# George B. Mertzios 🖂 🖻

Department of Computer Science, Durham University, UK

#### Hendrik Molter 🖂 🗈

Technische Universität Berlin, Faculty IV, Algorithmics and Computational Complexity, Germany

### Malte Renken $\square$

Technische Universität Berlin, Faculty IV, Algorithmics and Computational Complexity, Germany

#### Paul G. Spirakis 🖂 🗈

Department of Computer Science, University of Liverpool, UK Computer Engineering & Informatics Department, University of Patras, Greece

#### Philipp Zschoche 🖂 🗈

Technische Universität Berlin, Faculty IV, Algorithmics and Computational Complexity, Germany

#### — Abstract

In a temporal network with discrete time-labels on its edges, entities and information can only "flow" along sequences of edges whose time-labels are non-decreasing (resp. increasing), i.e. along temporal (resp. strict temporal) paths. Nevertheless, in the model for temporal networks of [Kempe, Kleinberg, Kumar, JCSS, 2002], the individual time-labeled edges remain undirected: an edge  $e = \{u, v\}$  with time-label t specifies that "u communicates with v at time t". This is a symmetric relation between u and v, and it can be interpreted that the information can flow in either direction. In this paper we make a first attempt to understand how the direction of information flow on one edge can impact the direction of information flow on other edges. More specifically, naturally extending the classical notion of a transitive orientation in static graphs, we introduce the notion of a temporal transitive orientation and we systematically investigate its algorithmic behavior in various situations. An orientation of a temporal graph is called *temporally transitive* if, whenever u has a directed edge towards v with time-label  $t_1$  and v has a directed edge towards w with time-label  $t_2 \ge t_1$ , then u also has a directed edge towards w with some time-label  $t_3 \geq t_2$ . If we just demand that this implication holds whenever  $t_2 > t_1$ , the orientation is called *strictly* temporally transitive, as it is based on the fact that there is a strict directed temporal path from u to w. Our main result is a conceptually simple, yet technically quite involved, polynomial-time algorithm for recognizing whether a given temporal graph  $\mathcal{G}$  is transitively orientable. In wide contrast we prove that, surprisingly, it is NP-hard to recognize whether  $\mathcal{G}$  is strictly transitively orientable. Additionally we introduce and investigate further related problems to temporal transitivity, notably among them the temporal transitive completion problem, for which we prove both algorithmic and hardness results.

**2012 ACM Subject Classification** Theory of computation  $\rightarrow$  Graph algorithms analysis; Mathematics of computing  $\rightarrow$  Discrete mathematics

Keywords and phrases Temporal graph, transitive orientation, transitive closure, polynomial-time algorithm, NP-hardness, satisfiability.

Funding George B. Mertzios: Supported by the EPSRC grant EP/P020372/1.

Hendrik Molter: Supported by the German Research Foundation (DFG), project MATE (NI 369/17). Malte Renken: Supported by the German Research Foundation (DFG), project MATE (NI 369/17). Paul G. Spirakis: Supported by the NeST initiative of the School of EEE and CS at the University of Liverpool and by the EPSRC grant EP/P02002X/1.

# 1 Introduction

A temporal (or dynamic) network is, roughly speaking, a network whose underlying topology changes over time. This notion concerns a great variety of both modern and traditional networks; information and communication networks, social networks, and several physical systems are only few examples of networks which change over time [27,38,41]. Due to its vast applicability in many areas, the notion of temporal graphs has been studied from different perspectives under several different names such as *time-varying*, *evolving*, *dynamic*, and graphs over time (see [13–15] and the references therein). In this paper we adopt a simple and natural model for temporal networks which is given with discrete time-labels on the edges of a graph, while the vertex set remains unchanged. This formalism originates in the foundational work of Kempe et al. [28].

▶ **Definition 1** (Temporal Graph [28]). A temporal graph is a pair  $\mathcal{G} = (G, \lambda)$ , where G = (V, E) is an underlying (static) graph and  $\lambda : E \to \mathbb{N}$  is a time-labeling function which assigns to every edge of G a discrete-time label.

Mainly motivated by the fact that, due to causality, entities and information in temporal graphs can only "flow" along sequences of edges whose time-labels are non-decreasing (resp. increasing), Kempe et al. introduced the notion of a *(strict) temporal path*, or *(strict) time-respecting path*, in a temporal graph  $(G, \lambda)$  as a path in G with edges  $e_1, e_2, \ldots, e_k$  such that  $\lambda(e_1) \leq \ldots \leq \lambda(e_k)$  (resp.  $\lambda(e_1) < \ldots < \lambda(e_k)$ ). This notion of a temporal path naturally resembles the notion of a *directed* path in the classical static graphs, where the direction is from smaller to larger time-labels along the path. Nevertheless, in temporal paths the individual time-labeled edges remain undirected: an edge  $e = \{u, v\}$  with time-label  $\lambda(e) = t$  can be abstractly interpreted as "u communicates with v at time t". Here the relation "communicates" is symmetric between u and v, i.e. it can be interpreted that the information can flow in either direction.

In this paper we make a first attempt to understand how the direction of information flow on one edge can impact the direction of information flow on other edges. More specifically, naturally extending the classical notion of a transitive orientation in static graphs [24], we introduce the notion of a *temporal transitive orientation* and we thoroughly investigate its algorithmic behavior in various situations. Imagine that v receives information from u at time  $t_1$ , while w receives information from v at time  $t_2 \ge t_1$ . Then w indirectly receives information from u through the intermediate vertex v. Now, if the temporal graph correctly records the transitive closure of information passing, the directed edge from u to w must exist and must have a time label  $t_3 \geq t_2$ . In such a transitively oriented temporal graph, whenever an edge is oriented from a vertex u to a vertex w with time-label t, we have that every temporal path from u to w arrives no later than t, and that there is no temporal path from w to u. Different notions of temporal transitivity have also been used for automated temporal data mining [40] in medical applications [39], text processing [45]. Furthermore, in behavioral ecology, researchers have used a notion of orderly (transitive) triads A-B-C to quantify dominance among species. In particular, animal groups usually form dominance hierarchies in which dominance relations are transitive and can also change with time [33].

