On Parameterized Complexity of Binary Networked Public Goods Game

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Abstract

In the Binary Networked Public Goods game, every player needs to decide if she participates in a public project whose utility is shared equally by the community. We study the problem of computing if there exists a pure strategy Nash equilibrium (PSNE) in such games. The problem is already known to be NP-complete. We provide fine-grained analysis of this problem under the lens of parameterized complexity theory. We consider various natural graph parameters and show either W[1]-hardness or exhibit an FPT algorithm. We finally exhibit some special graph classes, for example path, cycle, bi-clique, complete graph, etc., which always have a PSNE if the utility function of the players are fully homogeneous.

1 Introduction

In a public goods game, players need to decide if they contribute in a public project and, if yes, then how much. The outcome of such public projects is typically shared equally by all the players. Public goods games are effective in modeling tension between individual cost vs community well beings [Kollock(1998), Santos et al.(2008)]. Among some well-explored variants of the above game, the *networked* public goods game is prominent where we assume a network structure on the players and the utility of individual players depend on the action of their neighbors only [Bramoullé, Kranton et al.(2007)].

An important class of networked public goods game is the *binary* networked public goods (BNPG for short) game where players only need to decide if they participate in the public project or not [Galeotti et al.(2010)]. Although seems restricted, such games are still powerful enough to model various important real world application scenarios. For some motivating examples, let us think of an air-borne virus pandemic like the Covid-19 pandemic where individuals need to decide whether to wear a ask or not.

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While individuals may feel uncomfortable while wearing a mask, the benefits of herd immunity, if achieved by a large fraction of population wearing a mask, will be shared by the entire community. Indeed, there are reports that a considerable fraction of population refuse to wear a mask during Covid-19 pandemic [Buchwald(2020), Wong(2020)]. Another important application is whether to report a crime or not. While individuals who report crimes may be at risk, the benefit of having lower crimes will be enjoyed by the entire community. The general observation at many places is that crimes are often substantially under-reported [Morgan and Truman(2017)].

Computing a pure strategy Nash equilibrium (PSNE) in any game is a fundamental question. Studying equilibriums guides us to predict how players will act in a strategic setting. For BNPG games, it is already known that computing if there exists a PSNE is NP-complete [Yu et al.(2020)]. We explore BNPG games under both general utility functions and *fully homogeneous* utility functions where all the players have the same cost and the same externality.

1.1 Contribution

We study parameterized complexity of the problem of computing if there exists a PSNE (we call this problem Exists-PSNE) in BNPG games.

- ▷ The first parameter that we consider is the maximum degree of the graph. Since most applications of BNPG games involve human being as players, maximum degree (the number of neighbors) is often some small number which makes it suitable to consider it as a parameter. However, we show that the Exists-PSNE problem is para-NP-hard parameterized by maximum degree even for fully homogeneous BNPG games [Theorems 1 and 5]. Graphs involving human beings often have a small diameter. The para-NP-hardness with respect to the diameter as the parameter follows easily from the NP-completeness proof of [Yu et al.(2020)] [Observation 2].

- ▷ Treewidth has often turned out to be a useful parameter to have a fixed-parameter-tractable (FPT) algorithm. We show that the Exists-PSNE problem is W[1]-hard parameterized by treewidth even for fully homogeneous BNPG games [Theorems 3 and 4]. We complement this hardness result by designing an FPT algorithm for our problem for a class of BNPG games where players are never indifferent between participation and non-participation (we call such games *strict*) [Theorem 6].

- ▷ We know that the Exists-PSNE problem is polynomial time solvable for trees and cliques. We show that the Exists-PSNE problem is FPT parameterized by distance from tree [Theorem 7], which is known as circuit rank, and from complete graphs [Theorem 8].

▷ For BNPG games, the number of participating players and the number of non-participating players are also natural parameters. We prove W[2]-hardness for both the parameters [Theorems 7 and 8].

Finally we consider fully homogeneous BNPG games on special graph classes. We show that a PSNE always exists for a path [Theorem 9], complete graph, cycle, and bi-clique [Theorem 10].

1.2 Related Work

The work of [Yu et al.(2020)] is the immediate predecessor of our work where the authors initiate the algorithmic question of computing a PSNE in BNPG games. Our work broadly belongs to the field of graphical games where there is a graph structure on the players and a player's utility depends only on the actions of her neighbors [Kearns(2007)]. A central question in graphical games is to find complexity of the problem of computing an equilibrium [Elkind et al.(2006), Daskalakis et al.(2009), Gottlob et al.(2005)]. Network public goods games are special case of graphical games where the utility of players depends only on the sum of the efforts put in by neighbors and the cost of her action. Many different models of the network public goods game have been explored which are fine-tuned to different applications. Important examples of such applications include economics, research collaboration, social influence, etc. [Burt(1987), Valente(1995), Conley and Udry(2010), Valente(2005)]. The BNPG model is closely related to that proposed in [Bramoullé, Kranton et al.(2007)]. There are however two qualitative distinctions (a)[Bramoullé, Kranton et al.(2007)] focuses on the continuous investment model whereas BNPG model focuses on binary investment decisions and (b)[Bramoullé, Kranton et al.(2007)] assume homogeneous concave utilities whereas BNPG model considers a more general setting. Supermodular network games [Manshadi and Johari(2009)] and best-shot games (which is actually a special case of BNPG game) [Dall'Asta et al.(2011)], etc. [Galeotti et al.(2010), Komarovsky et al.(2015), Levit et al.(2018)] are some of the variations which are to graphical games. In the model of Supermodular network games, each agent's payoff is a function of the aggregate action of its neighbors and it exhibits strategic complementarity. An important example of supermodular games on graphs are technology adoption games. These games have been studied in the social network literature [Kleinberg(2007), Immorlica et al.(2007), Morris(2000)].

2 Preliminaries

For a set $\mathcal{X}$, we denote its power set by $2^{\mathcal{X}}$. We denote the set $\{1, \ldots, n\}$ by $[n]$. For 2 sets $\mathcal{X}$ and $\mathcal{Y}$, we denote the set of functions from $\mathcal{X}$ to $\mathcal{Y}$ by

$\mathcal{Y}^{\mathcal{X}}$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph with n vertices. An edge between $u, v \in \mathcal{V}$ is represented by $\{u, v\}$. In a graph $\mathcal{G}$, we denote the degree of any vertex v by $d(v)$. For a subset $\mathcal{U} \subseteq \mathcal{V}$ of vertices (respectively a subset $\mathcal{F} \subseteq \mathcal{E}$ of edges), we denote the subgraph induced by $\mathcal{U}$ (respectively $\mathcal{F}$) by $\mathcal{G}[\mathcal{U}]$ (respectively $\mathcal{G}[\mathcal{F}]$). A Binary Networked Public Goods (BNPG for short) game can be defined on $\mathcal{G}$ as follows. The set of players is $\mathcal{V}$. The strategy set of every player is $\{0, 1\}$. For a strategy profile $(x_v)_{v\in\mathcal{V}} \in \{0, 1\}^{|\mathcal{V}|}$, the utility $U_w((x_v)_{v\in\mathcal{V}})$ of player (without abusing the notation much) $w \in \mathcal{V}$ is defined as follows. We denote the number of neighbors of w in $\mathcal{G}$ who plays 1 in $(x_v)_{v\in\mathcal{V}}$ by n_w; that is, $n_w = |\{u \in \mathcal{V} : \{u, w\} \in \mathcal{E}, x_u = 1\}|$.

$$U_w((x_v)_{v\in\mathcal{V}}) = U_w(x_w, n_w) = g_w(x_w + n_w) - c_w \cdot x_w$$

where $g_w : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{R}^+$ is a non-decreasing function in x and $c_w \in \mathbb{R}^+$ is a constant. We denote a BNPG game by $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), (g_v)_{v\in\mathcal{V}}, (c_v)_{v\in\mathcal{V}})$. For any number $n \in \mathbb{N} \cup \{0\}$ and function $g : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{R}^+$, we define $\Delta g(n) = g(n + 1) - g(n)$. In general, every player $w \in \mathcal{V}$ has a different mapping function $g_w(.)$ and hence we call this version of the game a *heterogeneous* BNPG game. If not mentioned otherwise, by BNPG game, we refer to a heterogeneous BNPG game. In this paper, we also study the following three special cases — (i) *homogeneous:* $g_w = g$ for all $w \in \mathcal{V}$ (ii) *fully homogeneous:* homogeneous and $c_w = c$ for all $w \in \mathcal{V}$ and (iii) strict: for every player $w \in \mathcal{V}$, we have $U_w(x_w = 0, x_{-w}) \neq U_w(x_w = 1, x_{-w})$ for every strategy profile x_{-w} of other players.

Observation 1. *A BNPG game is strict if and only if*

$$\Delta g_w(k) \neq c_w, \ \forall w \in \mathcal{V}, \forall k \in \{0, 1, \ldots, d(w)\}$$

A strategy profile $(x_v)_{v\in\mathcal{V}}$ is called a *pure-strategy Nash Equilibrium (PSNE)* of a BNPG game if we have

$$U_v(x_v, x_{-v}) \geqslant U_v(x'_v, x_{-v}) \ \forall x'_v \in \{0, 1\}, \forall v \in \mathcal{V}$$

For a player w in a BNPG game $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), (g_v)_{v\in\mathcal{V}}, (c_v)_{v\in\mathcal{V}})$, we define her best response function $\beta_w : \{0, 1, \ldots, n-1\} \longrightarrow 2^{\{0,1\}} \setminus \{\emptyset\}$ as follows. For every $k \in \{0, 1, \ldots, n-1\}$ and $a \in \{0, 1\}$, we have $a \in \beta_w(k)$ if and only if, for every strategy profile x_{-w} of players other than w where exactly k players in the neighborhood of w play 1, we have $U_w(x_w = a, x_{-w}) \geqslant U_w(x_w = a', x_{-w})$ for all $a' \in \{0, 1\}$. The following lemma proves that, for every best response functions β_w, there is a function $g_w : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{R}^+$ and constant c_w such that β_w is the best response function; moreover such g_w and c_w can be computed in polynomial time.

