{s-1}) \in \mathrm{A}^s$, $0 < s < \omega$. Then for every A-tie $\tau = (\varphi, \Phi, \boldsymbol{\Gamma})$ we have:

$$\tau \text{ is satisfiable in } \sum\nolimits_{\mathrm{NPO}}^{\boldsymbol{a}} \mathcal{F} \text{ iff } \tau \text{ is satisfiable in } \bigsqcup\nolimits_{\leq \#\varphi} \sum\nolimits_{\mathrm{Tr}(\#\varphi,\#\varphi)}^{\boldsymbol{a}} \mathcal{F}. \qquad (14)$$

This can be proven by induction of the length of $\boldsymbol{a}$ with the help of the following lemma (see the proof of [Sha18, Theorem 5.2] for the details):

Lemma 30. *Let* $\mathcal{F}$ *be a class of* A*-frames,* $a \in \mathrm{A}$*. Then every frame in* $\sum_{\mathrm{NPO}}^{a} \bigsqcup \mathcal{F}$ *is isomorphic to a frame in* $\sum_{\mathrm{NPO}}^{a} \mathcal{F}$*.*

Proof. By Theorem 1(3c), a sum of form $\sum_{i\in \mathsf{I}}^{a} \bigsqcup_{j\in J_i} \mathsf{F}_{ij}$ is isomorphic to the sum $\sum_{(i,j)\in \sum_{k\in \mathsf{I}} (J_k,\varnothing)}^{a} \mathsf{F}_{ij}$. It remains to observe that if $\mathsf{I} = (I,<) \in \mathrm{NPO}$ and $(J_i)_{i\in I}$ is a family of non-empty sets, then $\sum_{\mathsf{I}} (J_i, \varnothing) \in \mathrm{NPO}$. ☐

In view of Proposition 4, from (14) we obtain

Corollary 31. *Let* $\mathcal{F}$ *be a class of* A*-frames,* $s < \omega$*,* $\boldsymbol{a} = (a_0, \ldots a_{s-1}) \in \mathrm{A}^s$*. Then for every* A*-formula* φ *we have:*

$$\varphi \textit{ is satisfiable in } \sum\nolimits_{\mathrm{NPO}}^{\boldsymbol{a}} \mathcal{F} \textit{ iff } \varphi \textit{ is satisfiable in } \sum\nolimits_{\mathrm{Tr}(\#\varphi,\#\varphi)}^{\boldsymbol{a}} \mathcal{F}.$$

Remark 4. The operation of iterated sum can result in very tangled structures. It this remark, we expand the formal definition of this operation.

Consider a tree $(T,<)$ such that every maximal chain in T is of length s, $0 < s < \omega$. Let I be the set of maximal elements of $(T,<)$, $J = T \setminus I$, and for $i \in J$, let $\mathrm{sc}(i)$ be the set of immediate successors of i. A structure $\mathsf{T} = (T, <, \{S_i\}_{i\in J})$ such that S_i is a binary relation on $\mathrm{sc}(i)$ is called an *indexing tree*. If also $(\mathrm{sc}(i), S_i) \in \mathcal{I}$ for every $i \in J$, then T is an *indexing tree for* $\mathcal{I}$.

For a sequence $\boldsymbol{a} = (a_0, \ldots, a_{s-1}) \in \mathrm{A}^s$, and a family $(\mathsf{F}_i)_{i\in I}$ of A-frames, we define *the* $\boldsymbol{a}$*-sum of* $(\mathsf{F}_i)_{i\in I}$ *over the indexing tree* T, in symbols $\sum_{\mathsf{T}}^{\boldsymbol{a}} \mathsf{F}_i$, as the following A-frame $(W, (R_a)_{a\in\mathrm{A}})$. The domain of this structure is the disjoint union of the domains of F_i:

$$W = \{(i, w) \mid i \text{ is maximal in } \mathsf{T} \text{ and } w \text{ is in } \mathsf{F}_i\}$$

To define the relations R_a, consider $(i, w), (j, v) \in W$. If $i = j$, then, as usual, we put $(i, w)R_a(j, v)$ iff $wR_{i,a}v$, where $R_{i,a}$ denotes the a-th relation in F_i. Assume that $i \neq j$. Let $k = \inf\{i, j\}$, h be the height of k in T, that is the number of elements in $\{l \in T \mid l < k\}$. There are unique immediate successors i' and j' of k such that $i \leq i'$ and $j \leq j'$. We put $(i, w)R_a(j, v)$ iff $i'S_kj'$ and a is the h-th element of $\boldsymbol{a}$.

