Network Robustness via Global k-cores

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ABSTRACT

Network robustness is a measure a network's ability to survive adversarial attacks. But not all parts of a network are equal. K-cores, which are dense subgraphs, are known to capture some of the key properties of many real-life networks. Therefore, previous work has attempted to model network robustness via the stability of its k-core. However, these approaches account for a single core value and thus fail to encode a global network resilience measure. In this paper, we address this limitation by proposing a novel notion of network resilience that is defined over all cores. In particular, we evaluate the stability of the network under node removals with respect to each node's initial core. Our goal is to compute robustness via a combinatorial problem: find b most critical nodes to delete such that the number of nodes that fall from their initial cores is maximized. One of our contributions is showing that it is NP-hard to achieve any polynomial factor approximation of the given objective. We also present a fine-grained complexity analysis of this problem under the lens of parameterized complexity theory for several natural parameters. Moreover, we show two applications of our notion of robustness: measuring the evolution of species and characterizing networks arising from different domains.

KEYWORDS

Network optimization, k-core, network robustness

1 INTRODUCTION

Networks model many real-world complex systems. An important aspect of these networks is their robustness or resilience. Robustness quantifies a network's capability to resist failures that might affect its functionalities. These network failures often lead to a considerable of economic losses. As an example, a snowy weather in 2008 caused a major power grid failure in China [29].

The study of network resilience via stability of the k-core structure [24] has been a popular topic in recent literature. Bhawalkar et al. [1] propose maximizing the initial k-core size to prevent network unravelling. The resilience of k-core have also been studied

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(a) Random   **(b) High Degree**
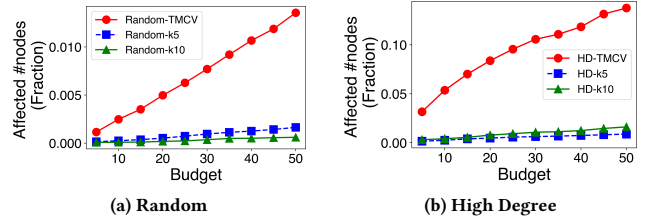
Figure 1: Fraction (F_v) of the entire node set is affected by deleting (a) random and (b) high degree nodes. The red line shows the result by our objective (see TMCV in Def. 4), whereas, the other ones show the effect inside fixed k-cores ($k = 5$ and $k = 10$) in a co-authorship network (CondMat in Sec. 3). Our objective has a global and larger impact on the entire network.

under critical node/edge deletion to increase or maintain users' engagement in social networks [17, 28, 30] and to prevent failures in technological networks [12]. Consider an example of a P2P network where the users who benefit from the network should also share their resources with other users. This follows a k-core model and in this case the network owner has to be aware of the critical nodes to maintain the resource sharing process uninterrupted.

The aforementioned studies suffer from a local notion of network stability as they aim to modify the k-core for a given value of k. We address this limitation by proposing a novel combinatorial problem over k-cores: *find b (budget) critical nodes whose deletion will remove the maximum number of nodes from their initial core*. The number of nodes staying in their core after removal of those critical nodes quantifies the stability of the network. Thus, a network is more (less) robust or resilient if a larger set of nodes are unaffected (affected).

Figure 1 shows an example of the global effect of our formulation. We show how the nodes get affected (i.e., fall from their initial core) under node deletion via two strategies when (a) random and (b) high degree nodes are selected. The y-axis shows the fraction of the total nodes that get affected. Our formulated objective (red line) captures a global robustness notion and the number of affected nodes are much larger than in the individual cores (denoted by blue and green for 5-core and 10-core respectively).

1.1 Contributions

We study a novel combinatorial problem, *Total Minimization of Coreness via Vertex deletion* (TMCV), which aims to measure network robustness based on the maximum number of nodes that fall from their initial core after b number of nodes are deleted. Besides showing strong inapproximability result, we present fine-grained parameterized complexities of the problem for several parameters. Table 3 (in Section 4) summarizes the main theoretical results.

Additionally, we propose a few heuristics to solve TMCV and evaluate their performance on real datasets. These heuristics nicely capture interesting structural properties of networks from different genres (e.g., social, co-authorship). Furthermore, we apply our proposed network robustness measure to understand the evolution of species. Zitnik et al. [32] has shown that evolution is related to a network robustness measure based on network connectivity. Intuitively, more genetic changes in a species would result in a more resilient protein-protein interaction network of the same. In Section 3, we show significant correlation between our proposed resilience/robustness measure and the evolution dynamics of species.

Our main contributions are as follows:

▷ We propose a novel network robustness problem (TMCV) based on the coreness of nodes under deletion of nodes.

▷ We show that it is NP-hard to achieve any polynomial factor approximation for TMCV (Thm. 4).

▷ We study the parameterized complexity of our problem for several natural parameters. We show that TMCV is $W[2]$-hard (Thm. 1) parameterized by the budget b and para-NP-hard parameterized by the degeneracy (Cor. 2) of the graph and the maximum degree (Thm. 3) individually.

▷ We propose several heuristics that capture interesting structural properties of networks from different genres. Furthermore, we show how we can apply our network robustness measure to understand the evolution of species.

Organization of the paper. The paper is organized as follows: Section 1.2 describes the related work. We define our network robustness problem in Section 2. We show how to apply our network robustness measure to capture interesting structural properties of networks as well as to understand the evolution of species in Section 3. Finally, Section 4 demonstrates all the theoretical results.

1.2 Related Work

Understanding robustness of a network via the stability of its k-core has recently received a significant amount of attention. The major goal in this line of work is to measure the resilience of the k-core of a network under its modifications. Zhang et al. [28] first propose the collapsed k-core problem that aims to minimize the k-core by deleting b critical vertices. The edge version of this problem has been recently addressed with efficient heuristics [17, 31]. Another related paper [12] measures the stability of k-core under random edge/node deletions. These studies only focus on the k-core robustness, i.e., the effect on the nodes inside the k-core. On the contrary, we propose a novel and generalized version of these problems. Our robustness measure captures the affected nodes in different cores (i.e., any k) upon a budget number of node deletions.