Lemma 1. *Let $\beta : \{0, 1, \ldots, n-1\} \longrightarrow 2^{\{0,1\}} \setminus \{\emptyset\}$ any function. Then we can compute in polynomial (in n) time a function $g : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{R}^+$ and constant c such that β is the corresponding best response function.*

We call a function $f : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{R}^+$ *sub-additive* if $f(x+y) \leqslant f(x) + f(y)$ for every $x, y \in \mathbb{N} \cup \{0\}$ and *additive* if $f(x+y) = f(x) + f(y)$. We call a BNPG game $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), (g_v)_{v\in\mathcal{V}}, (c_v)_{v\in\mathcal{V}})$ sub-additive (respectively additive) if g_v is sub-additive (respectively additive) for every $v \in \mathcal{V}$. We denote $\mathcal{O}(f(n)\text{poly}(n))$ by $\mathcal{O}^*(f(n))$.

3 Results

We first study the problem of computing if there exists a PSNE in BNPG games; we call this problem Exists-PSNE.

3.1 Hardness Results

In this subsection, we present our hardness results. The Exists-PSNE problem is already known to be NP-complete [Yu et al.(2020)]. We strengthen this result significantly in Theorem 1. We use the NP-complete problem $(3, \text{B2})$-SAT to prove some of our hardness results [Berman et al.(2004)]. The $(3, \text{B2})$-SAT problem is the 3-SAT problem restricted to formulas in which each clause contains exactly three literals, and each variable occurs exactly twice positively and twice negatively.

Theorem 1. *The* Exists-PSNE *problem is* NP*-complete for sub-additive strict BNPG games even if the underlying graph is* 3*-regular and the number of different utility functions is* 2*. In particular, the* Exists-PSNE *problem parameterized by (maximum degree, number of different utility functions) is para-*NP*-hard even for sub-additive strict BNPG games.*

Proof. The Exists-PSNE problem clearly belongs to NP. To show its NP-hardness, we reduce from the $(3, \text{B2})$-SAT problem. The high-level idea of our proof is as follows. For every clause in $(3, \text{B2})$-SAT instance, we create a vertex in the Exists-PSNE instance. Also, for every literal we create a vertex in the Exists-PSNE instance. We add the set of edges and define the best-response functions in such a way that all the clause vertices play 1 in any PSNE and a set of literal vertices play 1 in a PSNE if and only if there is a satisfying assignment where the same set of literal vertices is assigned TRUE. We now present our construction formally.

Let $(\mathcal{X} = \{x_i : i \in [n]\}, \mathcal{C} = \{C_j : j \in [m]\})$ be an arbitrary instance of $(3, \text{B2})$-SAT. We define a function $f : \{x_i, \bar{x}_i : i \in [n]\} \longrightarrow \{a_i, \bar{a}_i : i \in [n]\}$ as $f(x_i) = a_i$ and $f(\bar{x}_i) = \bar{a}_i$ for $i \in [n]$ and consider the following instance $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), (g_v)_{v\in\mathcal{V}}, (c_v)_{v\in\mathcal{V}})$ of Exists-PSNE.

$$\begin{aligned}\mathcal{V} &= \{a_i, \bar{a}_i : i \in [n]\} \cup \{y_j : j \in [m]\}\\ \mathcal{E} &= \{\{y_j, f(l_1^j)\}, \{y_j, f(l_2^j)\}, \{y_j, f(l_3^j)\} :\\ C_j &= (l_1^j \vee l_2^j \vee l_3^j), j \in [m]\} \cup \{\{a_i, \bar{a}_i\} : i \in [n]\}\end{aligned}$$

We observe that the degree of every vertex in $\mathcal{G}$ is 3. We now define $(g_v)_{v\in\mathcal{V}}$ and $(c_v)_{v\in\mathcal{V}}$.

$$\forall j \in [m],\ c_{y_j} = 4, g_{y_j}(x) = \begin{cases} 1000 & \text{if } x = 0\\ 1003 & \text{if } x = 1\\ 1008 & \text{if } x = 2\\ 1013 & \text{if } x = 3\\ 1018 & \text{if } x = 4 \end{cases}$$

$$\forall i \in [n],\ c_{a_i} = c_{\bar{a}_i} = 4,$$

$$g_{a_i}(x) = g_{\bar{a}_i}(x) = \begin{cases} 1000 & \text{if } x = 0\\ 1005 & \text{if } x = 1\\ 1010 & \text{if } x = 2\\ 1015 & \text{if } x = 3\\ 1018 & \text{if } x = 4 \end{cases}$$

It follows form the definition that both the above functions are sub-additive. Also one can easily verify that the above functions give the following best-response functions for the players.

$$\forall i \in [n], \beta_{a_i}(k) = \beta_{\bar{a}_i}(k) = \begin{cases} 1 & \text{if } k \leqslant 2\\ 0 & \text{otherwise} \end{cases}$$

$$\forall j \in [m], \beta_{y_j}(k) = \begin{cases} 0 & \text{if } k = 0\\ 1 & \text{otherwise} \end{cases}$$

From the best-response functions, it follows that the game is strict. We now claim that the above BNPG game has a PSNE if and only if the (3, B2)-SAT instance is a YES instance.

For the "if" part, suppose the (3, B2)-SAT instance is a YES instance. Let $g : \{x_i : i \in [n]\} \longrightarrow \{\text{TRUE}, \text{FALSE}\}$ be a satisfying assignment of the (3, B2)-SAT instance. We consider the following strategy profile for the BNPG game.

- ▷ $\forall j \in [m], z_{y_j} = 1$
- ▷ $\forall i \in [n], z_{a_i} = 1$ if and only if $g(x_i) = \text{TRUE}$

$\triangleright$ $\forall i \in [n], z_{\bar{a}_i} = 0$ if and only if $g(x_i) = \text{TRUE}$

We observe that, since g is a satisfying assignment, the player y_j for every $j \in [m]$ has at least one neighbor who plays 1 and thus y_j does not have any incentive to deviate (form playing 1). For $i \in [n]$ such that $g(x_i) = \text{TRUE}$, the player a_i has at least one neighbor, namely $\bar{a}_i$, who plays 0 and thus a_i does not have any incentive to deviate (from playing 1); on the other hand the player $\bar{a}_i$ has all her neighbor playing 1 and thus she is happy to play 0. Similarly, for $i \in [n]$ such that $g(x_i) = \text{FALSE}$, both the players a_i and $\bar{a}_i$ have no incentive to deviate. This proves that the above strategy profile is a PSNE.

For the "only if" part, let $(s(a_i)_{i\in[n]}, s(\bar{a}_i)_{i\in[n]}, s(y_j)_{j\in[m]})$ be a PSNE for the BNPG game. We claim that $s(y_j) = 1$ for every $j \in [m]$. Suppose not, then there exists a $t \in [m]$ such that $s(y_t) = 0$. Let the literals in clause C_t be l_1^t, l_2^t, l_3^t. Then $s(f(l_i^t)) = 0, \forall i \in [3]$ otherwise the player y_t will deviate form 0 and play 1. But then the player $f(l_1^t)$ will deviate to 1 as y_t plays 0 which is a contradiction. We now claim that we have $s(a_i) \neq s(\bar{a}_i)$ for every $i \in [n]$. Suppose not, then there exists an $\lambda \in [n]$ such that $s(a_\lambda) = s(\bar{a}_\lambda)$. If $s(a_\lambda) = s(\bar{a}_\lambda) = 1$, then both the players a_λ and $\bar{a}_\lambda$ has incentive to deviate to 0. On the other hand, if $s(a_\lambda) = s(\bar{a}_\lambda) = 0$, then both the players a_λ and $\bar{a}_\lambda$ has incentive to deviate to 1. This proves the claim. We now consider the assignment $g : \{x_i : i \in [n]\} \longrightarrow \{\text{TRUE}, \text{FALSE}\}$ defined as $g(x_i) = \text{TRUE}$ if and only if $s(a_i) = 1$ for every $i \in [n]$. We claim that g is a satisfying assignment for the $(3, \text{B2})$-SAT formula. Suppose not, then g does not satisfy a clause, say $C_\gamma, \gamma \in [m]$. Then the player y_γ has incentive to deviate to 0 as none of its neighbors play 1 which is a contradiction. $\square$

We next consider the diameter (Δ) of the graph as our parameter. Observation 2 follows from the fact that the reduced instance in the NP-completeness proof of EXISTS-PSNE for BNPG games in [Yu et al.(2020)] has diameter 2.

Observation 2. *The* EXISTS-PSNE *problem for BNPG games is* NP*-complete even for graphs of diameter at most* 2. *In particular, the* EXISTS-PSNE *problem for BNPG games is para-*NP*-hard parameterized by diameter.*

We next consider the problem of deciding the existence of a PSNE in a BNPG game where the number of players playing 0 (respectively 1) in the PSNE is at most k_0 (respectively k_1). Obviously there is a brute force XP algorithm for this problem which runs in time $\mathcal{O}^*\left(n^{k_0}\right)$ (respectively $\mathcal{O}^*\left(n^{k_1}\right)$). We show that the problem of deciding the existence of a PSNE in a BNPG game where the number of players playing 0 (respectively 1) in the PSNE is at most k_0 (respectively k_1) is W[2]-hard parameterized by k_0(respectively k_1). For this, we reduce from the DOMINATING SET problem parameterized by the size of dominating set that we are looking for which is known to be $\mathsf{W}[2]$-hard.

Theorem 2. *The problem of computing if there exists a PSNE where at most k_0 (respectively k_1) players play 0 (respectively 1) is $\mathsf{W}[2]$-hard parameterized by k_0 (respectively k_1) even for fully homogeneous BNPG games.*

Proof. We first prove the result for the parameter k_1. Let $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), k)$ an arbitrary instance of Dominating Set. We consider the following BNPG game on the same graph $\mathcal{G}$. We now describe the best-response functions $\beta_v(\cdot)$ for $v \in \mathcal{V}$.

$$\beta_v(k') = \begin{cases} 1 & \text{if } k' = 0 \\ \{0, 1\} & \text{otherwise} \end{cases}$$

We observe that, since every player has the same best-response function, the BNPG game is fully homogeneous. We claim that the above BNPG game has a PSNE having at most k players playing 1 if and only if the Dominating Set instance is a yes instance.