The class $\sum_{\mathcal{I}}^{*} \mathcal{F}$ of *iterated sums of frames in* $\mathcal{F}$ *over frames in* $\mathcal{I}$ consists of such structures where T is an indexing tree for $\mathcal{I}$ and all F_i are in $\mathcal{F}$. One can see that for a fixed $\boldsymbol{a}$, the elements of $\sum_{\mathcal{I}}^{*} \mathcal{F}$ are (up to isomorphisms) $\boldsymbol{a}$-iterated sums $\sum_{\mathcal{I}}^{\boldsymbol{a}} \mathcal{F}$.

5.2 Lexicographic sums

The sum operation does not change the signature. In many cases (see below) it is convenient to characterize a polymodal logic via the following modification of a-sums.

Definition 6. Let $\mathsf{I} = (I, S)$ be a unimodal frame, $(\mathsf{F}_i)_{i\in I}$ a family of A-frames, $\mathsf{F}_i = (W_i, (R_{i,a})_{a\in\mathrm{A}})$. The *lexicographic sum* $\overset{\text{lex}}{\sum}_{\mathsf{I}} \mathsf{F}_i$ is the $(1 + \mathrm{A})$-frame $\left(\bigsqcup_{i\in I} W_i, S^{\text{lex}}, (R_a)_{a<N}\right)$, where

$$\begin{array}{rcl} (i, w)S^{\text{lex}}(j, u) & \text{iff} & iSj, \\ (i, w)R_a(j, u) & \text{iff} & i = j \ \&\ wR_{i,a}u. \end{array}$$

For a class $\mathcal{F}$ of A-frames and a class $\mathcal{I}$ of 1-frames, we define $\overset{\text{lex}}{\sum}_{\mathcal{I}} \mathcal{F}$ as the class of all sums $\overset{\text{lex}}{\sum}_{\mathsf{I}} \mathsf{F}_i$, where $\mathsf{I} \in \mathcal{I}$ and all F_i are in $\mathcal{F}$.

Remark that unlike in the case of sums considered in previous sections, lexicographic sums depend on reflexivity of the indexing frame.

Recall that for an A-frame $\mathsf{F} = (W, (R_i)_{i\in\mathrm{A}})$, $\mathsf{F}^{(\forall)}$ is the $(1+\mathrm{A})$-frame $(W, W \times W, (R_i)_{i\in\mathrm{A}})$. Let $\mathsf{F}^{(\varnothing)}$ be the $(1+\mathrm{A})$-frame $(W, \varnothing, (R_a)_{a<\mathrm{A}})$; for a class $\mathcal{F}$ of frames, let $\mathcal{F}^{(\varnothing)} = \{\mathsf{F}^{(\varnothing)} \mid \mathsf{F} \in \mathcal{F}\}$.

The following is immediate from the definitions:

Proposition 32. *For a unimodal frame* $\mathsf{I} = (I, S)$ *and a family* $(\mathsf{F}_i)_{i\in I}$ *of* A*-frames,*

$$\overset{\text{lex}}{\sum}_{\mathsf{I}} \mathsf{F}_i = \sum_{\mathsf{I}'} \mathsf{F}'_i = \sum\nolimits_{\mathsf{I}}^{0} \mathsf{F}'_i,$$

where I' *is the* $(1 + \mathrm{A})$*-frame* $(I, S, (\varnothing)_{\mathrm{A}})$*, and for* $i \in I$, $\mathsf{F}'_i = \mathsf{F}_i^{(\forall)}$ *whenever* i *is reflexive in* I*, and* $\mathsf{F}'_i = \mathsf{F}_i^{(\varnothing)}$ *otherwise.*

Proposition 33. *Let* $\mathcal{F}$ *be a class of* A*-frames.*

1. *If* $\mathcal{I}$ *is a class of irreflexive 1-frames, then* $\overset{\text{lex}}{\sum}_{\mathcal{I}} \mathcal{F} = \sum_{\mathcal{I}}^{0} \mathcal{F}^{(\varnothing)}$.

2. *If* $\mathcal{I}$ *is a class of reflexive 1-frames, then* $\overset{\text{lex}}{\sum}_{\mathcal{I}} \mathcal{F} = \sum_{\mathcal{I}}^{0} \mathcal{F}^{[\forall]}$.

Proof. Immediate from Proposition 32. □

Given a class $\mathcal{F}$ of frames in an alphabet A and a class of unimodal frames $\mathcal{I}$, let 0-*iterated lexicographic sums* be elements of $\mathcal{F}$, and for $n < \omega$, let $(n+1)$-*iterated lexicographic sums* be lexicographic sums of n-iterated sums. The class of n-iterated lexicographic sums of frames in $\mathcal{F}$ over frames in $\mathcal{I}$ is denoted by $\overset{\text{lex}^n}{\sum}_{\mathcal{I}}\mathcal{F}$.