Other related but orthogonal literature studies the maximization of the k-core in networks via different mechanisms. One such example is maximization of the k-core by making a few nodes outside the k-core as anchors to prevent unraveling in social networks [1, 3]. The other example involves adding edges with nodes outside of the k-core [30]. Another related paper [15] discusses parameterized algorithms for the collapsed k-core problem [28]. In this paper, we

(a) Initial   (b) Only 3-core   (c) Delete v (d) Delete w
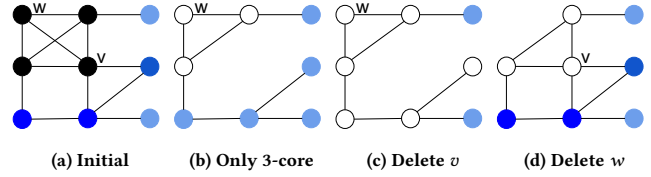
Figure 2: (a) Initial graph: four nodes are in 3-core, three in 2-core and two nodes are in 1-core. (b) Considering just 3-core, deleting a removes all other three nodes from 3-core. (c) In our problem TMCV **(Def. 4), all the six empty nodes got affected after deleting v. (d) In** TMCV**, all three empty nodes got affected on deleting w.**

discuss parameterized complexity for a different problem along with inapproximabilty results.

Network robustness: Previous work has also studied the ability of a network to sustain various types of attacks or failures and termed it as network robustness. An extensive survey of different robustness measures for undirected and unweighted networks is conducted by Ellens et al. [8]. The measures discussed in this survey are mainly based on network connectivity and shortest path distances along with some others based on Laplacian eigenvalues. Another survey [13] provides nine widely used robustness measures and studies their sensitivity. The network robustness models vary depending on the application. In a recent work, Lordan et al. [14] have identified the set of removed nodes that maximizes the size of the largest connected component of a network in an optimal manner. Recently, Zitnik et al. [32] show that a connectivity-based robustness of protein-protein interaction networks is a good predictor of the extent of evolution of a species. Here, we consider a different robustness measure.

Our proposed problem is also related to network design problems. These problems aim to optimize network properties or processes under network modifications. Examples include diameter [6], node centrality [5, 19] shortest path [7, 16, 20, 21], and influence spread [10, 11, 18, 26] improvement. However, our objective is different from the ones considered by network design studies.

2 PROBLEM DEFINITION

Let $G(V, E)$ be an undirected and unweighted graph with sets of vertices V ($|V| = n$) and edges E ($|E| = m$). We denote the degree of vertex u in G by $d(u, G)$. An induced subgraph, $H = (V_H, E_H)$ of G is the following: if $u, v \in V_H$ and $(u, v) \in E$ then $(u, v) \in E_H$. The k-core [24] of a network is defined as follows.

DEFINITION 1. k**-Core:** *The k-core of a graph G, denoted by $S_k(G) = (V_k(G), E_k(G))$, is defined as a maximal induced subgraph where each vertex has degree at least k.*

DEFINITION 2. ***Coreness:*** *The coreness of a node v in graph G, denoted by $C(v, G)$, is defined as the maximum k where $v \in S_k(G)$ and $v \notin S_l(G)$ for any $l > k$.*

DEFINITION 3. ***Degeneracy:*** *The degeneracy of a graph G, denoted by $D(G)$, is defined as the largest k where $S_k(G)$ is non-empty.*

EXAMPLE 1. *Consider the initial graph in Figure 2a as an example. The degeneracy of the graph is 3.*

We denote the modified graph G after deleting a set B consisting of b vertices (nodes) as $G \setminus B$. Deleting a vertex reduces the degree

Dataset	Type	$|V|$	$|E|$	D
Enron	Email	36692	183831	43
GrQc	Co-authorship	4158	13422	43
CondMat	Co-authorship	21363	91286	25
Facebook	Social	4039	88234	115
g+	Social	23628	39194	12
BrightKite	Social	58228	214078	52

Table 1: Statistics of Datasets: D **denotes degeneracy, i.e., the maximum core.**

of its neighbours and possibly their coreness. This reduction in coreness might propagate to other vertices. Let us define an affected node as follows: a node v is affected if $C(v, G) > C(v, G \setminus B)$. The example in Figure 2c shows that deleting a node (e.g. node v) can affect the neighbours and propagate to other non-neighbor nodes. Next we define the coreness minimization problem.

DEFINITION 4. ***Total Minimization of Coreness via Vertex deletion (***TMCV***): Given a graph*** $G = (V, E)$, *candidate vertices* $\Gamma \subseteq V$ *and budget* b, *find the set* $B \subset \Gamma$ *of nodes to be removed such that* $|B| \leqslant b$ *and* $f(B) = |\{v \in V \setminus B : C(v, G) > C(v, G \setminus B)\}|$ *is maximized.*

Note that the objective minimizes the number of unaffected nodes. Intuitively, a network is more robust if its value of f is small.

EXAMPLE 2. *Figure 2a shows an example of the initial graph. The TMCV objective is explained in Figures 2c and 2d. In Figure 2c, when v is deleted, all the remaining three nodes in the 3-core fall into 2-core and all the three nodes in the 2-core move to 1-core. Thus, five nodes are affected, i.e., $f(\{v\}) = 5$. In Figure 2d, by deleting the node w, only the nodes that were in the 3-core are affected, i.e., $f(\{w\}) = 3$. The empty nodes in Figures 2b, 2c and 2d are the affected ones—i.e., with reduced coreness.*

3 EMPIRICAL RESULTS

In this section, we motivate the TMCV problem using real applications and simple heuristics. We show that our problem can be used to characterize different types of networks and to understand the relationship between protein-protein interaction (PPI) networks and the evolution of species.

3.1 Robustness and Characterization of Networks

We evaluate our robustness measure using different networks and show interesting properties of those via a few heuristics. We measure the performance of each algorithm by a disruption measure F which is defined by the fraction of nodes getting affected (reduction of initial coreness) due to the deletion of the nodes in the solution set generated by each algorithm. A network is more robust if it has a lower value of F. We denote the modified graph G as $\mathbb{G}_v^*$ after deleting a set B consist of b vertices (nodes). Formally,

$$F(B) = \frac{f(B)}{|V|} = \frac{|\{v \in V : C(v, G) > C(v, \mathbb{G}_v^*)\}|}{|V|} \quad (1)$$

Datasets: We use six real datasets from different genres in our experiments. Table 1 and Figure 4a describe the statistics and the

core distributions of the datasets, respectively. The datasets are available in [22] and online[1].

3.1.1 *Heuristics.* We describe the heuristics below.

Random: This algorithm chooses b nodes randomly from the set of all nodes in the graph. The random strategy has been used in the past to enhance network robustness [25].