For the "if" part, suppose the Dominating Set instance is a yes instance and $\mathcal{W} \subseteq \mathcal{V}$ be a dominating set for $\mathcal{G}$ of size at most k. We claim that the strategy profile $\bar{x} = ((x_v = 1)_{v \in \mathcal{W}}, (x_v = 0)_{v \in \mathcal{V} \setminus \mathcal{W}})$ is a PSNE for the BNPG game. To see this, we observe that every player in $\mathcal{V} \setminus \mathcal{W}$ has at least 1 neighbor playing 1 and thus she has no incentive to deviate. On the other hand, since 1 is always a best-response strategy for every player irrespective of what others play, the players in $\mathcal{W}$ also do not have any incentive to deviate. Hence, $\bar{x}$ is a PSNE.

For the "only if" part, let $\bar{x} = ((x_v = 1)_{v \in \mathcal{W}}, (x_v = 0)_{v \in \mathcal{V} \setminus \mathcal{W}})$ be a PSNE for the BNPG game where $|\mathcal{W}| \leqslant k$ (that is, at most k players are playing 1). We claim that $\mathcal{W}$ forms a dominating set for $\mathcal{G}$. Indeed this claim has to be correct, otherwise there exists a vertex $w \in \mathcal{V} \setminus \mathcal{W}$ which does not have any neighbor in $\mathcal{W}$ and consequently, the player w has incentive to deviate to 1 from 0 as $n_w = 0$ which is a contradiction.

Let $d(v)$ denote the degree of v in $\mathcal{G}$. To prove the result for the parameter k_0, we use the following best-response function.

$$\beta_v(k') = \begin{cases} 0 & \text{if } k' = d(v) \\ \{0, 1\} & \text{otherwise} \end{cases}$$

We observe that, since every player has the same best-response function, the BNPG game is fully homogeneous. We claim that the above BNPG game has a PSNE having at most k players playing 0 if and only if the Dominating Set instance is a yes instance.

For the "if" part, suppose the Dominating Set instance is a yes instance and $\mathcal{W} \subseteq \mathcal{V}$ be a dominating set for $\mathcal{G}$ of size at most k. We claim that the strategy profile $\bar{x} = ((x_v = 0)_{v \in \mathcal{W}}, (x_v = 1)_{v \in \mathcal{V} \setminus \mathcal{W}})$ is a PSNE for

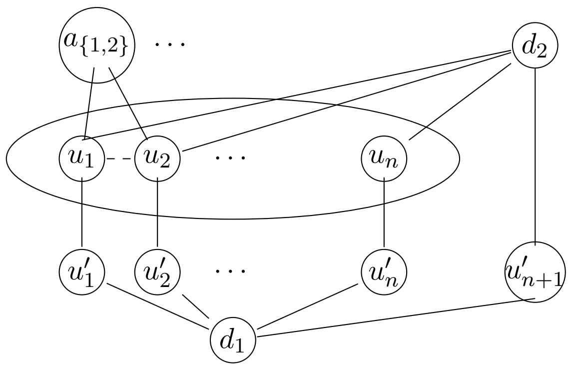

Figure 1: Graph $\mathcal{H}$ in the proof of Theorem 3.

the BNPG game. To see this, we observe that every player $w \in \mathcal{V} \setminus \mathcal{W}$ has at least 1 neighbor playing 0 and thus thus she has no incentive to deviate as $n_w < d(v)$. On the other hand, since 0 is always a best-response strategy for every player irrespective of what others play, the players in $\mathcal{W}$ also do not have any incentive to deviate. Hence, $\bar{x}$ is a PSNE.

For the "only if" part, let $\bar{x} = ((x_v = 0)_{v \in \mathcal{W}}, (x_v = 1)_{v \in \mathcal{V} \setminus \mathcal{W}})$ be a PSNE for the BNPG game where $|\mathcal{W}| \leqslant k$ (that is, at most k players are playing 0). We claim that $\mathcal{W}$ forms a dominating set for $\mathcal{G}$. Indeed this claim has to be correct, otherwise there exists a vertex $w \in \mathcal{V} \setminus \mathcal{W}$ which does not have any neighbor in $\mathcal{W}$ and consequently, the player w has incentive to deviate to 0 from 1 as $n_w = d(v)$ which is a contradiction. □

Corollary 1. *Assuming Exponential Time Hypothesis (ETH), there is no $f(k_1)n^{o(k_1)}$ (resp. $f(k_0)n^{o(k_0)}$) algorithm to decide if there exists a PSNE in a BNPG game where at most k_1 (resp. k_0) players play 1 (resp. 0).*

Proof. Under the assumption of ETH, there is no $f(k)n^{o(k)}$ algorithm to decide if there exists a dominating set of size atmost k. Since the parameteric transformation in the Theorem 2 is linear, the corollary follows. □

We next consider treewidth as parameter. Problems on graphs which are easy for trees are often fixed-parameter-tractable with respect to treewidth as parameter. We show that this is not the case for our problem. Towards that, we use the General Factor problem which is known to be W[1]-hard for the parameter treewidth [Szeider(2011)].

Definition 1 (General Factor)**.** *Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a set $K(v) \subseteq \{0, ..., d(v)\}$ for each $v \in \mathcal{V}$, compute if there exists a subset $\mathcal{F} \subseteq \mathcal{E}$ such that, for each vertex $v \in \mathcal{V}$, the number of edges in $\mathcal{F}$ incident on v is an element of $K(v)$. We denote an arbitrary instance of this problem by $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), (K(v))_{v \in \mathcal{V}})$.*

Theorem 3. *The* Exists-PSNE *problem for the BNPG game parameterized by treewidth is* W[1]*-hard.*

Proof. To prove W[1]-hardness, we reduce from General Factor to BNPG game. Let $\left(\mathcal{G} = (\{v_i : i \in [n]\}, \mathcal{E}'), (K(v_i))_{i\in[n]}\right)$ be an arbitrary instance of General Factor. The high level idea of our construction is as follows. For each vertex and edge in the graph $\mathcal{G}$ associated with General Factor instance, we add a node in the graph $\mathcal{H}$ (where the BNPG game is defined) associated with Exists-PSNE problem instance. On top of that we add some extra nodes and edges in $\mathcal{H}$ and appropriately define the best response functions of every player in $\mathcal{H}$ so that a set of nodes in $\mathcal{H}$ corresponding to a set $\mathcal{F}$ of edges belonging to $\mathcal{G}$ play 1 in a PSNE if and only if $\mathcal{F}$ makes General Factor an yes instance. We now formally present our construction.

We consider a BNPG game on the following graph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$. See Figure 1 for a pictorial representation of $\mathcal{H}$.

$$\begin{aligned}
\mathcal{V} = &\{u_i : i \in [n]\} \cup \{a_{\{i,j\}} : \{v_i, v_j\} \in \mathcal{E}'\}\\
&\cup \{u'_i : i \in [n+1]\} \cup \{d_1, d_2\}\\
\mathcal{E} = &\{\{u_i, a_{\{i,j\}}\}, \{u_j, a_{\{i,j\}}\} : \{v_i, v_j\} \in \mathcal{E}'\}\\
&\cup \{\{u_i, u'_i\} : i \in [n]\}\\
&\cup \{\{d_1, u'_i\}, \{d_2, u_i\} : i \in [n]\}\\
&\cup \{\{d_1, u'_{n+1}\}, \{d_2, u'_{n+1}\}\}
\end{aligned}$$

Let the treewidth of $\mathcal{G}$ be τ. Create a graph $\mathcal{G}'$ by adding the vertices d_1, d_2 and the set of edges $\{\{d_1, v_i\}, \{d_2, v_i\} : i \in [n]\} \cup \{d_1, d_2\}$ to the graph $\mathcal{G}$. The treewidth of $\mathcal{G}'$ is at most $\tau + 2$. We claim that the treewidth of $\mathcal{H}$ is at most $\tau + 3$. To see this, we begin with any nice tree decomposition (see [Cygan et al.(2015)] which talks about introduce edge vertices) of $\mathcal{G}'$ and replace v_i with u_i for every $i \in [n]$. Let $\mathcal{S} = \{u'_i : i \in [n+1]\} \cup \{a_{\{i,j\}} : \{v_i, v_j\} \in \mathcal{E}'\}$. $\forall u' \in \mathcal{S}$, add the vertex u' in the bag which introduces the edge $\{u, v\}$ (note that we look at that tree decomposition where no two edges are introduced by the same bag and such a tree decomposition exists for any graph) where u and v are neighbors of u' in $\mathcal{H}$. This results in a valid tree decomposition for $\mathcal{H}$ and hence, the treewidth of $\mathcal{H}$ is at most $\tau + 3$.

We now describe the best-response functions of the vertices in $\mathcal{H}$ to complete the description of the BNPG game.

$$\forall i \in [n], \beta_{u_i}(k) = \begin{cases} 1 & \text{if } k - 1 \in K(v_i) \\ 0 & \text{otherwise} \end{cases}$$

$$\forall \{v_i, v_j\} \in \mathcal{E}', \beta_{a_{\{i,j\}}}(k) = \{0, 1\} \; \forall k \in \mathbb{N} \cup \{0\}$$

$$\forall i \in [n+1], \beta_{u'_i}(k) = \begin{cases} 1 & \text{if } k = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\beta_{d_1}(k) = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n \\ 0 & \text{otherwise} \end{cases}, \beta_{d_2}(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

We claim that the above BNPG game has a PSNE if and only if the GENERAL FACTOR instance is a YES instance.

For the "if" part, suppose the GENERAL FACTOR instance is a YES instance. Then there exists a subset $\mathcal{F} \subseteq \mathcal{E}'$ such that the degree $d(v_i) \in K(v_i)$ for all $i \in [n]$ in $\mathcal{G}[\mathcal{F}]$. We consider the strategy profile $\bar{x} = (x_v)_{v \in \mathcal{V}}$.

$$\forall i \in [n], x_{u_i} = x_{u'_i} = 1, x_{u'_{n+1}} = 0,$$

$$\forall \{v_i, v_j\} \in \mathcal{E}', x_{a_{\{i,j\}}} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in \mathcal{F} \\ 0 & \text{otherwise} \end{cases}, x_{d_1} = 1, x_{d_2} = 0$$

Now we argue that $\bar{x}$ is a PSNE for the BNPG game. Clearly no player $a_{\{i,j\}}, \{v_i, v_j\} \in \mathcal{E}'$ deviates as both 0 and 1 are her best-responses irrespective of the action of their neighbors. The player d_1 does not deviate as she has exactly n neighbors playing 1. The player $u'_i, i \in [n]$ does not deviate as she has exactly 2 neighbors playing 1. The player u'_{n+1} does not deviate as she has exactly 1 neighbor playing 1. The player d_2 does not deviate as she has at least 1 neighbors playing 1. Note that $\forall i \in [n]$, the number of neighbors of u_i playing 1 excluding u'_i and d_2 (which in this case is $n_{u_i} - 1$ as $x_{d_2} = 0, x_{u'_i} = 1$) is same as the number of edges in $\mathcal{F}$ which are incident on v_i in $\mathcal{G}$. Hence, $\forall i \in [n]$, the player u_i does not deviate as $(n_{u_i} - 1) \in K(v_i)$. Hence, $\bar{x}$ is a PSNE.