The next fact is the iterated version of the above proposition. For a class $\mathcal{F}$ of A-frames and an alphabet B, let $\mathcal{F}^{(\varnothing)_{\mathrm{B}}}$ be the class of (B + A)-frames $(W, (\varnothing)_{\mathrm{B}}, (R_a)_{a\in \mathrm{A}})$ such that $(W, (R_a)_{a\in \mathrm{A}}) \in \mathcal{F}$. Likewise, let $\mathcal{F}^{(\forall)_{\mathrm{B}}}$ be the class of frames $(W, (S_a)_{a\in \mathrm{B+A}})$ such that $S_a = W \times W$ for $a \in \mathrm{B}$, and $(W, (S_{\mathrm{B}+a})_{a\in \mathrm{A}}) \in \mathcal{F}$.

Proposition 34. *Let* $\mathrm{A} \leq \omega$, $0 < n < \omega$, *and let* $\mathcal{F}$ *be a class of* A*-frames.*

1. *If* $\mathcal{I}$ *is a class of irreflexive 1-frames, then* $\overset{\text{lex}^n}{\sum}_{\mathcal{I}}\mathcal{F} = \sum_{\mathcal{I}}^{(0,\dots,n-1)} \mathcal{F}^{(\varnothing)_n}$.
2. *If* $\mathcal{I}$ *is class of reflexive 1-frames, then* $\overset{\text{lex}^n}{\sum}_{\mathcal{I}}\mathcal{F} = \sum_{\mathcal{I}}^{(0,\dots,n-1)} \mathcal{F}^{(\forall)_n}$.

Proof. Follows from Propositon 32 by a straightforward induction on n. □

Propositions 33 and 34 allow to expand our results on the finite model property and complexity to the logics of (iterated) lexicographic sums.

5.3 Japaridze's polymodal logic

In this section we show that Japaridze's polymodal provability logic GLP is decidable in PSpace. First, this theorem was proven in [Sha08]. Here we provide a version of the proof based on Theorem 15.

GLP is a normal modal logic in the language ML(ω). This system was introduced in [Jap86] and plays an important role in proof theory (see, e.g., [Bek04]). GLP is known to be Kripke incomplete, so we cannot directly apply our tools to analyze it. However, in [Bek10], L. Beklemishev introduced a modal logic J, a Kripke complete approximation of GLP. Semantically, J is characterised as the logic of frames called *stratified*, or *hereditary partial orderings* ([Bek10, Section 3]). They are defined as follows.

Definition 7. For $\mathrm{A} \leq \omega$, let S_{A} be $(\{0\}, (\varnothing)_{\mathrm{A}})$, a singleton A-frame with empty relations. For $\mathrm{A} < \omega$, let $\mathcal{J}(\mathrm{A})$ be the class of A-iterated lexicographic sums

$$\overset{\text{lex}^{\mathrm{A}}}{\sum}_{\mathrm{NPO}}\{\mathsf{S}_\omega\} = \underbrace{\overset{\text{lex}}{\sum_{\mathrm{NPO}}} \dots \overset{\text{lex}}{\sum_{\mathrm{NPO}}}}_{\mathrm{A\ times}} \{\mathsf{S}_\omega\}.$$

The class $\mathcal{J}$ of *hereditary partial orderings* is the class $\bigcup_{\mathrm{A}<\omega} \mathcal{J}(\mathrm{A})$. The *logic* J is defined as the logic of the class $\mathcal{J}$.

In [Bek10], it was shown that GLP is polynomial time many-to-one reducible to J: there exists a polynomial-time computable $f : \mathrm{ML}(\omega) \to \mathrm{ML}(\omega)$ such that $\varphi \in \mathrm{GLP}$ iff $f(\varphi) \in \mathrm{J}$ (for an explicit description of f, see [BFDJ14, Lemma 3.4]).

Our aim is to show that J is in PSpace. For our purposes, it is more convenient to work with a-sums. Since ${\mathsf{S}_\omega}^{(\varnothing)_{\mathrm{A}}} = \mathsf{S}_\omega$, from Proposition 34.1 we have:

Proposition 35. *For every* $\mathrm{A} < \omega$, $\mathcal{J}(\mathrm{A}) = \sum_{\mathrm{NPO}}^{0} \dots \sum_{\mathrm{NPO}}^{\mathrm{A}-1}\{\mathsf{S}_\omega\}$

Formally, for every $\mathrm{A} < \omega$, each frame in $\mathcal{J}(\mathrm{A})$ has infinitely many relations. Next facts allow us to consider frames of finite signatures. Let

$$\hat{\mathcal{J}}(\mathrm{A}) = \sum\nolimits_{\mathrm{NPO}}^{0} \cdots \sum\nolimits_{\mathrm{NPO}}^{\mathrm{A}-1} \{\mathsf{S}_{\mathrm{A}}\}.$$

Proposition 36. *For every* $\mathrm{A} < \omega$, *we have:*

1. $\mathcal{J}(\mathrm{A})^{\restriction \mathrm{A}} = \hat{\mathcal{J}}(\mathrm{A})$;
2. *If* $\mathrm{A} > 0$, *then every frame in* $\mathcal{J}^{\restriction \mathrm{A}}$ *is isomorphic to a frame in* $\hat{\mathcal{J}}(\mathrm{A})$.