High Degree (HD): It chooses top b nodes according to their degree. Coreness is related to degree and the nodes in higher core usually contribute to the coreness in the lower core. So, this strategy uses degree as a proxy of the coreness. Intuitively, the algorithm should work well with the presence of sensitive nodes, i.e., when the degree of a node is equal to its individual coreness.

Affected Size	0.1	0.2	0.3	0.4	0.5
Enron	3	7	16	33	61
GrQc	10	26	49	79	121
CondMat	22	66	132	229	385
Facebook	1	1	2	2	3
g+	1	3	6	9	14
BrightKite	10	53	169	438	1076

Table 2: The effect of AHDR on different networks: The cell (Enron, 0.1) as 3 represents that only 3 nodes to be deleted to achieve $F = 0.1$**, i.e., to affect 10% of the Enron network.**

High Disruption (HDR): The algorithm chooses top b nodes according to their "strength" in making nodes fall from their corresponding k-core. This strategy is more related with our objective function compared to the random and degree based heuristics. However, it requires the computation of the "strength" for each node. The running time of this algorithm is $O(n^2 + nm)$.

Adaptive High Disruption (AHDR): It chooses the best node in each iteration for the budget number of iterations. However, in each step one needs to recompute the strength of the nodes given that already chosen nodes are deleted from the graph. A naive implementation of this strategy would take $O(bn(n + m))$ time, where b is the budget. However, we are able to optimize this approach based on a few observations.

Observation 1. *Deletion of a node v might reduce coreness of another node u only when $C(v, G) \geqslant C(u, G)$. There will not be any effect in deleting v on u if $C(v, G) < C(u, G)$.*

Observation 2. *Based on the previous observation, node v can be pruned from the candidate set Γ if $C(u, G) > C(v, G), \forall u \in N(v)$ where $N(v)$ denotes the set of neighbors of v.*

3.1.2 *Results of different heuristics on real networks.* We vary the budget and evaluate the performance of each heuristic in all the datasets (Table 1). Figure 3 shows the results. Budget is the percentage of the total number of nodes in the network. A few interesting results are as follows: (a) AHDR is the most effective heuristic. The closest baseline, HDR, directly computes the effect of edge removal on the TMCV objective. AHDR, unlike others, considers the disruption in the network in an adaptive manner. (b) The efficacy of AHDR is more prominent in the co-authorship networks (CondMat and GrQc). The co-authorship networks often consist of small cliques and thus high degree (HD) or one shot strength computation (HDR) might not be effective to choose the critical nodes.

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[1]https://snap.stanford.edu/data
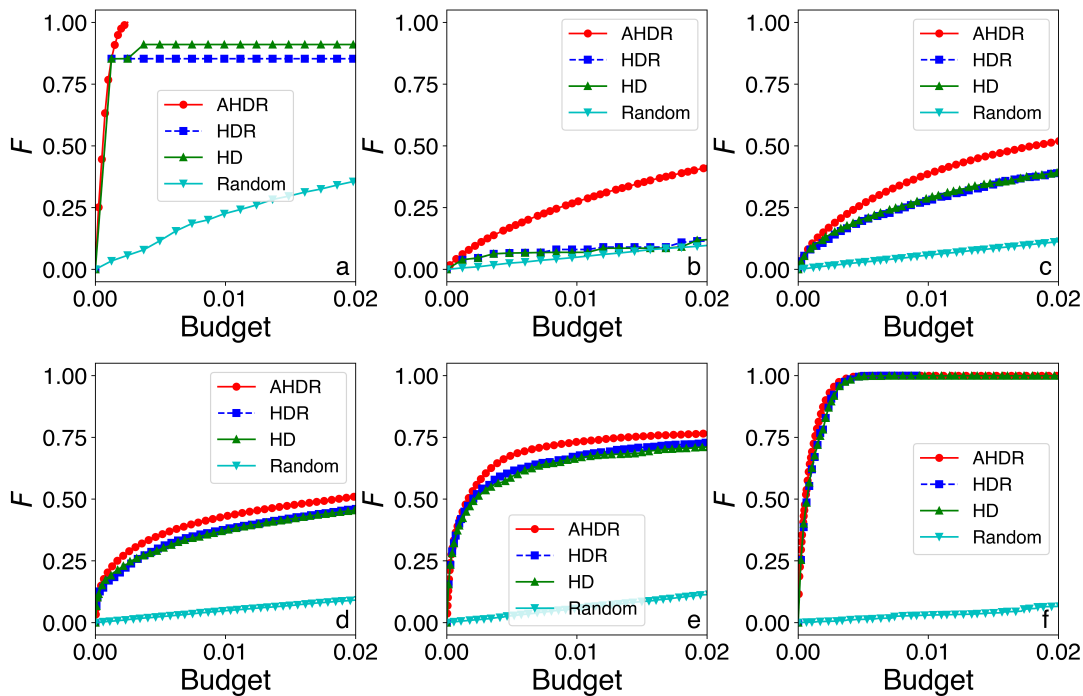
Figure 3: The performance of different heuristics varying the budget, b (percentage of total nodes to delete) on real datasets: (a) Facebook, (b) GrQc, (c) CondMat, (d) BrightKite, (e) Enron, (f) g+.

Simple network properties such as density plays an important role in robustness. Facebook is a dense graph and the best heuristic, AHDR produces $F = 1$ only with an extremely low budget (10 nodes). We further emphasize how robust the individual networks are by showing the number of nodes needed to be deleted by AHDR to affect a large portion of the network in Table 2. The dense structure makes the cores very sensitive and a node removal has high impact. Another interesting observation is that the graphs with the highest (Facebook) and the lowest (g+) densities are easy to disrupt compared to others. This suggests that the robustness does not entirely depend on the density of the graph.