For the "only if" part, let $\bar{x} = (x_v)_{v \in \mathcal{V}}$ be a PSNE of the BNPG game. We claim that we have $x_{d_1} = 1, x_{u_i} = x_{u'_i} = 1, \forall i \in [n], x_{u'_{n+1}} = 0, x_{d_2} = 0$. To prove this, we consider all cases for $(x_{u_i})_{i \in [n]}$.

(i) Case – $\forall i \in [n]$ $x_{u_i} = 1$: We have $x_{d_2} = 0$ as $n_{d_2} > 0$ otherwise d_2 would deviate. This implies that $x_{u'_{n+1}} = 0$ since $n_{u'_{n+1}} \leqslant 1$(as $x_{d_2} = 0$). Now we consider the following sub-cases (according to the values of x_{d_1} and $x_{u'_i}, i \in [n]$):

- $\triangleright$ ($x_{d_1} = 1, \exists k \in [n]$ such that $x_{u'_k} = 0$). Here $x_{u'_k}$ will then deviate to 1 as $n_{u'_k} = 2$. Hence, it is not a PSNE.
- $\triangleright$ ($x_{d_1} = 1, \forall i \in [n]$ $x_{u'_i} = 1$). This is exactly what we claim thus we have nothing to prove in this case.
- $\triangleright$ ($x_{d_1} = 0, \exists k \in [n]$ such that $x_{u'_k} = 1$). Here $x_{u'_k}$ will then deviate to 0 as $n_{u'_k} = 1$. Hence, it is not a PSNE.
- $\triangleright$ ($x_{d_1} = 0, \forall i \in [n]$ $x_{u'_i} = 0$). The player d_1 will deviate to 1 as $n_{d_1} = 0$. Hence, it is not a PSNE.

(ii) Case – $\exists k_1, k_2 \in [n]$ such that $x_{u_{k_1}} = 1$ and $x_{u_{k_2}} = 0$: We have $x_{d_2} = 0$ as $n_{d_2} > 0$ otherwise d_2 would deviate. This implies that $x_{u'_{n+1}} = 0$ since $n_{u'_{n+1}} \leqslant 1$ (as $x_{d_2} = 0$). Now we consider the following sub-cases (according to the values of x_{d_1} and $x_{u'_i}, i \in [n]$):

- ▷ ($x_{d_1} = 1, \forall i \in [n]\ x_{u'_i} = 0$). Here u'_{k_1} will deviate to 1 as $n_{u'_{k_1}} = 2$ and hence, it is not a PSNE.
- ▷ ($x_{d_1} = 1, \forall i \in [n]\ x_{u'_i} = 1$). Here u'_{k_2} will deviate to 0 as $n_{u'_{k_2}} = 1$ and hence, it is not a PSNE.
- ▷ ($x_{d_1} = 1$, $\exists i, j \in [n]$ such that $x_{u'_i} = 1$ and $x_{u'_j} = 0$). Here d_1 will deviate to 0 as $0 < n_{d_1} < n$ (there are at least 2 neighbours of d_1 which play 0 and at least 1 neighbour of d_1 which plays 1). Hence, it is not a PSNE.
- ▷ ($x_{d_1} = 0, \forall i \in [n]\ x_{u'_i} = 0$). Here d_1 will deviate to 1 as $n_{d_1} = 0$ and hence, it is not a PSNE.
- ▷ ($x_{d_1} = 0, \exists i \in [n]$ such that $x_{u'_i} = 1$). Here u'_i will deviate to 0 as $n_{u'_i} \leqslant 1$ and hence, it is not a PSNE.

(iii) Case – $\forall i \in [n]\ x_{u_i} = 0$: For every $i \in [n]$, we must have $x_{u'_i} = 0$ so that u'_i doesn't deviate. We have the following sub-cases (according to the values of x_{d_1}, x_{d_2} and $x_{u'_{n+1}}$):

- ▷ ($x_{d_1} = 0, x_{u'_{n+1}} = 0$). Here d_1 deviates to 1 as $n_{d_1} = 0$ and hence, it is not a PSNE.
- ▷ ($x_{d_1} = 0, x_{u'_{n+1}} = 1$). Here u'_{n+1} deviates to 0 as $n_{u'_{n+1}} \leqslant 1$ and hence, it is not a PSNE.
- ▷ ($x_{d_1} = 1, x_{u'_{n+1}} = 0, x_{d_2} = 0$). Here d_2 deviates to 1 as $n_{d_2} = 0$ and hence, it is not a PSNE.
- ▷ ($x_{d_1} = 1, x_{u'_{n+1}} = 0, x_{d_2} = 1$). Here u'_{n+1} deviates to 1 as $n_{u'_{n+1}} = 2$ and hence, it is not a PSNE.
- ▷ ($x_{d_1} = 1, x_{u'_{n+1}} = 1, x_{d_2} = 0$).Here u'_{n+1} deviates to 0 as $n_{u'_{n+1}} = 1$ and hence, it is not a PSNE.
- ▷ ($x_{d_1} = 1, x_{u'_{n+1}} = 1, x_{d_2} = 1$).Here d_2 deviates to 0 as $n_{d_2} > 0$ and hence, it is not a PSNE.

So if $\bar{x} = (x_v)_{v \in \mathcal{V}}$ is a PSNE of the BNPG game, then we have $x_{d_1} = 1$, $\forall i \in [n], x_{u'_i} = 1, x_{u'_{n+1}} = 0, \forall i \in [n]\ x_{u_i} = 1, x_{d_2} = 0$. Now consider the set $\mathcal{F} = \{\{v_i, v_j\} : x_{a_{\{i,j\}}} = 1, \{v_i, v_j\} \in \mathcal{E}'\}$. Note that $\forall i \in [n]$, the number of neighbors of u_i playing 1 excluding u'_i and d_2 (which in this case is $n_{u_i} - 1$ as $x_{d_2} = 0, x_{u'_i} = 1$) is same as the number of edges in $\mathcal{F}$ which are incident on v_i in $\mathcal{G}$. Since $\forall i, n_{u_i} - 1 \in K(v_i)$, the number of edges in $\mathcal{F}$ incident on v_i in General Factor instance is an element of $K(v_i)$. Hence the General Factor instance is a yes instance. □

We next consider fully homogeneous BNPG games and show the following by reducing from the Exists-PSNE problem on heterogeneous BNPG games.

Theorem 4. *(i) The* Exists-PSNE *problem for fully homogeneous BNPG games is* NP*-complete even if the diameter of the graph is at most* 4.

(ii) The Exists-PSNE *problem for fully homogeneous BNPG games is* W[1]*-hard with respect to the parameter treewidth of the graph.*

Proof. We first present a reduction from the Exists-PSNE problem on heterogeneous BNPG games to the Exists-PSNE problem on fully homogeneous BNPG games. Let $(\mathcal{G} = (\mathcal{V} = \{v_i : i \in [n]\}, \mathcal{E}), (g_v)_{v\in\mathcal{V}}, (c_v)_{v\in\mathcal{V}})$ be any heterogeneous BNPG game. The high-level idea of our construction is as follows. In the constructed BNPG instance on a graph $\mathcal{H}$, we take a copy of $\mathcal{G}$, add some special set of nodes and some set of edges. We define the best-response function in such a way that the nodes in $\mathcal{H}$ corresponding to nodes in $\mathcal{G}$ do not have their set of "possible n_v values" overlapping with any other node in any PSNE for $\mathcal{H}$. This allows us to make the nodes (excluding the special set of nodes which always play 1 in PSNE) in $\mathcal{H}$ behave in the same way as the corresponding nodes in $\mathcal{G}$ if PSNE exists for any instance. This implies both instance are equivalent. We now formally present our proof.

We now consider the following graph $\mathcal{H} = (\mathcal{V}', \mathcal{E}')$.

$$\begin{aligned}
\mathcal{V}' &= \{u_i : i \in [n]\} \cup_{i\in[n]} \mathcal{V}_i, \text{where} \\
\mathcal{V}_i &= \{a^i_j : j \in [2 + (i-1)n]\}, \forall i \in [n] \\
\mathcal{E}' &= \{\{u_i, u_j\} : \{v_i, v_j\} \in \mathcal{E}\} \cup_{i\in[n]} \mathcal{E}_i, \text{where} \\
\mathcal{E}_i &= \{\{a^i_j, u_i\} : j \in [2 + (i-1)n]\}, \forall i \in [n]
\end{aligned}$$

Let us define $f(x) = \lfloor \frac{x-2}{n} \rfloor + 1, h(x) = x - 2 - (f(x) - 1)n$. We now define best-response strategies β for the fully homogeneous BNPG game on $\mathcal{H}$.

$$\beta(k) = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = 1 \\ \{0,1\} & \text{if } \Delta g_{v_{f(k)}}(h(k)) = c_{v_{f(k)}}, k > 1 \\ 1 & \text{if } \Delta g_{v_{f(k)}}(h(k)) > c_{v_{f(k)}}, k > 1 \\ 0 & \text{if } \Delta g_{v_{f(k)}}(h(k)) < c_{v_{f(k)}}, k > 1 \end{cases}$$

This finishes description of our fully homogeneous BNPG game on $\mathcal{H}$. We now claim that there exists a PSNE in the heterogeneous BNPG game on $\mathcal{G}$ if and only if there exists a PSNE in the fully homogeneous BNPG game on $\mathcal{H}$.