Proof. From Definition 1 it is immediate that for alphabets $\mathrm{B} \leq \mathrm{C}$ and classes of C-frames $\mathcal{F}, \mathcal{I}$,

$$\left(\sum_{\mathcal{I}} \cdots \sum_{\mathcal{I}} \mathcal{F} \right)^{\restriction \mathrm{B}} = \sum_{\mathcal{I}^{\restriction \mathrm{B}}} \cdots \sum_{\mathcal{I}^{\restriction \mathrm{B}}} \left(\mathcal{F}^{\restriction \mathrm{B}} \right). \tag{15}$$

Clearly, ${\mathsf{S}_{\omega}}^{\restriction \mathrm{A}}$ is S_{A}. Now we obtain (1) from Proposition 35 and Definition 4.

To prove (2), let us fix $\mathrm{B} < \omega$ and show that every frame in $\mathcal{J}(\mathrm{B})^{\restriction \mathrm{A}}$ is isomorphic to a frame in $\hat{\mathcal{J}}(\mathrm{A})$.

First, observe that $\mathcal{J}(\mathrm{B}+1)$ contains copies of all frames in $\mathcal{J}(\mathrm{B})$. Indeed, every frame in the class $\sum_{\mathrm{NPO}}^{0} \cdots \sum_{\mathrm{NPO}}^{\mathrm{B}-1} \{\mathsf{S}_{\omega}\}$ is isomorphic to a frame in $\sum_{\mathrm{NPO}}^{0} \cdots \sum_{\mathrm{NPO}}^{\mathrm{B}-1} \sum_{\mathrm{NPO}}^{\mathrm{B}} \{\mathsf{S}_{\omega}\}$, since $\sum_{\mathrm{NPO}}^{\mathrm{B}} \{\mathsf{S}_{\omega}\}$ contains a copy of S_{ω}. Now the case $\mathrm{B} < \mathrm{A}$ follows.

The case $\mathrm{A} < \mathrm{B}$ is more interesting. Consider the class $\mathcal{G} = \sum_{\mathrm{NPO}}^{\mathrm{A}-1} \cdots \sum_{\mathrm{NPO}}^{\mathrm{B}-1} \{\mathsf{S}_{\omega}\}$. By (15), we have:

$$\mathcal{J}(\mathrm{B})^{\restriction \mathrm{A}} = (\sum\nolimits_{\mathrm{NPO}}^{0} \cdots \sum\nolimits_{\mathrm{NPO}}^{\mathrm{B}-1} \{\mathsf{S}_{\omega}\})^{\restriction \mathrm{A}} = \sum\nolimits_{\mathrm{NPO}}^{0} \cdots \sum\nolimits_{\mathrm{NPO}}^{\mathrm{A}-2} \mathcal{G}^{\restriction \mathrm{A}}.$$

By (15) again, $\mathcal{G}^{\restriction \mathrm{A}}$ can be represented as sums of disjoin unions of singletons: $\mathcal{G}^{\restriction \mathrm{A}} = \sum_{\mathrm{NPO}}^{\mathrm{A}-1} \bigsqcup \cdots \bigsqcup \{\mathsf{S}_{\mathrm{A}}\}$. By Lemma 30, every frame in $\mathcal{G}^{\restriction \mathrm{A}}$ is isomorphic to a frame in the class $\sum_{\mathrm{NPO}}^{\mathrm{A}-1} \{\mathsf{S}_{\mathrm{A}}\}$. It follows that $\hat{\mathcal{J}}(\mathrm{A})$ contains copies of all frames in $\mathcal{J}(\mathrm{B})^{\restriction \mathrm{A}}$. ☐

Proposition 37. *For every* $\mathrm{A} < \omega$, $\mathrm{J} \cap \mathrm{ML}(\mathrm{A}) = \mathrm{Log}(\hat{\mathcal{J}}(\mathrm{A}))$.

Proof. By Proposition 36.1, $\hat{\mathcal{J}}(\mathrm{A}) \subseteq \mathcal{J}^{\restriction \mathrm{A}}$, so if φ is satisfiable in $\hat{\mathcal{J}}(\mathrm{A})$, then it is satisfiable in $\mathcal{J}$. Conversely, if φ is satisfiable in $\mathcal{J}$, then it is satisfiable in $\mathcal{J}^{\restriction \mathrm{A}}$, and hence — in $\hat{\mathcal{J}}(\mathrm{A})$ by Proposition 36.2. ☐

It is trivial that the satisfiability problem on the singleton $\mathsf{S}_{\mathrm{A}}^{(\forall)}$ is in PSpace (in fact, it is in NP). From Corollary 29 and Proposition 36.1, we obtain

Corollary 38. *For every* $\mathrm{A} < \omega$, $\mathrm{Sat}\, \hat{\mathcal{J}}(\mathrm{A})^{(\forall)} \in \mathrm{PSpace}$.