3.1.3 Synthetic vs real networks. Figure 4c shows the impact of the best performing heuristic, AHDR, in co-authorship (CondMat), social (g+) and synthetic ($|V| = 20,000$) networks. BA-d2 (BA-d4) and ER-d2 (ER-d4) represent the synthetic network structures from two well-studied models: (a) Barabasi-Albert and (b) Erdos-Renyi, respectively, with average node degree 2 (degree 4). Note that all of these six networks have similar number of nodes. The k-core distributions of these networks is shown in Fig. 4b. The goal is to compare the robustness of different networks while applying the same algorithm (e.g., AHDR). We observe that the random network, ER, is the most robust or difficult to break. As the edges are present uniformly across the network, node deletions do not have large affect on the network structure. This is true even with higher density ER graphs (see ER-d2 and ER-d4). Comparing BA and ER, BA is less robust to node removals as a few nodes have high degree and might be part of several cores. On the other hand, the real networks are less robust than both these synthetic networks. Even if the co-authorship network is denser than the social network

(g+), the structure of g+ is less robust and ADHD can affect more than 80% of the network by only removing 50 nodes.
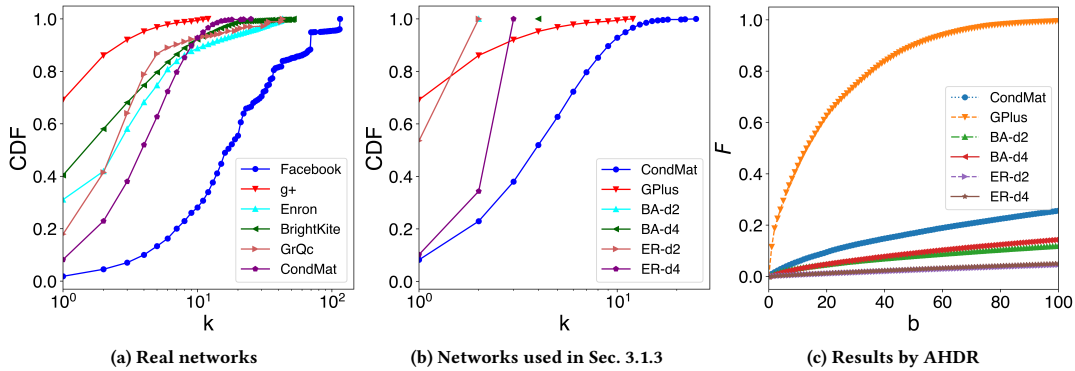
3.2 Robustness and Evolution

In the last section, we have applied network robustness as a tool to characterize different types of networks (email, co-authorship and social). Here, we use robustness to compare multiple networks of the same type. Protein-protein interaction (PPI) networks capture how proteins interact to perform various biological functions (e.g., DNA replication, energy production). These networks are relevant in biological and biomedical applications, specially in the study of new treatments for complex diseases, such as cancer and autoimmune disorders [23]. Recently, it has been shown that the structure PPI networks is also related to the evolution of species [32]. In particular, evolution was shown to be positively correlated with network resilience. In this section, we evaluate how k-core robustness can help us to better understand this relationship.

Dataset: For this study, we apply a subset of the Tree of Life dataset[2], which combines PPI networks and an evolution score—based on the depth in the phylogenetic tree—for 63 species. The species selected were those with at least 1,000 publications in the NCBI PubMed and belonging to the *Bacteria* and *Archaea* domains.

Baseline: We compare our resilience measure against the one applied in [32]. More specifically, their approach measures how fragmented the network becomes after the removal of a fraction α of nodes selected at random. Once a node is removed, all its edges are also removed from the network. The fragmentation of G_α is

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[2]http://snap.stanford.edu/tree-of-life

(a) Real networks   (b) Networks used in Sec. 3.1.3   (c) Results by AHDR

Figure 4: (a) The core distributions in the real networks in Table 1. (b) The core distributions of the real and synthetic networks used in the experiments in Sec. 3.1.3. (c) The performance of the best heuristic (AHDR) varying the number (budget, b) of nodes in co-authorship, social and synthetic networks of similar sizes.

measured based on a modified version of the Shannon divergence of the resulting connected components $\{V_1, V_2, \dots V_K\}$:

$$H(G_\alpha) = \frac{1}{\log n} \sum_{k=1}^{K} p_k \log p_k$$

where $p_k = |V_k|/n$ and the $1/\log n$ factor enables comparing graphs with different sizes.

The overall resilience of a network G is computed as the area under the curve produced varying α from 0 to 1:

$$Resilience_{rand}(G) = 1 - \int_0^1 H(G_\alpha)d\alpha$$

K-core Resilience: We propose a resilience metric similar to the one defined above but replacing the Shannon entropy by the fraction of nodes out of their k-core:

$$Resilience_{core}(G) = 1 - \int_0^1 F(B_\alpha)d\alpha$$

where $F(B_\alpha)$ is the fraction of nodes affected after $\alpha|V|$ nodes are removed from G.

Similar to [32], we also apply our measures only to the largest connected component of each network. Moreover, we emphasize two key differences between our resilience metric ($Resilience_{core}$) and $Resilience_{rand}$. First, ours takes into account the k-core instead of the connected components in the graph. Second, we do not remove nodes at random, but as to maximize $F(B_\alpha)$.

Figure 5 shows the correlation between $Resilience_{rand}$ and the evolution of species. Notice that the measures have a weak correlation, with a Pearson's coefficient of 0.0325 and a p-value of 0.80. As a consequence, we are unable to reject the hypothesis that the variables are in fact uncorrelated. Notice that we consider a subset of the species from [32]—with only the domains *Bacteria* and *Archaea*. Still, one would expect the correlation between evolution and resilience to also hold within these domains.

In Figure 6, we show the correlation between $Resilience_{core}$ (our metric) and evolution. Compared to Figure 5, we notice that our resilience measure has a stronger correlation with evolution of the species. In particular, the Pearson's coefficient for the correlation is 0.2366 with a small p-value of 0.06. This is a strong evidence that
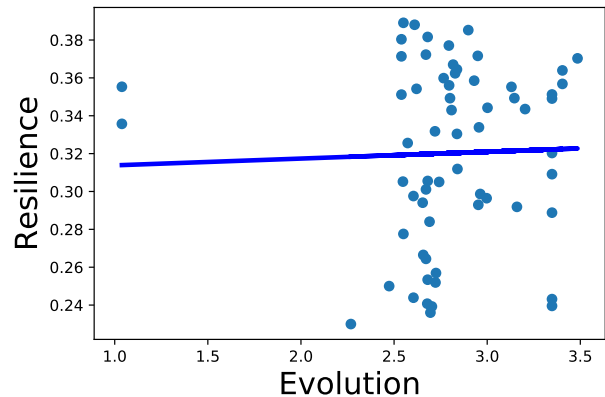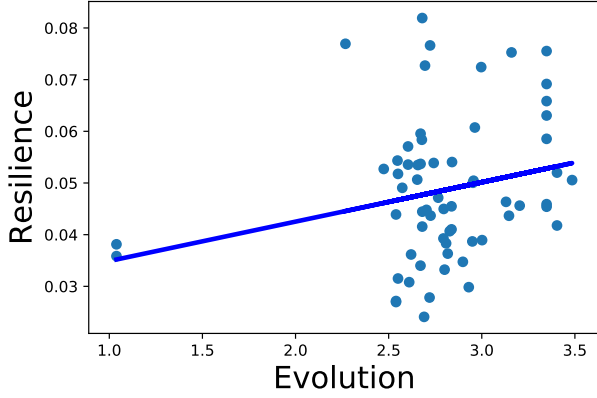