For the "only if" part, let $x^* = (x^*_v)_{v \in \mathcal{V}}$ be a PSNE in the heterogeneous BNPG game on $\mathcal{G}$. We now consider the following strategy profile $\bar{y} = (y_v)_{v \in \mathcal{V}'}$ for players in $\mathcal{H}$.

$$\forall i \in [n] y_{u_i} = x^*_{v_i}; y_w = 1 \text{ for other vertices } w$$

Clearly the players in $\cup_{i \in [n]} \mathcal{V}_i$ do not deviate as their degree is 1 and $\beta(0) = \beta(1) = 1$. In $\bar{y}$, we have $n_{u_i} = n_{v_i} + 2 + (i-1)n \geqslant 2$ and $n_{v_i} \leqslant n-1$ for every $i \in [n]$. If $x^*_{v_i} = 1$, then we have $\Delta g_{v_i}(n_{v_i}) \geqslant c_{v_i}$. We have $f(n_{u_i}) = i$ and $h(n_{u_i}) = n_{v_i}$. This implies that $\Delta g_{v_{f(n_{u_i})}}(h(n_{u_i})) \geqslant c_{v_{f(n_{u_i})}}$. So u_i does not deviate as 1 is the best-response. If $x^*_{v_i} = 0$, then we have $\Delta g_{v_i}(n_{v_i}) \leqslant c_{v_i}$. This implies that $\Delta g_{v_{f(n_{u_i})}}(h(n_{u_i})) \leqslant c_{v_{f(n_{u_i})}}$. So u_i does not deviate as 0 is the best-response. Hence, $\bar{y}$ is a PSNE.

For the "if" part, suppose there exists a PSNE $(x^*_v)_{v \in \mathcal{V}'}$ in the fully homogeneous BNPG game on $\mathcal{H}$. Clearly $x^*_v = 1$ for all $v \in \cup_{i \in [n]} \mathcal{V}_i$ as $n_v \leqslant 1$. Now we claim that the strategy profile $\bar{x} = (x_{v_i} = x^*_{u_i})_{i \in [n]}$ forms a PSNE for the heterogeneous BNPG game on $\mathcal{G}$. We observe that if $x^*_{u_i} = 1$, then $\Delta g_{v_{f(n_{u_i})}}(h(n_{u_i})) \geqslant c_{v_{f(n_{u_i})}}$ for $i \in [n]$. This implies that $\Delta g_{v_i}(n_{v_i}) \geqslant c_{v_i}$. So $x_{v_i} = 1$ is the best-response for v_i in $\mathcal{G}$ and hence, she does not deviate. Similarly, If $x^*_{u_i} = 0$, then $\Delta g_{v_{f(n_{u_i})}}(h(n_{u_i})) \leqslant c_{v_{f(n_{u_i})}}$. This implies that $\Delta g_{v_i}(n_{v_i}) \leqslant c_{v_i}$. So $x_{v_i} = 0$ is the best-response for $v_i \in \mathcal{V}$ and hence, it won't deviate. Hence, $\bar{x}$ is a PSNE in the heterogeneous BNPG game on $\mathcal{G}$.

We now prove the two statements in the theorem as follows.

(i) We observe that, if the diameter of $\mathcal{G}$ is at most 2, then the diameter of $\mathcal{H}$ is at most 4. Hence, the result follows from Observation 2.

(ii) We observe that the treewidth of $\mathcal{H}$ is at most 1 more than the treewidth of $\mathcal{G}$. Hence, the result follows from Theorem 3.

□

We next show that the EXISTS-PSNE problem for fully homogeneous BNPG games is para-NP-hard parameterized by the maximum degree of the graph again by reducing from heterogeneous BNPG games.

Theorem 5. *The* EXISTS-PSNE *problem for fully homogeneous BNPG games is* NP*-complete even if the maximum degree* Δ *of the graph is at most* 9.

Proof. The high-level idea in this proof is the same as the proof of Theorem 4. The only difference is that we add the special nodes in a way such that the maximum degree in the instance of fully homogeneous BNPG game is upper bounded by 9.

Formally, we consider an instance of a heterogeneous BNPG game on a graph $\mathcal{G} = (\mathcal{V} = \{v_i : i \in [n]\}, \mathcal{E})$ such that there are only 2 types of utility functions $U_1(x_v, n_v) = g_1(x_v + n_v) - c_1 x_v, U_2(x_v, n_v) = g_2(x_v + n_v) - c_2 x_v$, and the degree of any vertex is at most 3; we know from Theorem 1 that it is an NP-complete instance. Let us partition $\mathcal{V}$ into $\mathcal{V}_1$ and $\mathcal{V}_2$ such that the utility function of the players in $\mathcal{V}_1$ is $U_1(\cdot)$ and the utility function of the players in $\mathcal{V}_2$ is $U_2(\cdot)$. We now construct the graph $\mathcal{H} = (\mathcal{V}', \mathcal{E}')$ for the instance of the fully homogeneous BNPG game.

$$\begin{aligned}
\mathcal{V}' &= \{w_i : i \in [n]\} \cup \mathcal{W}_1 \cup \mathcal{W}_2, \text{where} \\
\mathcal{W}_i &= \{a_j^k : j \in [2 + 4(i-1)], v_k \in \mathcal{V}_i\} \ \forall i \in [2] \\
\mathcal{E}' &= \{\{w_i, w_j\} : \{v_i, v_j\} \in \mathcal{E}\} \cup \mathcal{E}_1 \cup \mathcal{E}_2, \text{where} \\
\mathcal{E}_i &= \{\{a_j^k, w_k\} : j \in [2 + 4(i-1)], v_k \in \mathcal{V}_i\} \ \forall i \in [2]
\end{aligned}$$

We define two functions — $f(x) = \lfloor \frac{x-2}{4} \rfloor + 1$ and $h(x) = x - 2 - 4(f(x) - 1)$. We now define best-response functions for the players in $\mathcal{H}$.

$$\beta(k) = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = 1 \\ \{0, 1\} & \Delta g_{f(k)}(h(k)) = c_{f(k)}, k > 1 \\ 1 & \Delta g_{f(k)}(h(k)) > c_{f(k)}, k > 1 \\ 0 & \Delta g_{f(k)}(h(k)) < c_{f(k)}, k > 1 \end{cases}$$

This finishes description of our fully homogeneous BNPG game on $\mathcal{H}$. We now claim that there exists a PSNE in the heterogeneous BNPG game on $\mathcal{G}$ if and only if there exists a PSNE in the fully homogeneous BNPG game on $\mathcal{H}$. We note that degree of any node in $\mathcal{G}$ is at most 9.

For the "only if" part, let $x^* = (x_v^*)_{v \in \mathcal{V}}$ be a PSNE in the heterogeneous BNPG game on $\mathcal{G}$. We now consider the following strategy profile $\bar{y} = (y_v)_{v \in \mathcal{V}'}$ for players in $\mathcal{H}$.

$$\forall i \in [n] y_{w_i} = x_{v_i}^*; y_b = 1 \text{ for other vertices } b$$

We now claim that the players in $\mathcal{V}'$ also does not deviate. Clearly the players in $\cup_{i \in [2]} \mathcal{W}_i$ do not deviate as their degree is 1 and $\beta(0) = \beta(1) = 1$. If $v_k \in \mathcal{V}_i$, then in $\bar{y}$ we have $n_{w_k} = n_{v_k} + 2 + 4(i-1) \geqslant 2$ for every $i \in [2]$ and $n_{v_k} \leqslant 3$ as the maximum degree in $\mathcal{G}$ is 3. If $x_{v_k}^* = 1$, then we have $\Delta g_i(n_{v_k}) \geqslant c_i$. We have $f(n_{w_k}) = i$ and $h(n_{w_k}) = n_{v_k}$. This implies that $\Delta g_{f(n_{w_k})}(h(n_{w_k})) \geqslant c_{f(n_{w_k})}$. So w_k does not deviate as 1 is the best-response. If $x_{v_k}^* = 0$, then we have $\Delta g_i(n_{v_k}) \leqslant c_i$. This implies that $\Delta g_{f(n_{w_k})}(h(n_{w_k})) \leqslant c_{f(n_{w_k})}$. So w_k does not deviate as 0 is the best-response. Hence, $\bar{y}$ is a PSNE.

For the "if" part, suppose there exists a PSNE $(x_v^*)_{v \in \mathcal{V}'}$ in the fully homogeneous BNPG game on $\mathcal{H}$. Clearly $x_w^* = 1$ for all $w \in \cup_{i \in [2]} \mathcal{W}_i$ as $n_w \leqslant 1$. Now we claim that the strategy profile $\bar{x} = (x_{v_k} = x_{w_k}^*)_{k \in [n]}$ forms a PSNE for the heterogeneous BNPG game on $\mathcal{G}$. We observe that if $x_{w_k}^* = 1$ and $v_k \in \mathcal{V}_i$, then $\Delta g_{f(n_{w_k})}(h(n_{w_k})) \geqslant c_{f(n_{w_k})}$ for $k \in [n]$. This implies that $\Delta g_i(n_{v_k}) \geqslant c_i$. So $x_{v_k} = 1$ is the best-response for $v_k \in \mathcal{V}_i$ and hence, she does not deviate. Similarly, If $x^*_{w_k} = 0$ and $v_k \in \mathcal{V}_i$, then $\Delta g_{f(n_{w_k})}(h(n_{w_k})) \leqslant c_{f(n_{w_k})}$. This implies that $\Delta g_i(n_{v_k}) \leqslant c_i$. So $x_{v_k} = 0$ is the best-response for $v_k \in \mathcal{V}_i$ and hence, it won't deviate. Hence, $\bar{x}$ is a PSNE in the heterogeneous BNPG game on $\mathcal{G}$. □

3.2 Algorithmic Results

We now present our algorithmic results. Our first result is an FPT algorithm for the Exists-PSNE problem for strict games when parameterized by the size of a minimum vertex cover. The high-level idea of our algorithm is the following. For every possible set of response of players in the minimum vertex cover of the instance graph $\mathcal{G}$, there is only one best response for the players not in the minimum vertex cover. This is due to fact that we are looking at a strict BNPG game and the set of nodes not in minimum vertex cover form an independent set. So we can brute force over all possible set of responses of players in the vertex cover of the instance graph $\mathcal{G}$ and check the existence of PSNE.

Theorem 6. *There is a $\mathcal{O}^*(2^{vc(\mathcal{G})})$ time algorithm for the* Exists-PSNE *problem for strict BNPG games.*

Proof. Let $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), (g_v)_{v\in\mathcal{V}}, (c_v)_{v\in\mathcal{V}})$ be any instance of Exists-PSNE for BNPG games. We compute a minimum vertex cover $\mathcal{S} \subset \mathcal{V}$ in time $\mathcal{O}^*(2^{\mathrm{vc}(\mathcal{G})})$ [Cygan et al.(2015)]. The idea is to brute force on the strategy profile of players in $\mathcal{S}$ and assign actions of other players based on their best-response functions. For every strategy profile $x_{\mathcal{S}} = (x_v)_{v\in\mathcal{S}}$, we do the following.