Hence, every fragment of J with finitely many modalities is in PSpace. The above corollary does not directly imply that $\mathrm{Sat}\, \mathcal{J}$ is in PSpace. But the latter fact is a corollary of Theorem 15 and the observations below.

Lemma 39. *Let* $\mathrm{A} < \omega$. *An* A*-formula* φ *is satisfiable in* J *iff* φ *is satisfiable in* $\sum_{\mathrm{Tr}(\#\varphi, \#\varphi)}^{(0,\ldots,\mathrm{A}-1)} \{\mathsf{S}_{\mathrm{A}}\}$.

Proof. Immediate from Proposition 37 and Corollary 31. ☐

Consider a formula φ in the language ML_{ω}. Let $a_0 < \cdots < a_{N-1}$ be the increasing sequence of indices of all occurring in φ modalities. Put $N(\varphi) = N$. Let $\hat{\varphi}$ be the result of replacing a_b-modalities by b-modalities in φ. Note that $\hat{\varphi}$ is an $N(\varphi)$-formula, and $N(\varphi) < \#\varphi = \#\hat{\varphi}$.

ALGORITHM 2: Decision procedure for CSat $\mathcal{S}(h,b,a,\mathrm{A})$

CSatJ $(\varphi, \mathbf{v}, \mathbf{U}, h, b, a, \mathrm{A})$:*boolean*
  Input: An A-tie $(\varphi, \mathbf{v}, \mathbf{U})$; positive integers h, b; $a \leq \mathrm{A} < \omega$.
  if $a = \mathrm{A}$ **then return** ($(\varphi, \mathbf{v}, \mathbf{U})$ is satisfiable in S_{A});
  if CSatJ$(\varphi, \mathbf{v}, \mathbf{U}, \mathrm{A}, \mathrm{A}, a+1, \mathrm{A})$ **then return** true;
  if $h > 1$ **then**
    for k *such that* $1 \leq k \leq b$ **do for** $\mathbf{u}, \mathbf{v}_0, \ldots, \mathbf{v}_{k-1} \in 2^{\#\varphi}$ *such that* $\mathbf{v} = \mathbf{u} + \sum_{i<k} \mathbf{v}_i$ **do**
      if CSatJ$(\varphi, \mathbf{u}, \mathbf{U} +^a \sum_{i<k} \mathbf{v}_i, \mathrm{A}, \mathrm{A}, a+1, \mathrm{A})$ **then**
        if $\bigwedge_{i<k}$ CSatJ$(\varphi, \mathbf{v}_i, \mathbf{U}, h-1, b, a, \mathrm{A})$ **then return** true;
  return false.

Lemma 40. *[BFDJ14, Lemma 3.5].* $\varphi \in \mathrm{J}$ *iff* $\hat{\varphi} \in \mathrm{J}$.

Theorem 41. J *is in* PSpace.

Proof. We describe a decision procedure for the conditional satisfiability on iterated sums of trees. For positive h, b and $a \leq \mathrm{A} < \omega$, we define sums $\mathcal{S}(h,b,a,\mathrm{A})$ as follows: if $a = \mathrm{A}$, let $\mathcal{S}(h,b,a,\mathrm{A})$ denote $\{\mathsf{S}_{\mathrm{A}}\}$ for all h, b; if $a < \mathrm{A}$, let $\mathcal{S}(h,b,a,\mathrm{A})$ denote $\sum^{a}_{\mathrm{Tr}(h,b)} \sum^{(a+1,\ldots,\mathrm{A}-1)}_{\mathrm{Tr}(\mathrm{A},\mathrm{A})} \{\mathsf{S}_{\mathrm{A}}\}$. In particular, $\mathcal{S}(h,b,\mathrm{A}-1,\mathrm{A})$ is $\sum^{\mathrm{A}-1}_{\mathrm{Tr}(h,b)}\{\mathsf{S}_{\mathrm{A}}\}$, and $\mathcal{S}(h,b,0,\mathrm{A}) = \sum^{(0,\ldots,\mathrm{A}-1)}_{\mathrm{Tr}(\mathrm{A},\mathrm{A})}\{\mathsf{S}_{\mathrm{A}}\}$. Using Theorem 15, we describe the procedure CSatJ (Algorithm 2) that decides whether a given A-tie $(\varphi, \mathbf{v}, \mathbf{U})$ is satisfiable $\mathcal{S}(h,b,a,\mathrm{A})$:

Lemma 42. *Let* $a \leq \mathrm{A} < \omega$, $(\varphi, \mathbf{v}, \mathbf{U})$ *an* A-*tie,* $0 < h, b < \omega$. *Then*

$$(\varphi, \mathbf{v}, \mathbf{U}) \textit{ is satisfiable in } \mathcal{S}(h,b,a,\mathrm{A}) \textit{ iff } \mathrm{CSatJ}(\varphi, \mathbf{v}, \mathbf{U}, h, b, a, \mathrm{A}) \textit{ returns } \mathrm{true}.$$

Proof. By induction on $\mathrm{A} - a$. The case $\mathrm{A} = a$ is trivial: $\mathcal{S}(h,b,\mathrm{A},\mathrm{A})$ consists of a single singleton S_{A}. If $a < \mathrm{A}$, then $\mathcal{S}(h,b,a,\mathrm{A})$ is $\sum^{a}_{\mathrm{Tr}(h,b)} \mathcal{S}(\mathrm{A},\mathrm{A},a+1,\mathrm{A})$. By induction hypothesis, CSatJ$(\tau, \mathrm{A}, \mathrm{A}, a+1, \mathrm{A})$ decides whether a tie τ is satisfiable in $\mathcal{S}(\mathrm{A},\mathrm{A},a+1,\mathrm{A})$. Now the induction step follows from Theorem 15. ☐

We have from Lemmas 39 and 40:

Lemma 43. φ *is satisfiable in* J *iff* $\hat{\varphi}$ *is satisfiable in* $\mathcal{S}(\#\varphi, \#\varphi, 0, \#\varphi)$.

Hence, by Proposition 11, we obtain

$$\varphi \text{ is satisfiable in J iff } \exists \mathbf{v} \in 2^{\#\varphi}\, (\mathbf{v}(0) = 1 \,\&\, (\hat{\varphi}, \mathbf{v}, \mathbf{0}) \in \mathrm{CSat}\, \mathcal{S}(\#\varphi, \#\varphi, 0, \#\varphi)),$$

where $\mathbf{0}$ represents the empty condition.

Let us estimate the amount of space used by CSatJ on the input $(\hat{\varphi}, \mathbf{v}, \mathbf{0}, \#\varphi, \#\varphi, 0, \#\varphi)$. On each recursive call, either the argument a increases by 1, or a does not change and the parameter h decreases by 1. Since $1 \leq h \leq \#\varphi$ and $0 \leq a \leq \#\varphi$, we obtain that the depth of recursion is bounded by $\#\varphi \cdot (\#\varphi + 1)$. At each call CSatJ needs $O(n^2)$ space to store new variables. Hence, to check the satisfiability of φ in $\mathcal{J}$ we need $O(n^4)$ space. This completes the proof of the theorem. ☐

Corollary 44. *Japaridze's polymodal provability logic* GLP *is decidable in* PSpace.

Remark 5. It is immediate that GLP and J are PSpace-hard (e.g., it follows from the fact that the 1-modal fragments of these logics are the logic GL). From [CR03], it follows that one-variable fragment of GLP is PSpace-hard. In [Pak14], it was shown that even the constant (closed) fragment of GLP is PSpace-hard.

5.4 Refinements and lexicographic products of modal logics

In this paragraph we discuss algorithmic properties of logics obtained via the *refinement* and the *lexicographic product* operations.

The operation of *refinement of modal logics* was introduced in [BR10].

Definition 8. Let $\mathsf{F} = (W, R)$ be a preorder, $\mathrm{sk}F = (\overline{W}, \leq)$ its skeleton. Consider a family $(\mathsf{F}_C)_{C\in\overline{W}}$ of A-frames such that $\mathrm{dom}(\mathsf{F}_C) = C$ for all $C \in \overline{W}$. The *refinement* of F *by* $(\mathsf{F}_C)_{C\in\overline{W}}$ is the $(1 + N)$-frame $(W, R, (R_a^{\triangleright})_{a\in\mathrm{A}})$, where

$$R_a^{\triangleright} \subseteq \bigcup_{C\in\overline{W}} C \times C \qquad \text{for all } a \in \mathrm{A}, \tag{16}$$

$$(W, (R_a^{\triangleright})_{a\in\mathrm{A}}){\restriction}C = \mathsf{F}_C \qquad \text{for all } C \in \overline{W}. \tag{17}$$

For a class $\mathcal{I}$ of preorders and a class $\mathcal{G}$ of N-frames let $\mathrm{Ref}(\mathcal{I}, \mathcal{F})$ be the class of all refinements of frames from $\mathcal{I}$ by frames in $\mathcal{F}$. For logics $L_1 \supseteq \mathrm{S4}, L_2$, we put $\mathrm{Ref}(L_1, L_2) = \mathrm{Log}\,\mathrm{Ref}(\mathrm{Fr}\,L_1, \mathrm{Fr}\,L_2)$.

In [BR10] it was shown that in many cases the refinement operation preserves the finite model property and decidability.