Figure 5: Correlation (coefficient=0.0325,p-value=0.80) between the evolution of species (x) and resilience (y) measured using $Resilience_{rand}$, which is based on random node removals [32]. The plot and correlation values show that there is not a strong correlation between the measures.

our notion of k-core resilience is able to capture relevant structural properties of PPI networks. Species that are further (or deeper) in the tree of life present a more robust network. More importantly, this relationship is even stronger when we consider a targeted attack, instead of random, to the k-core structure of the network.

4 THEORETICAL RESULTS

The evaluation of our robustness measure relied on simple heuristics. However the question of finding an optimal algorithm still needs to be addressed. In this section, we evaluate the hardness of the TMCV problem. The theoretical results show that there is no polynomial time algorithm even to achieve a constant factor approximation for the TMCV problem. From a parameterized perspective, we show that there is no fixed parameter tractable (FPT) algorithm for a few natural parameters such as the degeneracy, budget and the maximum degree of a node. The TMCV problem is either para-NP-hard or $W[2]$-hard for these parameters. However, for the size of the candidate set, there exists a FPT algorithm. We summarize our main results in Table 3.

Figure 6: Correlation (coefficient=0.2366,p-value=0.06) between evolution of species (x) and our resilience (y) measure based on k-cores. Our approach shows a significantly stronger correlation between the measures, which is an evidence that evolution induces PPI networks with a more resilient k-core structure.

Cond./Param.	Results
b	$W[2]$-hard (Theorem 1)
D	para-NP-hard (Corollary 2)
Δ	para-NP-hard (Theorem 3)
$|\Gamma|$	FPT (Observation 3)
$D(G) = 1$	Poly (Theorem 5)
$D(G) \geqslant 3$	NP-hard to approximate (Thm. 4)

Table 3: Summary of our hardness and parameterized complexity results for the TCMV problem. We denote the budget by b, the degeneracy (maximum coreness over all vertices) of the graph by $D(G)$ or D, the maximum degree of any vertex by Δ, and the candidate set by Γ.

4.1 Parameterized Complexity Results

Our first result shows that the TMCV problem is $W[2]$-hard parameterized by b. The proof involves an fpt-reduction from the well-known $W[2]$-hard SET COVER problem parameterized by the size of set cover [2]. The SET COVER problem is defined as follows.

DEFINITION 5 (SET COVER). *Given an universe $\mathcal{U}$, a collection $\mathcal{S}$ of subsets of $\mathcal{U}$, and a positive integer r, compute if there exists a subcollection $\mathcal{W} \subseteq \mathcal{S}$ such that (i) $|\mathcal{W}| \leqslant r$ and (ii) $\cup_{A \in \mathcal{W}} A = \mathcal{U}$.*

THEOREM 1. *The TMCV problem is $W[2]$-hard parameterized by b for $k \geqslant 3$.*

PROOF. Let $(\mathcal{U} = \{u_1, u_2, ..., u_n\}, \mathcal{S} = \{S_1, S_2, ..., S_m\}, r)$ be an instance of the SET COVER problem. We define a corresponding TMCV problem instance via constructing a graph $\mathcal{G}$ as follows.

For each $S_i \in \mathcal{S}$ we create a clique of four vertices $(P_{i,1}, \cdots, P_{i,4})$. For each $u_j \in \mathcal{U}$, we create a cycle of m vertices $Q_{j,1}, Q_{j,2}, \cdots, Q_{j,m}$ with edges $(Q_{j,1}, Q_{j,2}), \cdots, (Q_{j,m-1}, Q_{j,m}), (Q_{j,m}, Q_{j,1})$. We also create a clique of four vertices $(R_{j,1}, \cdots, R_{j,4})$ for each $u_j \in \mathcal{U}$. Furthermore, edge $(P_{i,1}, Q_{j,i})$ will be added to $\mathcal{E}[\mathcal{G}]$ if $u_j \in S_i$. Additionally, if $u_j \notin S_i$, edge $(Q_{j,i}, R_{j,1})$ will be added to $\mathcal{E}$. The
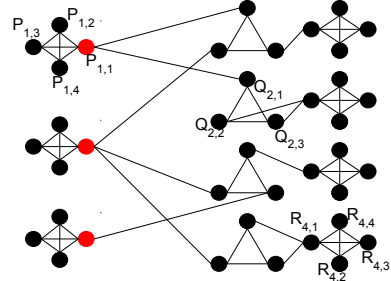
Figure 7: An example of the construction for parameterized hardness of TMCV: reduction from Set Cover where $U = \{u_1, u_2, u_3, u_4\}, S = \{S_1, S_2, S_3\}, S_1 = \{u_1, u_2\}, S_2 = \{u_1, u_3, u_4\}, S_3 = \{u_3\}$.

candidate set, $\Gamma_v = \{P_{i,1} | \forall i = 1, 2, ..., m\}$. Fig. 7 illustrates our construction for sets $S_1 = \{u_1, u_2\}, S_2 = \{u_1, u_3, u_4\}$ and $S_3 = \{u_3\}$. We formally describe our TMCV instance as follows.

$$V[\mathcal{G}] = \{P_{i,t} : S_i \in \mathcal{S}, t \in [4]\} \cup V_1, \text{where}$$
$$V_1 = \{Q_{j,i} : u_j \in \mathcal{U}, S_i \in \mathcal{S}\} \cup \{R_{j,t} : u_j \in \mathcal{U}, t \in [4]\}$$
$$\mathcal{E}[\mathcal{G}] = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5, \text{where}$$
$$E_1 = \{(P_{i,s}, P_{i,t}) : i \in [m]; s, t \in [4], s \neq t\}$$
$$E_2 = \{(R_{j,s}, R_{j,t}) : j \in [n]; s, t \in [4], s \neq t\}$$
$$E_3 = \{(Q_{j,i}, Q_{j,i+1}) : i \in [m-1], j \in [n]\}$$
$$E_4 = \{(P_{i,1}, Q_{j,i}) | u_j \in S_i, i \in [m], j \in [n]\}$$
$$E_5 = \{(Q_{j,i}, R_{j,1}) | u_j \notin S_i, i \in [m], j \in [n]\}$$
$$\Gamma = \{P_{i,1} | i \in [m]\}$$
$$b = r$$

For any integer z, we denote the set $\{1, 2, ..., z\}$ by $[z]$. We now claim that the set cover instance is a YES instance if and only if there exists a subset $B \subseteq \Gamma$ with $|B| \leqslant b$ and $|\{v \in V \setminus B : C(v, \mathcal{G}) > C(v, \mathcal{G} \setminus B)\}| \geqslant 4b + mn$.