1. For $w \in \mathcal{V} \setminus \mathcal{S}$, let n_w be the number of neighbors of w (they can only be in $\mathcal{S}$) who play 1. We define $x_w = 1$ if $\Delta g_w(n_w) > c_w$ and 0 if $\Delta g_w(n_w) < c_w$. This is well-defined since $\Delta g_w(n_w) \neq c_w$ as the game is strict.

2. If $(x_v)_{v\in\mathcal{V}}$ forms a PSNE, then output yes. Otherwise we discard the current $x_{\mathcal{S}}$.

If the above the procedure does not output yes for any $x_{\mathcal{S}}$, then we output no. The correctness of the algorithm is immediate. Since the computation for every guess of $x_{\mathcal{S}}$ can be done in polynomial time and the number of such guesses is $2^{\mathcal{S}} = 2^{\mathrm{vc}(\mathcal{G})}$, it follows that the running time of our algorithm is $\mathcal{O}^*(2^{\mathrm{vc}(\mathcal{G})})$. □

We next show that the Exists-PSNE problem for BNPG games, parameterized by both maximum degree and diameter, is fixed-parameter-tractable. This complements our hardness result of Observation 2.

Observation 3. *The* EXISTS-PSNE *problem for BNPG games is FPT with respect to the parameter* (Δ, d).

Proof. Let $\mathcal{G}$ be any graph with diameter d and maximum degree Δ. A simple breadth-first search based argument proves that the number of vertices n of a graph is at most Δ^d. Hence, we can run a brute force search in time $O(2^{O(\Delta^d)} \cdot O(\Delta^{d \cdot O(1)}))$ by checking every strategy profile and conclude whether there is any PSNE or not. $\square$

Our next result shows that we can always find a PSNE for additive BNPG games in $\mathcal{O}(n)$ time.

Observation 4. *There exists an* $\mathcal{O}(n)$ *time algorithm to find a PSNE in an additive BNPG game.*

Proof. $\forall x \geqslant 0, \forall i \in [n]$, $g_{v_i}(x+1) - g_{v_i}(x) = g_{v_i}(1)$. This implies for a player v_i, the best response doesn't depend on the responses of its neighbours and solely depends on $g_{v_i}(1)$. Hence if $g_{v_i}(1) \geqslant c_{v_i}$ then we assign the response of player v_i as 1 and 0 otherwise. This will make sure that no player v_i deviates. So calculating the PSNE takes $\mathcal{O}(n)$ $\square$

We next consider circuit rank and distance from complete graph as parameter. These parameters can be thought of distance from tractable instances (namely tree and complete graph). They are defined as follows.

Definition 2. *Let the number of edges and number of vertices in a graph* $\mathcal{G}$ *be* m *and* n *respectively. Then* d_1 *(circuit rank) is defined to be* $m - n + c$ *(*c *is the number of connected components in the graph) and* d_2 *(distance from complete graph) is defined to be* $\frac{n(n-1)}{2} - m$.

Yu et al. presented an algorithm for the EXISTS-PSNE problem on trees in [Yu et al.(2020)]. It turns out that their algorithm can be easily modified to prove the following.

Observation 5. *Given a BNPG game on a tree* $\mathcal{T} = (\mathcal{V}, \mathcal{E})$*, a subset of vertices* $\mathcal{U} \subseteq \mathcal{V}$ *and a strategy profile* $(x_u)_{u \in \mathcal{U}} \in \{0,1\}^{\mathcal{U}}$*, there is a polynomial time algorithm for computing if there exists a PSNE* $(y_v)_{v \in \mathcal{V}} \in \{0,1\}^{\mathcal{V}}$ *for the BNPG game such that* $x_u = y_u$ *for every* $u \in \mathcal{U}$.

Theorem 7. *There is an algorithm running in time* $\mathcal{O}^*(4^{d_1})$ *for the* EXISTS-PSNE *problem in BNPG games where* d_1 *is the circuit rank of the input graph.*

Proof. Let $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), (g_v)_{v \in \mathcal{V}}, (c_v)_{v \in \mathcal{V}})$ be any instance of EXISTS-PSNE for BNPG games. Let the graph $\mathcal{G}$ have c connected components namely, $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1), \ldots, \mathcal{G}_c = (\mathcal{V}_c, \mathcal{E}_c)$. For every $i \in [c]$, we compute if there exists a PSNE in $\mathcal{G}_i$; clearly there is a PSNE in $\mathcal{G}$ if and only if there is a PSNE in $\mathcal{G}_i$ for every $i \in [n]$. Hence, in the rest of the proof, we focus on the algorithm to compute a PSNE in $\mathcal{G}_i$. We compute a minimum spanning tree $\mathcal{T}_i$ in the connected component $\mathcal{G}_i$. Let $\mathcal{E}_i' \subset \mathcal{E}_i$ be the set of edges which are not part of $\mathcal{T}_i$; let $|\mathcal{E}_i'| = d_1^i$ and $\mathcal{V}_i' = \{v_1^i, v_2^i, \ldots, v_l^i\} \subseteq \mathcal{V}_i$ be the set of vertices which are endpoints of at least one edge in $\mathcal{E}_i'$. Of course, we have $|\mathcal{V}_i'| = \ell \leqslant 2d_1^i$. For every tuple $t = (x_v')_{v \in \mathcal{V}_i'} \in \{0,1\}^l$, we do the following.

(i) For each $v \in \mathcal{V}_i'$, let n_v^t be the number of neighbours of v in $\mathcal{G}_i[\mathcal{E}_i']$ (subgraph of $\mathcal{G}_i$ containing the set of nodes $\mathcal{V}_i$ and the set of edges $\mathcal{E}_i'$) who play 1 in t. We now define g_v^t for every player $v \in \mathcal{V}$ as follows.

$$g_v^t(k) = \begin{cases} g_v(k + n_v^t) & \text{if } v \in \mathcal{V}_i' \\ g_v(k) & \text{otherwise} \end{cases}$$

(ii) We now compute if there exists a PSNE $(y_v)_{v \in \mathcal{V}_i} \in \{0,1\}^{\mathcal{V}_i}$ in the BNPG game $(\mathcal{T}_i, (g_v^t)_{v \in \mathcal{V}_i}, (c_v)_{v \in \mathcal{V}_i})$ such that $y_v = x_v'$ for every $v \in \mathcal{V}_i'$; this can be done in polynomial time due to Observation 5. If such a PSNE exists, then we output YES.

If we fail to find a PSNE for every choice of tuple t, then we output NO. The running time of the above algorithm (for $\mathcal{G}_i$) is $\mathcal{O}^*\left(2^{|\mathcal{V}_i'|}\right)$. Hence the overall running time of our algorithm is $\mathcal{O}^*\left(\sum_{i=1}^{c} 2^{|\mathcal{V}_i'|}\right) \leqslant \mathcal{O}^*\left(2^{2d_1}\right) = \mathcal{O}^*\left(4^{d_1}\right)$. We now argue correctness of our algorithm. We observe that it is enough to argue correctness for one component.

In one direction, let $x^* = (x_v^*)_{v \in \mathcal{V}_i}$ be a PSNE in the BNPG game $(\mathcal{G}_i, (g_v)_{v \in \mathcal{V}_i}, (c_v)_{v \in \mathcal{V}_i})$. We now claim that $(x_v^*)_{v \in \mathcal{V}_i}$ is also a PSNE in the BNPG game on $(\mathcal{T}_i, (g_v^t)_{v \in \mathcal{V}_i}, (c_v)_{v \in \mathcal{V}_i})$ where $t = (x_v^*)_{v \in \mathcal{V}_i'}$. Let $n_v^{\mathcal{G}_i}$ and $n_v^{\mathcal{T}_i}$ be the number of neighbors of $v \in \mathcal{V}_i$ in $\mathcal{G}_i$ and $\mathcal{T}_i$ respectively who play 1 in x^*. With n_v^t defined as above, we have $n_v^{\mathcal{G}_i} = n_v^{\mathcal{T}_i} + n_v^t$ for $v \in \mathcal{V}_i'$ and $n_v^{\mathcal{G}_i} = n_v^{\mathcal{T}_i}$ for $v \in \mathcal{V}_i \setminus \mathcal{V}_i'$. Hence, we have $\Delta g_v^t(n_v^{\mathcal{T}_i}) = \Delta g_v(n_v^{\mathcal{T}_i} + n_v^t) = \Delta g_v(n_v^{\mathcal{G}_i})$ for $v \in \mathcal{V}_i'$ and $\Delta g_v^t(n_v^{\mathcal{T}_i}) = \Delta g_v(n_v^{\mathcal{G}_i})$ for $v \in \mathcal{V}_i \setminus \mathcal{V}_i'$. If $x_v^* = 1$ where $v \in \mathcal{V}_i$, then $\Delta g_v(n_v^{\mathcal{G}_i}) \geqslant c_v$ and thus we have $\Delta g_v^t(n_v^{\mathcal{T}_i}) \geqslant c_v$. Hence, v does not deviate in $\mathcal{T}_i$. Similarly, if $x_v^* = 0$ where $v \in \mathcal{V}_i$, then $\Delta g_v(n_v^{\mathcal{G}_i}) \leqslant c_v$ and thus we have $\Delta g_v^t(n_v^{\mathcal{T}_i}) \leqslant c_v$. Hence, v does not deviate in $\mathcal{T}_i$. Hence $(x_v^*)_{v \in \mathcal{V}}$ is also a PSNE in BNPG game $(\mathcal{T}_i, (g_v^t)_{v \in \mathcal{V}_i}, (c_v)_{v \in \mathcal{V}_i})$ where $t = (x_v^*)_{v \in \mathcal{V}_i'}$ (which means our Algorithm returns YES).