Refinements can be considered as sums according to the following fact:

Proposition 45. *The refinement of* F *by frames* $(\mathsf{F}_C)_{C\in\mathrm{sk}\mathsf{F}}$ *is isomorphic to the sum* $\sum^0_{C\in\mathrm{sk}\mathsf{F}} \mathsf{F}_C^{[\forall]}$.

Proof. The isomorphism is given by $w \mapsto (C, w)$, where $w \in C$. □

In view of Theorems 2 and 8, this observation provides another way to prove the finite model property of refinements (see [Sha18, Section 5.2] for more details). Moreover, representation of refinements as sums allows to obtain complexity results according to our theorems in Section 4.

Let us illustrate this with the logic $\mathrm{Ref}(\mathrm{S4}, \mathrm{S4})$. In [BR10], it was proven that $\mathrm{Ref}(\mathrm{S4}, \mathrm{S4})$ is decidable and that $\mathrm{Ref}(\mathrm{S4}, \mathrm{S4}) = \mathrm{Log}\,\mathrm{Ref}(\mathrm{QO}_f, \mathrm{QO}_f)$, where QO_f is the class of all finite non-empty preorders. The latter fact in combination with above proposition yields

$$\mathrm{Ref}(\mathrm{S4}, \mathrm{S4}) = \mathrm{Log} \sum\nolimits^0_{\mathrm{Tr}_f} {\mathrm{QO}_f}^{[\forall]}.$$

Since the satisfiability problem for the class ${\mathrm{QO}_f}^{(\forall)}$ is in PSpace ([Hem96]; see also Example 3), from Theorem 9 (and Corollary 26) we obtain

Corollary 46. $\mathrm{Ref}(\mathrm{S4}, \mathrm{S4})$ *is* PSpace-*complete.*

A related operation is the *lexicographic product of modal logics* introduced in [Bal09] by Ph. Balbiani.

Definition 9. Consider frames $\mathsf{I} = (I, S)$ and $\mathsf{F} = (W, (R_a)_{a\in\mathrm{A}})$. Their *lexicographic product* $\mathsf{I} \leftthreetimes \mathsf{F}$ is the $(1 + \mathrm{A})$-frame $(I \times W, S^{\leftthreetimes}, (R_a^{\leftthreetimes})_{a\in\mathrm{A}})$, where

$$(i, w)S^{\leftthreetimes}(j, u) \quad \text{iff} \quad iSj,$$

$$(i, w)R_a^{\leftthreetimes}(j, u) \quad \text{iff} \quad i = j \;\&\; wR_au.$$

In other words, $\mathsf{I} \leftthreetimes \mathsf{F}$ is the lexicographic sum $\sum^{\mathrm{lex}}_{\mathsf{I}} \mathsf{F}_i$, where $\mathsf{F}_i = \mathsf{F}$ for all i in I.

For a class $\mathcal{I}$ of 1-frames and a class $\mathcal{G}$ of A-frames, the class $\mathcal{I} \leftthreetimes \mathcal{F}$ is the class of all products $\mathsf{I} \leftthreetimes \mathsf{F}$ such that $\mathsf{I} \in \mathcal{I}$ and $\mathsf{F} \in \mathcal{F}$. For logics L_1, L_2, we put $L_1 \leftthreetimes L_2 = \mathrm{Log}\,(\mathrm{Fr}\,L_1 \leftthreetimes \mathrm{Fr}\,L_2)$.

As examples, consider the logics $\mathrm{S4} \leftthreetimes \mathrm{S4}$ and $\mathrm{GL} \leftthreetimes \mathrm{S4}$ and show that they are in PSpace (PSpace-complete). In [Sha18], it was shown that $\mathrm{S4} \leftthreetimes \mathrm{S4}$ is equal to $\mathrm{Ref}(\mathrm{S4}, \mathrm{S4})$. Also, it was shown that that

$$\mathrm{GL} \leftthreetimes \mathrm{S4} = \mathrm{Log}\,(\mathrm{Tr}_f \leftthreetimes \mathrm{QO}_f) = \mathrm{Log} \sum\nolimits^0_{\mathrm{Tr}_f} \sum\nolimits^1_{\mathrm{Tr}_f} \mathcal{C}$$

where $\mathcal{C}$ is the class of finite frames of form $(C, \varnothing, C \times C)$ (in terms of lexicographic sums, $\sum^0_{\mathrm{Tr}_f} \sum^1_{\mathrm{Tr}_f} \mathcal{C}$ is the class $\overset{\mathrm{lex}}{\sum}_{\mathrm{Tr}_f} \mathrm{QO}_f$); see [Sha18, Theorem 15] for the proof. Hence, this logic is in PSpace by Corollary 29.

Corollary 47. *The lexicographic products* $\mathrm{S4} \leftthreetimes \mathrm{S4}$*,* $\mathrm{GL} \leftthreetimes \mathrm{S4}$ *are* PSpace*-complete.*

This result contrasts with the undecidability results for *modal products* of transitive logics [GKWZ03], [GKWZ05].