In one direction, let us assume that the SET COVER instance is a YES instance. By renaming, let $S' = \{S_1, \ldots, S_r\}$ form a set cover of the instance. We delete the nodes in the set $V' = \{P_{i,1} | i \in [r]\}$ in the graph $\mathcal{G}$. We claim that, by deleting the nodes in the set V', every node in $\{P_{i,t} | S_i \in S', t \in [4]\} \cup \{Q_{j,i} | j \in [n], i \in [m]\}$, i.e. $4b + mn$ nodes will go to the 2-core. We first observe that deletion of $P_{i,1}$ will move the other three nodes $(P_{i,2}, P_{i,3}, P_{i,4})$ to the 2-core. Also, for any $j \in [n]$, if any connection $(Q_{j,i}, P_{i,1})$ gets deleted because of deletion of $P_{i,1}$, all the m nodes in the set $\{Q_{j,t} | t \in [m]\}$ will go to 2-core. Since S' forms a set cover for $\mathcal{U}$, at least one connection $(Q_{j,l}, P_{l,1})$ where $j \in [n]$ and some $P_{l,1} \in V'$ will get removed. Thus a total of mn nodes will go to the 2-core. Hence, the TMCV instance is a YES instance.

For the other direction, we assume that there exists a subset $V' = \{P_{i,1} | i \in [b]\}$ (by renaming) of size b of the set Γ such that deletion of V' would make at least $4b + nm$ nodes fall from the 3-core. We claim that $S' = \{S_i : i \in [b]\}$ ($b = r$) forms a set cover for $\mathcal{U}$. Suppose it is not, then at most $n - 1$ connections among $(Q_{j,l}, P_{l,1})$ for $j \in [n]$ and some $P_{l,1} \in V'$ will get deleted. Thus, at most $(n-1)m$ nodes will go into the 2-core making it a total of $b + (n-1)m$ nodes falling from 3-core. Hence, this is a contradiction and S' is a set cover. □

From the proof of Theorem 1, we obtain the following corollary. This follows from the observation that the SET COVER problem remains NP-complete even if the size of each subset is 3 and every element of the universe belongs to exactly 2 subsets [9].

COROLLARY 2. *The TMCV problem is para-NP-hard parameterized by D.*

PROOF. From the above construction, if we begin with an instance of SET COVER where the size of each subset is 3 and every element of the universe belongs to exactly 2 subsets, then we observe that the TMCV problem is also NP-hard when $D = 3$. □

We next consider maximum degree of the graph as parameter. By reducing from the Exact Cover problem [4], we show next the TMCV problem is para-NP-hard parameterized by the maximum degree of the input graph. The Exact Cover problem is the SET COVER problem where every set contains exactly 3 elements from the universe. However, we use a special case of the Exact Cover problem where the elements are exactly in two subsets. This special case is known to be NP-complete [4].

THEOREM 3. *The TMCV problem is para-NP-hard parameterized by the maximum degree (Δ) in the graph.*

PROOF. To prove our claim, we show a parameterized reduction from the special case of the Exact Cover problem which is known to be NP-hard. The problem is a SET COVER problem where each subset has exactly 3 elements and each element belongs to exactly two subsets. Let $(\mathcal{U} = \{u_1, u_2, ..., u_n\}, \mathcal{S} = \{S_1, S_2, ..., S_m\}, r)$ be an instance of the mentioned problem. We define a corresponding TMCV problem instance via constructing a graph $\mathcal{G}$ as follows.

We follow a similar reduction as in Theorem 1. For each $S_i \in \mathcal{S}$ we create a clique of four vertices $(P_{i,1}, \cdots, P_{i,4})$ for each $S_i \in \mathcal{S}$. For each $u_j \in \mathcal{U}$, we create two nodes $Q_{j,1}$ and $Q_{j,2}$ (we know each element belongs to exactly two subsets, one node is corresponding to the first subset and the second one is for the second in an arbitrary order) and an edge $(Q_{j,1}, Q_{j,2})$ between them. We also create a clique of four vertices $(R_{j,1}, \cdots, R_{j,4})$ for each $u_j \in \mathcal{U}$. Furthermore, edge $(P_{i,1}, Q_{j,i})$ will be added to $\mathcal{E}[\mathcal{G}]$ if $u_j \in S_i$. Additionally, two edges $(Q_{j,1}, R_{j,1})$ and $(Q_{j,2}, R_{j,1})$ will be added to $\mathcal{E}$. Clearly the reduction takes polynomial time. The candidate set, $\Gamma = \{P_{i,1} | \forall i = 1, 2, ..., m\}$. Note that the maximum degree in the graph is constant, i.e. $\Delta = 6$.

Initially in $\mathcal{G}$, all vertices are in the 3-core. We claim that a set $S' \subset S$, with $|S'| \leqslant r$, is a cover iff $f(B) = 4b + 2n$ where $B = \{P_{i,1} | S_i \in S'\}$.

Let us assume that the Exact Cover instance is a YES instance and, by renaming, the collection $S' = \{S_1, \ldots, S_r\}$ forms a valid set cover of the instance. We delete the nodes in the set $V' = \{P_{i,1} | i \in [r]\}$ in the graph $\mathcal{G}$. We claim that by deleting the nodes in the set V', every node in $\{P_{i,t} | S_i \in S', t \in [4]\} \cup \{Q_{j,i} | j \in [n], i \in \{1, 2\}\}$, i.e. $4b + 2n$ nodes will go in 2-core. We first observe that deletion of $P_{i,1}$ will make the other three nodes $P_{i,2}$ in 2-core. Deletion of b such nodes will lead $4b$ nodes falling into 2-core. Also, for any $j \in [n]$, if any connection $(Q_{j,i}, P_{i,1}$ gets deleted because of deletion of $P_{i,1}$, both nodes in the set $\{Q_{j,t} | t \in [2]\}$ will go to 2-core. Since S' forms a set cover for $\mathcal{U}$, at least one connection $(Q_{j,l}, P_{l,1})$ for all $j \in [n]$

and some $P_{l,1} \in V'$ will get removed. Thus a total of $2m$ nodes will go in 2-core. Hence, the TMCV instance is a YES instance.