In the other direction, let $(x_v^*)_{v \in \mathcal{V}_i}$ be the PSNE in BNPG game on $(\mathcal{T}_i, (g_v^t)_{v \in \mathcal{V}_i}, (c_v)_{v \in \mathcal{V}_i})$ where $t = (x_v^*)_{v \in \mathcal{V}_i'}$ (which means our Algorithm returns YES). We claim that $(x_v^*)_{v \in \mathcal{V}_i}$ is also a PSNE in BNPG game $(\mathcal{G}_i, (g_v)_{v \in \mathcal{V}_i}, (c_v)_{v \in \mathcal{V}_i})$. If $x_v^* = 1$ for $v \in \mathcal{V}_i$, then $\Delta g_v^t(n_v^{\mathcal{T}_i}) \geqslant c_v$. This implies that $\Delta g_v(n_v^{\mathcal{G}_i}) \geqslant c_v$ and thus v does not deviate in $\mathcal{T}_i$. Similarly, if $x_v^* = 0$ for $v \in \mathcal{V}_i$, then $\Delta g_v^t(n_v^{\mathcal{T}_i}) \leqslant c_v$. This implies that $\Delta g_v(n_v^{\mathcal{G}_i}) \leqslant c_v$ and thus v does not deviate in $\mathcal{T}_i$. Hence $(x_v^*)_{v\in\mathcal{V}}$ is also a PSNE in BNPG game on $(\mathcal{G}_i, (g_v)_{v\in\mathcal{V}_i}, (c_v)_{v\in\mathcal{V}_i})$. $\square$

We next present our FPT algorithm parameterized by d_2. It turns out that the algorithm of [Yu et al.(2020)] for the Exists-PSNE problem can be easily modified to solve a more general problem.

Observation 6. *Given a BNPG game on a complete graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and an integer k, there is a polynomial time algorithm for computing if there exists a PSNE for the BNPG game where exactly k players play 1 and returns such a PSNE if it exists.*

On a high-level, the idea of our algorithm in Theorem 8 is the same as in Theorem 7.

Theorem 8. *There is an algorithm running in time $\mathcal{O}^*(4^{d_2})$ for the* Exists-PSNE *problem in BNPG games where d_2 is the number of edges one needs to add in the input graph to make it a complete graph.*

Proof. Let $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), (g_v)_{v\in\mathcal{V}}, (c_v)_{v\in\mathcal{V}})$ be any instance of Exists-PSNE for BNPG games. If $d_2 \geqslant \frac{n}{2}$, then iterating over all possible strategy profiles takes time $\mathcal{O}^*(2^n) \leqslant \mathcal{O}^*(4^{d_2})$. So allow us to assume for the rest of the proof that $d_2 < \frac{n}{2}$. Let us define $\mathcal{V}' = \{u \in \mathcal{V} : \exists v \in \mathcal{V}, v \neq u, \{u, v\} \notin \mathcal{E}\}$; we have $|\mathcal{V}'| \leqslant 2d_2$.

For every strategy profile $y = (y_u)_{u\in\mathcal{V}'}$, we do the following. For each $v \in \mathcal{V} \setminus \mathcal{V}'$, let n'_v be the number of neighbors of v in $\mathcal{V}'$ who play 1 in y. We now define $g'_v(\ell) = g_v(\ell + n'_v)$ for every $\ell \in \mathbb{N} \cup \{0\}$ and every player $v \in \mathcal{V} \setminus \mathcal{V}'$. For every $k \in \{0, \ldots, |\mathcal{V} \setminus \mathcal{V}'|\}$, we compute (using the algorithm in Observation 6) if there exists a PSNE $x^k = (x_v^k)_{v\in\mathcal{V}\setminus\mathcal{V}'}$ in the BNPG game $(\mathcal{G}[\mathcal{V} \setminus \mathcal{V}'], (g'_v)_{v\in\mathcal{V}\setminus\mathcal{V}'}, (c_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$ where exactly k players play 1. If x^k exists, then we output yes if $((y_u)_{u\in\mathcal{V}'}, (x_v^k)_{v\in\mathcal{V}\setminus\mathcal{V}'})$ forms a PSNE in the BNPG game $(\mathcal{G} = (\mathcal{V}, \mathcal{E}), (g_v)_{v\in\mathcal{V}}, (c_v)_{v\in\mathcal{V}})$.

If the above procedure fails to find a PSNE, then we output no. The running time of the above algorithm is $\mathcal{O}^*\left(2^{|\mathcal{V}'|}\right) \leqslant \mathcal{O}^*\left(4^{d_2}\right)$. We now argue correctness of our algorithm.

Clearly, if the algorithm outputs yes, then there exists a PSNE for the input game. On the other hand, if there exists a PSNE $((y_u)_{u\in\mathcal{V}'}, (x_v)_{v\in\mathcal{V}\setminus\mathcal{V}'}) \in \{0, 1\}^{\mathcal{V}}$ in the input game, then let us consider the iteration of our algorithm with the guess $(y_u)_{u\in\mathcal{V}'}$. Let the number of players playing 1 in $(x_v)_{v\in\mathcal{V}\setminus\mathcal{V}'}$ be k. If $x_v = 1$ where $v \in \mathcal{V}\setminus\mathcal{V}'$, then $\Delta g_v(n'_v + k - 1) \geqslant c_v$ and thus we have $\Delta g'_v(k-1) \geqslant c_v$. Similarly, if $x_v = 0$ where $v \in \mathcal{V}\setminus\mathcal{V}'$, then $\Delta g_v(n'_v + k) \leqslant c_v$ and thus we have $\Delta g'_v(k) \leqslant c_v$. Hence we observe that $(x_v)_{v\in\mathcal{V}\setminus\mathcal{V}'}$ forms a PSNE in the BNPG game $(\mathcal{G}[\mathcal{V} \setminus \mathcal{V}'], (g'_v)_{v\in\mathcal{V}\setminus\mathcal{V}'}, (c_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$. Let $(x'_v)_{v\in\mathcal{V}\setminus\mathcal{V}'}$ be the PSNE of the BNPG game $(\mathcal{G}[\mathcal{V} \setminus \mathcal{V}'], (g'_v)_{v\in\mathcal{V}\setminus\mathcal{V}'}, (c_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$ where exactly k players play 1 returned by the algorithm in Observation 6. We observe that every player in $\mathcal{V}'$ has the same number of neighbors playing 1 in both the strategy profiles $((y_u)_{u\in\mathcal{V}'}, (x_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$ and $((y_u)_{u\in\mathcal{V}'}, (x'_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$. So no player in $\mathcal{V}'$ will deviate in the strategy profile $((y_u)_{u\in\mathcal{V}'}, (x'_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$. If $x'_v = 1$ where $v \in \mathcal{V} \setminus \mathcal{V}'$, then $\Delta g'_v(k-1) \geqslant c_v$ and thus we have $\Delta g_v(n'_v + k - 1) \geqslant c_v$. Hence, v does not deviate in in the strategy profile $((y_u)_{u\in\mathcal{V}'}, (x'_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$. Similarly, if $x'_v = 0$ where $v \in \mathcal{V}\setminus\mathcal{V}'$, then $\Delta g'_v(k) \leqslant c_v$ and thus we have $\Delta g_v(n'_v+k) \leqslant c_v$. Hence, v does not deviate in in the strategy profile $((y_u)_{u\in\mathcal{V}'}, (x'_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$. Hence, $((y_u)_{u\in\mathcal{V}'}, (x'_v)_{v\in\mathcal{V}\setminus\mathcal{V}'})$ also forms a PSNE in the input BNPG game and thus the algorithm outputs YES. This concludes the correctness of our algorithm. □

3.3 Structural Results

We now show that a PSNE for fully homogeneous BNPG games always exists for some special graph classes. We begin with paths.

Theorem 9. *There is always a PSNE in a fully homogeneous BNPG game for paths. Moreover, we can find a PSNE in this case in $\mathcal{O}(n)$ time.*

Proof. Let the set of vertices in the input path $\mathcal{P}$ be $\mathcal{V} = \{v_1, \ldots, v_n\}$ and the set of edges $\mathcal{E} = \{\{v_i, v_{i+1}\} : i \in [n-1]\}$. Note that the possible values of n_v for any vertex v is $\mathcal{P}$ is $0, 1$ and 2. We show that there is a PSNE in $\mathcal{P}$ for all possible best response strategies. Let $S_i := \beta(i)$ (since the game is fully homogeneous, the best-response function is the same for all players) be the set of best responses of a player v if $n_v = i$. Let x_v be the response of a player $v \in \mathcal{V}$.

- ▷ If $0 \in S_0$, then $(x_v = 0)_{v\in\mathcal{V}}$ forms a PSNE as clearly no player would deviate as $n_v = 0$ for every player. So, allow us to assume for the rest of the proof that $S_0 = \{1\}$.
- ▷ If $1 \in S_i, \forall i \in \{1, 2\}$, then $(x_v = 1)_{v\in\mathcal{V}}$ forms a PSNE as the best response is 1 irrespective of the value of n_v and hence no player would deviate. So, allow us to assume that we have either $1 \notin S_1$ or $1 \notin S_2$.
- ▷ If $0 \in S_1, 0 \in S_2$, then $((x_{v_i} = 0)_{i\equiv 1 \pmod 2}, (x_{v_i} = 1)_{i\equiv 0 \pmod 2})$ forms a PSNE. If i is an odd integer then $n_{v_i} > 0$ and in this case one of the best response is 0 and hence v_i does not deviate. If i is an even integer then $n_{v_i} = 0$ and in this case one of the best response is 1 (recall $S_0 = \{1\}$) and hence v_i does not deviate.
- ▷ If $1 \in S_1, 0 \in S_2$, then $((x_{v_i} = 1)_{i\equiv 1 \pmod 2, i\neq n}, (x_{v_i} = 0)_{i\equiv 0 \pmod 2, i\neq n}, x_{v_n} = 1)$ forms a PSNE. If i is an odd integer and not equal to n, then $n_{v_i} \leqslant 1$ and in this case one of the best response is 1 and hence v_i does not deviate. If i is an even number and not equal to n then $n_{v_i} = 2$ and in this case one of the best response is 0 and hence v_i does not deviate. Note that $n_{v_n} \leqslant 1$ and hence in this case one of the best response is 1 and hence v_n does not deviate.
- ▷ If $0 \in S_1, 1 \in S_2$. In this we have two sub-cases:

- **n is a multiple of** 3: In this sub-case, $((x_{v_i} = 0)_{i \not\equiv 2 \pmod 3}, (x_{v_i} = 1)_{i \equiv 2 \pmod 3})$ forms a PSNE. If we have $i \not\equiv 2 \pmod 3$, then $n_{v_i} = 1$ and in this case, 0 is a best response and hence v_i does not deviate. If $i \equiv 2 \pmod 3$, then $n_{v_i} = 0$ and in this case one of the best response is 1 and hence v_i does not deviate.
- **n is not a multiple of 3**: In this sub-case, $((x_{v_i} = 0)_{i \not\equiv 1 \pmod 3}, (x_{v_i} = 1)_{i \equiv 1 \pmod 3})$ forms a PSNE. If $i \not\equiv 1 \pmod 3$, then $n_{v_i} = 1$ and in this case one of the best response is 0 and hence v_i does not deviate. If $i \equiv 1 \pmod 3$, then $n_{v_i} = 0$ and in this case one of the best response is 1 and hence v_i does not deviate.