6 Conclusion

In many cases, sum-like operations preserve the finite model property and decidability [BR10], [Sha18]. In this paper we showed that transferring results can be obtained for the complexity of the modal satisfiability problems on sums. In particular, it follows that for many logics PSpace-completeness is immediate from their semantic characterizations.

Let us indicate some further results and directions.

- Sums and products with linear indices

  In the linear case, *modal products* are typically (highly) undecidable [RZ01]. However, the modal satisfiability problem on the lexicographic squares of dense unbounded linear orders is in NP [BM13]. This positive result seems to be scalable according to the following observation: in many cases, φ is satisfiable in sums over linear (quasi)orders iff φ is satisfiable in such sums that the length of indices is bounded by $\#\varphi$. In this situation there is a stronger than $\leq^{\mathrm{PSpace}}_{\mathrm{T}}$ reduction between the sums and the summands.

- Further results on the finite model property and decidability of sums

  In many cases, the finite model property of a modal logic L can be obtained by a filtration method; in this case we say that L *admits filtration*. It follows from [BR10] that filtrations of refinements can be reconstructed from filtrations of components. This result can be extended for a more general setting of lexicographic sums.

  Also, the results obtained in [BR10] in a combination with Theorem 2 suggest the following conjecture: in the case of finitely many modalities, if $\mathrm{Log}\,\mathcal{F}^{(\forall)}$ has the finite model property and $\mathrm{Log}\,\mathcal{I}$ admits filtration, then the logic of $\sum_{\mathcal{I}} \mathcal{F}$ has the finite model property.

  Another fact that we announce relates to the property of local finiteness of logics: if both the logics of indices and summands are locally finite, then the logic of lexicographic sums (a fortiori, of lexicographic products) is locally finite.

- Axiomatization of sums

  For unimodal logics L_1, L_2, let $\overset{\mathrm{lex}}{\sum}_{L_1} L_2$ be the logic of the class $\overset{\mathrm{lex}}{\sum}_{\mathrm{Fr}\,L_1} \mathrm{Fr}\,L_2$. Consider the following 2-modal formulas:

$$\alpha = \Diamond_1 \Diamond_0 p \to \Diamond_0 p, \quad \beta = \Diamond_0 \Diamond_1 p \to \Diamond_0 p, \quad \gamma = \Diamond_0 p \to \Box_1 \Diamond_0 p$$

One can see that these formulas are valid in every lexicographic sum $\overset{\text{lex}}{\sum}_{\mathsf{I}}\mathsf{F}_i$ (and hence, in every product $\mathsf{I} \leftthreetimes \mathsf{F}$) of 1-frames. In many cases, these axioms provide a complete axiomatization of $\overset{\text{lex}}{\sum}_{L_1} L_2$. In particular, the logic $\overset{\text{lex}}{\sum}_{\mathrm{GL}}\mathrm{GL} = \mathrm{Log}(\overset{\text{lex}}{\sum}_{\mathrm{NPO}}\mathrm{NPO})$, the bimodal fragment of the logic J considered in Section 5.3, is the logic $\mathrm{GL} * \mathrm{GL} + \{\alpha, \beta, \gamma\}$ [Bek10] ($L_1 * L_2$ denotes the *fusion* of L_1 and L_2, $L + \Psi$ is for the least logic containing $L \cup \Psi$). Analogous results hold for various lexicographic products [Bal09]; e.g., $\mathrm{S4} \leftthreetimes \mathrm{S4} = \mathrm{S4} * \mathrm{S4} + \{\alpha, \beta, \gamma\}$. We announce the following results:

1. If $L_1 * L_2 + \{\alpha, \beta, \gamma\}$ is Kripke complete, and the class $\mathrm{Fr}\, L_1$ (considered as a class of models in the classical model-theoretic sense) is first-order definable without equality, then
$$\sum_{L_1} L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\}.$$
2. Assume that $\mathrm{Fr}\, L_2$ is closed under direct products and validates the property $\forall x \exists y\, xRy$, and that the class $\mathrm{Fr}\, L_1$ is first-order definable without equality. If the logic $L_1 * L_2 + \{\alpha, \beta, \gamma\}$ is Kripke complete, then
$$L_1 \leftthreetimes L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\}.$$

At the same time, no general axiomatization results are known for the logics of sums in the sense of Definition 1.

- Sum-based operations in the Kripke-incomplete case

  The operations we considered so far lead to Kripke complete logics. What could be definition of sums for modal algebras (models, general Kripke frames)? E..g., can we give a semantic characterization of an important Kripke-incomplete logic GLP by sums of form $\sum_{\mathrm{NPO}} \cdots \sum_{\mathrm{NPO}} \mathcal{C}$?

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