For the other direction, we assume that there exists a subset $V' = \{P_{i,1} | i \in [b]\}$ (by renaming) of nodes of size b of the set Γ such that in the graph deletion of which would make at least $4b + 2n$ nodes fall from the 3-core. We claim that $S' = \{S_i : i \in [b]\}$ ($b = r$) forms a set cover for $\mathcal{U}$. Suppose it is not, then at most $n - 1$ connections among $(Q_{j,l}, P_{l,1})$ for $j \in [n]$ and some $P_{l,1} \in V'$ will get deleted. Thus, at most $2(n - 1)$ nodes will go into the 2-core making it a total of $4b + 2(n - 1)$ nodes falling from 3-core. Hence, this is a contradiction and S' is a set cover. So, the TMCV problem is NP-hard when the maximum degree is constant ($\Delta = 6$). Thus, the TMCV problem is para-NP-hard parameterized by the maximum degree (Δ) in the graph. □

We conclude this section with the observation that the TMCV problem is fixed parameter tractable parameterized by $|\Gamma|$. The algorithm simply tries all possible subsets of $|\Gamma|$ of size at most b.

Observation 3. *There is an algorithm for the TMCV problem running in time $O(2^{|\Gamma|} poly(n))$. In particular, TMCV is fixed parameter tractable parameterized by $|\Gamma|$.*

4.2 Inapproximability and Algorithm for $D(G) = 1$

In this section, we discuss the traditional hardness spectrum of the TMCV problem. We show a strong inapproximability result—it is NP-hard to achieve any $m^{-l_1} n^{-l_2}$-factor approximation even when $D(G) \geqslant 3$ for any constants $l_1 > 1$ and $l_2 > 1$.

THEOREM 4. *The TCMV problem is NP-hard to approximate within any $m^{-l_1} n^{-l_2}$-factor approximation even when $D(G) \geqslant 3$ for any constants $l_1 > 1$ and $l_2 > 1$.*

PROOF. To prove our claim, first let us consider a reduction from the SET COVER problem. Let $(\mathcal{U} = \{u_1, u_2, ..., u_n\}, \mathcal{S} = \{S_1, S_2, ..., S_m\}, r)$ be an instance of the SET COVER problem. We define a corresponding TMCV problem instance via constructing a graph $\mathcal{G}$ as follows.

We create a clique of four vertices $(P_{i,1}, \cdots, P_{i,4})$ for each $S_i \in \mathcal{S}$. For each $u_j \in \mathcal{U}$, we create a cycle of m vertices $Q_{j,1}, Q_{j,2}, \cdots, Q_{j,m}$ with edges $(Q_{j,1}, Q_{j,2}), \cdots, (Q_{j,m-1}, Q_{j,m}), (Q_{j,m}, Q_{j,1})$.

We also create a vertex R along with a connected sub-graph on a set $\mathcal{T}$ of $10(m^{l_1} n^{l_2})^2$ vertices with degree exactly 3 (for example, we can take a perfect matching between two cycles on $|\mathcal{T}|/2$ vertices each). The node R is attached with exactly two vertices in $\mathcal{T}$. Furthermore, edge $(P_{i,1}, Q_{j,i})$ will be added to $\mathcal{E}[\mathcal{G}]$ if $u_j \in S_i$. Additionally, if $u_j \notin S_i$, edge $(Q_{j,i}, R)$ will be added to $\mathcal{E}$. Clearly the reduction takes polynomial time. The candidate set, $\Gamma_v = \{P_{i,1} | \forall i = 1, 2, ..., m\}$. Fig. 8 illustrates our construction for sets $S_1 = \{u_1, u_2\}, S_2 = \{u_1, u_2, u_4\}$ and $S_3 = \{u_3\}$.

Initially in $\mathcal{G}$, all vertices are in the 3-core. We claim that a set $S' \subset S$, with $|S'| \leqslant r$, is a cover iff $f(B) = 3r + mn + 1 + |\mathcal{T}|$ where $B = \{P_{i,1} | S_i \in S'\}$.

Let us assume that the SET COVER instance is a YES instance and, by renaming, $S' = \{S_1, \ldots, S_r\}$ forms a valid set cover of the instance. We delete the nodes in the set $V' = \{P_{i,1} | i \in [r]\}$ in the graph $\mathcal{G}$. We claim that by deleting the nodes in the set V', every
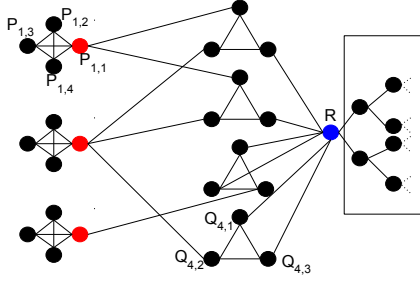
Figure 8: Example of construction to prove inapproxima-bility of TMCV: reduction from Set Cover where $U = \{u_1, u_2, u_3, u_4\}, S = \{S_1, S_2, S_3\}, S_1 = \{u_1, u_2\}, S_2 = \{u_1, u_2, u_4\}, S_3 = \{u_3\}$. The number of nodes in the rectangular box is $10(m^{l_1}n^{l_2})^2$ and the nodes have exactly degree 3.

node in $\{P_{i,t} | S_i \in S', t \in [4]\} \cup \{Q_{j,i} | j \in [n], i \in [m]\}$, i.e. $4b + mn$ nodes will go to the 2-core. We first observe that deletion of $P_{i,1}$ will make the other three nodes $P_{i,2}$ to 2-core. Also, for any $j \in [n]$, if any connection $(Q_{j,i}, P_{i,1})$ gets deleted because of deletion of $P_{i,1}$, all the m nodes in the set $\{Q_{j,t} | t \in [m]\}$ will go to 2-core. Since S' forms a set cover for $\mathcal{U}$, at least one connection $(Q_{j,l}, P_{l,1})$ for all $j \in [n]$ and some $P_{l,1} \in V'$ will get removed. Note that if all the nodes $Q_{j,i}; \forall j \in [n]$ and $\forall i \in [m]$ go to 2-core, the node R will go to 2-core and thus all the nodes in $\mathcal{T}$ will follow the same. Thus a total of $3r + mn + 1 + |\mathcal{T}|$ nodes will go to 2-core.