Since we have a PSNE for every possible best-response functions, we conclude that there is always a PSNE in a fully homogeneous BNPG game on paths. Also, we can find a PSNE in paths in $\mathcal{O}(n)$ time. □

Theorem 10. *There is always a PSNE in a fully homogeneous BNPG game for complete graphs, cycles, and bi-cliques. Moreover, we can find a PSNE in $\mathcal{O}(n)$ time.*

Proof. We divide the proof into 3 parts:

Part 1- Complete graph: We assume that the input graph $\mathcal{G}$ $(\mathcal{V},\mathcal{E})$ is a complete graph.Let the utility function for all the players $v \in \mathcal{V}$ be $U(x_v, n_v) = g(x_v + n_v) - c \cdot x_v$. If $\Delta g(n-1) \geqslant c$, then $(x_v = 1)_{v\in\mathcal{V}}$ is a PSNE. If $\Delta g(0) \leqslant c$, then $(x_v = 0)_{v\in\mathcal{V}}$ is a PSNE. If $\Delta g(n-1) < c$ and $\Delta g(0) > c$, then there should exist a $0 < k \leqslant n-1$ such that $\Delta g(k) \leqslant c$ and $\Delta g(k-1) \geqslant c$ otherwise both $\Delta g(n-1) < c$ and $\Delta g(0) > c$ can't simultaneously hold true. Now we claim that if there exists a $0 < k \leqslant n-1$ such that $\Delta g(k) \leqslant c$ and $\Delta g(k-1) \geqslant c$, then choosing any k players and making their response 1 and rest of players response as 0 would be PSNE. Any player w whose response is 1 has $n_w = k-1$ and since $\Delta g(k-1) \geqslant c$, w does not have any incentive to deviate. Similarly any player w' whose response is 0 has $n_{w'} = k$ and since $\Delta g(k) \leqslant c$, w' does not have any incentive to deviate. This concludes the proof of the theorem as we showed that there is a PSNE in all possible cases. Also clearly we can find a PSNE in $\mathcal{G}$ in $\mathcal{O}(n)$ time.

Part 2- Cycles: We assume that the input graph is a Cycle. Let the set of vertices in the input cycle $\mathcal{C}$ be $\mathcal{V} = \{v_1, \ldots, v_n\}$ and the set of edges $\mathcal{E} = \{\{v_i, v_{i+1}\} : i \in [n-1]\} \cup \{v_n, v_1\}$. Note that the possible values of n_v(number of neighbours of v choosing 1) for any vertex $v \in \mathcal{C}$ is $0, 1,$ and 2. We show that there is a PSNE in $\mathcal{C}$ for all possible best response strategies. Let $S_i := \beta(i)$ (since the game is fully homogeneous, the best-response function is the same for all players) denote the set of best responses of a player v if $n_v = i$. Let x_v denote the response of a player $v \in \mathcal{V}$.

- $\triangleright$ If $0 \in S_0$ then $x_v = 0$ for every player $v \in \mathcal{V}$ forms a PSNE as clearly no player would deviate as $n_v = 0$ for every player. So, allow us to assume that $S_0 = \{1\}$ in the rest of the proof.

- $\triangleright$ If $1 \in S_2$, then $x_v = 1$ for every player v in $\mathcal{V}$ forms a PSNE as clearly no player would deviate as $n_v = 2$ for every player. So, allow us to assume that $S_2 = \{0\}$ in the rest of the proof.

- $\triangleright$ If $0 \in S_1$, then $((x_{v_i} = 0)_{i \equiv 1 \pmod 2}, (x_{v_i} = 1)_{i \equiv 0 \pmod 2})$ forms a PSNE. If i is odd then $n_{v_i} > 0$ and in this case one of the best response is 0 and hence v_i does not deviate. If i is even number then $n_{v_i} = 0$ and in this case one of the best response is 1 (recall, we have $S_0 = \{1\}$) and hence v_i does not deviate.

- $\triangleright$ If $1 \in S_1$, then $((x_{v_i} = 1)_{i \equiv 1 \pmod 2}, (x_{v_i} = 0)_{i \equiv 0 \pmod 2})$ forms a PSNE. If i is an odd number then $n_{v_i} \leqslant 1$ and in this case one of the best response is 1 and hence v_i does not deviate. If i is an even number and then $n_{v_i} = 2$ and in this case one of the best response is 0 and hence v_i does not deviate.

Since we have a PSNE for every possible best-response functions, we conclude that there is always a PSNE in a fully homogeneous BNPG game on cycles. Also, we can find a PSNE in cycles in $\mathcal{O}(n)$ time.

Part 3- Bicliques: Let the input graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a biclique; $\mathcal{V}$ is partitioned into 2 sets namely $\mathcal{V}_1 = \{u_1, \ldots, u_{n_1}\}$ and $\mathcal{V}_2 = \{v_1, \ldots, v_{n_2}\}$ where $n_1 + n_2 = n$ and $\mathcal{E} = \{(u_i, v_j) : i \in [n_1], j \in [n_2]\}$. We show that there is a PSNE in $\mathcal{C}$ for all possible best response strategies. Let $S_i := \beta(i)$ (since the game is fully homogeneous, the best-response function is the same for all players) denote the set of best responses of a player v if $n_v = i$. Let x_v denote the response of a player $v \in \mathcal{V}$.

1. **If** $n_1 = n_2$: For this case we have the following sub-cases:

   - $\triangleright$ If $0 \in S_0$, then $(x_v = 0)_{v \in \mathcal{V}}$ forms a PSNE as clearly no player would deviate as $n_v = 0$ for every player. So, allow us to assume that $S_0 = \{1\}$.
   - $\triangleright$ If $1 \in S_{n_1}$, then $(x_v = 1)_{v \in \mathcal{V}}$ forms a PSNE as clearly no player would deviate as $n_v = n_1$ for every player. So, allow us to assume that $S_{n_1} = \{0\}$. However, then $((x_v = 0)_{v \in \mathcal{V}_1}, (x_v = 1)_{v \in \mathcal{V}_2})$ forms a PSNE. If $v \in \mathcal{V}_2$ then $n_v = 0$ and in this case one of the best response is 1 and hence v won't deviate. If $v \in \mathcal{V}_1$ then $n_v = n_1$ and in this case one of the best response is 0 and hence v won't deviate.

2. **If** $n_1 \neq n_2$: For this case we have the following sub-cases:

- ▷ If $0 \in S_0$ then $x_v = 0$ for every player v in $\mathcal{V}$ forms a PSNE as clearly no player would deviate as $n_v = 0$ for every player. So, allow us to assume that $S_0 = \{1\}$ for the rest of the proof.
- ▷ If $1 \in S_{n_1}$, $1 \in S_{n_2}$, then $(x_v = 1)_{v \in \mathcal{V}}$ forms a PSNE. If $v \in \mathcal{V}_1$, then $n_v = n_2$ and in this case one of the best response is 1 and hence v does not deviate. If $v \in \mathcal{V}_2$, then $n_v = n_1$ and in this case one of the best response is 1 and hence v does not deviate.
- ▷ If $0 \in S_{n_1}, 1 \in S_{n_2}$, then $((x_v = 1)_{:v \in \mathcal{V}_1}, (x_v = 0)_{v \in \mathcal{V}_2})$ forms a PSNE. If $v \in \mathcal{V}_1$ then $n_v = 0$ and in this case one of the best response is 1 (recall $S_0 = \{1\}$) and hence v does not deviate. If $v \in \mathcal{V}_2$ then $n_v = n_1$ and in this case one of the best response is 0 and hence v does not deviate.
- ▷ If $0 \in S_{n_1}, 0 \in S_{n_2}$, then $((x_v = 1)_{v \in \mathcal{V}_1}, (x_v = 0)_{v \in \mathcal{V}_2})$ forms a PSNE. If $v \in \mathcal{V}_1$ then $n_v = 0$ and in this case one of the best response is 1 (recall $S_0 = \{1\}$) and hence v does not deviate. If $v \in \mathcal{V}_2$ then $n_v = n_1$ and in this case one of the best response is 0 and hence v does not deviate.
- ▷ If $1 \in S_{n_1}, 0 \in S_{n_2}$, then $((x_v = 0)_{:v \in \mathcal{V}_1}, (x_v = 1)_{v \in \mathcal{V}_2})$ forms a PSNE. If $v \in \mathcal{V}_1$ then $n_v = n_2$ and in this case one of the best response is 0 and hence v does not deviate. If $v \in \mathcal{V}_2$ then $n_v = 0$ and in this case one of the best response is 1 (recall $S_0 = \{1\}$) and hence v does not deviate.

Since we have a PSNE for every possible best-response functions, we conclude that there is always a PSNE in a fully homogeneous BNPG game on biclique. Also, we can find a PSNE in biclique in $\mathcal{O}(n)$ time. □

4 Conclusion

We have studied parameterized complexity of the Exists-PSNE problem for the BNPG games with respect to various important graph parameters. With maximum degree and diameter as the parameters, we have shown that the Exists-PSNE problem is para-NP-hard even for fully homogeneous BNPG games. We have proved that our problem is W[1]-hard with respect to treewidth as the parameter. We have shown W[2]-hardness with respect to the number of players playing 1 and the number of players playing 0 as parameters.

On the positive side, we have demonstrated FPT algorithms parameterized by the size of a minimum vertex cover (for strict BNPG games), circuit rank, and the distance from complete graph. We have finally shown that BNPG games on some important graph classes, for example, complete graph, path, cycle, bi-clique, etc. always have a PSNE. Moreover, a PSNE can be found in $\mathcal{O}(n)$ time for these graph classes.

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