If there is no set cover of size r, then at most $n - 1$ connections among $(Q_{j,l}, P_{l,1})$ for $j \in [n]$ and some $P_{l,1} \in V'$ will get deleted. Thus, at most $(n-1)m$ nodes will go into the 2-core making it a total of $3r + (n - 1)m$ nodes falling from the 3-core. Note that the node R will be still in the 3-core and thus the nodes in set $\mathcal{T}$ will remain unaffected in the 3-core. Hence, a total of $3r + (n - 1)m$ nodes will go to 2-core. So, the multiplicative difference of fs corresponding to the yes instance $(3r + mn + 1 + |\mathcal{T}|)$ and no instance $(3r + (n-1)m)$ of the Set Cover problem is less than $m^{-l_1}n^{-l_2}$ and thus TMCV cannot be approximated within $m^{l_1}n^{l_2}$ factor unless $P \neq NP$. □

We show next that the TCMV problem is polynomial time solv-able if the degeneracy of the input graph is 1.

Theorem 5. *The TCMV problem is polynomial time solvable if $D(v, G) = 1$.*

Proof. Let $(G = (V, E), \Gamma \subseteq V, b)$ be any instance of TMCV such that $D(G) = 1$. Since $D(G) = 1$, it follows that G is a forest. Let $G = T_1 \cup \cdots \cup T_k$ for some integer k where T_i is a tree for every $i \in [k]$. We first describe a dynamic programming based algorithm for the TCMV problem that works for trees.

Let T be the input tree and $\Gamma_T \subseteq T$ the subset of vertices which can be deleted. We make the tree rooted at any node $r \in T$. At every node $x \in T$ and every integer $\ell \in [b] \cup \{0\}$, we store the following.

$A[x, \ell]$ = maximum number of isolated vertices in $\Gamma_T \cap T_x$
    by deleting at most ℓ vertices from $\Gamma_T \cap T_x$
    subject to the condition that x becomes isolated

$B[x, \ell]$ = maximum number of isolated vertices in $\Gamma_T \cap T_x$
    by deleting at most ℓ vertices from $\Gamma_T \cap T_x$ subject to
    the condition that x is neither isolated nor deleted

$C[x, \ell]$ = maximum number of isolated vertices in $\Gamma_T \cap T_x$
    by deleting at most ℓ vertices from $\Gamma_T \cap T_x$ subject to
    the condition that x is deleted

$D[x, \ell]$ = maximum number of isolated vertices in $\Gamma_T \cap T_x$
    by deleting at most ℓ vertices from $\Gamma_T \cap T_x$

From the definitions of $A[x, \ell], B[x, \ell], C[x, \ell]$, and $D[x, \ell]$, the following recurrences follow. Let the children and grandchildren of x be respectively $y_1, \ldots, y_i$ and $z_1, \ldots, z_j$ (j could be 0).

$$A[x, \ell] = \mathbb{1}(\ell \geqslant i) + \max\{D[z_1, \ell_1] + \cdots + D[z_j, \ell_j] :$$
$$\ell_1 + \cdots + \ell_j \leqslant \ell - i\}$$

$$B[x, \ell] = \max \bigcup_{\lambda=1}^{i} \Big\{ \max\{A[y_1, \ell_1] - 1, B[y_1, \ell_1], C[y_1, \ell_1]\} +$$
$$\ldots \max\{A[y_{\lambda-1}, \ell_{\lambda-1}] - 1, B[y_{\lambda-1}, \ell_{\lambda-1}], C[y_{\lambda-1}, \ell_{\lambda-1}]\}$$
$$+ \max\{A[y_\lambda, \ell_\lambda] - 1, B[y_\lambda, \ell_\lambda]\} +$$
$$\max\{A[y_{\lambda+1}, \ell_{\lambda+1}] - 1, B[y_{\lambda+1}, \ell_{\lambda+1}], C[y_{\lambda+1}, \ell_{\lambda+1}]\}$$
$$+ \ldots + \max\{A[y_i, \ell_i] - 1, B[y_i, \ell_i], C[y_i, \ell_i]\} :$$
$$\ell_1 + \cdots + \ell_i \leqslant \ell\Big\}$$

$$C[x, \ell] = \max\{D[y_1, \ell_1] + \cdots + D[y_i, \ell_i] :$$
$$\ell_1 + \cdots + \ell_i \leqslant \ell - 1\}$$

$$D[x, \ell] = \max\{A[x, \ell], B[x, \ell], C[x, \ell]\}$$

We make the convention that the maximum over an empty set is 0. For every leaf node x, we initialize $A[x, \ell], B[x, \ell], C[x, \ell]$, and $D[x, \ell]$ as follows.

$$A[x, \ell] = 1, \qquad\qquad \ell \geqslant 0$$
$$B[x, \ell] = -1, \qquad\qquad \ell \geqslant 0$$
$$C[x, \ell] = \begin{cases} -1, & \ell = 0 \\ 0, & \ell > 0 \end{cases}$$
$$D[x, \ell] = 1, \qquad\qquad \ell \geqslant 0$$

We observe that, given the tables at every descendant vertex of x, $A[x, \ell]$ can be computed by a standard dynamic programming based algorithm for the knapsack problem in time $O(j)$ [27]. Similarly, $B[x, \ell]$ and $C[x\ell]$ can be computed respectively in $O(i^2)$ and $O(i)$ time. Hence the running time of our algorithm is $O(n^2 b) = O(n^3)$. □

5 CONCLUSION

In this work we have introduced a novel network robustness mea-sure based on k-cores. More specifically, we have addressed the algorithmic problem that aims to maximize the number of nodes falling from their initial cores upon a given budget number of node deletions. We have characterized the hardness of the problem in both traditional and parameterized frameworks. Our problem is NP-hard to approximate by any constant, is $W[2]$-hard parameterized by the budget and is para-NP-hard for several other parameters such as degeneracy and maximum degree of the graph. We have also proposed a few heuristics and demonstrated their performance on several datasets. When applied to PPI networks, our approach

has allowed us to correlate network resilience and the evolution of species. In the future, we will apply our resilience metric to the entire PPI database from [32]. Moreover, we want to explore if there exist approximation algorithms for our problem in some relevant constrained cases beyond the ones considered here.

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