One natural motivation for our temporal transitivity notion comes from thinking about vertices u, v, w as journalists or lawyers. Suppose that v queried some important information from u (the information source) at time  $t_1$ , and afterwards, at time  $t_2 \ge t_1$ , w queried the important information from v (the intermediary). Then, in order to ensure the validity of the information received, w might want to verify it by *subsequently* querying the information directly from u at some time  $t_3 \ge t_2$ . Note that w might first receive the important information from u through various intermediaries, and using several channels of different lengths. Then, to maximize confidence about the information, w should query u for verification only after receiving the information from the latest of these indirect channels.

It is worth noting here that the model of temporal graphs given in Definition 1 has been also used in its extended form, in which the temporal graph may contain multiple time-labels per edge [35]. This extended temporal graph model has been used to investigate temporal paths [3,9,11,16,35,48] and other temporal path-related notions such as temporal analogues of distance and diameter [1], reachability [2] and exploration [1,3,20,21], separation [22,28,49], and path-based centrality measures [12,29], as well as recently non-path problems too such as temporal variations of coloring [37], vertex cover [4], matching [36], cluster editing [18], and maximal cliques [8,26,47]. However, in order to better investigate and illustrate the inherent combinatorial structure of temporal graphs given by Kempe et al. [28] with one time-label per edge [7,17,19]. Throughout the paper, whenever we assume multiple time-labels per edge we will state it explicitly; in all other cases we consider a single label per edge.

In static graphs, the transitive orientation problem has received extensive attention which resulted in numerous efficient algorithms. A graph is called *transitively orientable* (or a comparability graph) if it is possible to orient its edges such that, whenever we orient utowards v and v towards w, then the edge between u and w exists and is oriented towards w. The first polynomial-time algorithms for recognizing whether a given (static) graph G on nvertices and m edges is comparability (i.e. transitively orientable) were based on the notion of forcing an orientation and had running time  $O(n^3)$  (see Golumbic [24] and the references therein). Faster algorithms for computing a transitive orientation of a given comparability graph have been later developed, having running times  $O(n^2)$  [43] and  $O(n + m \log n)$  [30], while the currently fastest algorithms run in linear O(n+m) time and are based on efficiently computing a modular decomposition of G [31, 32]; see also Spinrad [44]. It is fascinating that, although all the latter algorithms compute a valid transitive orientation if G is a comparability graph, they fail to recognize whether the input graph is a comparability graph; instead they produce an orientation which is non-transitive if G is not a comparability graph. The fastest known algorithm for determining whether a given orientation is transitive requires matrix multiplication, currently achieved in  $O(n^{2.37286})$  time [5].

**Our contribution.** In this paper we introduce the notion of temporal transitive orientation and we thoroughly investigate its algorithmic behavior in various situations. An orientation of a temporal graph  $\mathcal{G} = (G, \lambda)$  is called temporally transitive if, whenever u has a directed edge towards v with time-label  $t_1$  and v has a directed edge towards w with time-label  $t_2 \ge t_1$ ,<sup>1</sup> then u also has a directed edge towards w with some time-label  $t_3 \ge t_2$ . If we just demand that this implication holds whenever  $t_2 > t_1$ , the orientation is called strictly temporally transitive, as it is based on the fact that there is a strict directed temporal path from u to w. Similarly, if we demand that the transitive directed edge from u to w has time-label  $t_3 > t_2$ , the orientation is called strongly (resp. strongly strictly) temporally transitive.

Although these four natural variations of a temporally transitive orientation seem superficially similar to each other, it turns out that their computational complexity (and their underlying combinatorial structure) varies massively. Indeed we obtain a surprising result in Section 3: deciding whether a temporal graph  $\mathcal{G}$  admits a *temporally transitive* orientation is solvable in polynomial time (Section 3.1), while it is NP-hard to decide whether it admits

<sup>&</sup>lt;sup>1</sup> That is, whenever there exists a (non-strict) directed temporal path from u to w arriving at time  $t_2$ 

a strictly temporally transitive orientation (Section 3.2). On the other hand, it turns out that, deciding whether  $\mathcal{G}$  admits a strongly or a strongly strictly temporal transitive orientation is (easily) solvable in polynomial time as they can both be reduced to 2SAT satisfiability.

Our main result is that, given a temporal graph  $\mathcal{G} = (G, \lambda)$ , we can decide in polynomial time whether  $\mathcal{G}$  can be transitively orientable, and at the same time we can output a temporal transitive orientation if it exists. Our algorithm, whose proof of correctness is technically quite involved, is conceptually simple and easy to implement. It is heavily based on the notion of forcing an orientation<sup>2</sup>, thus extending the classical polynomial-time algorithm for computing a transitive orientation in static graphs described by Golumbic [24]. However, in contrast to the case of static graphs, it turns out that, in temporal graphs, we cannot use simple orientation forcings to eliminate the condition that a *triangle* cannot be oriented cyclically. To resolve this issue, we prove that the recognition problem of temporally transitive graphs is equivalent to deciding the Boolean satisfiability of a mixed formula  $\phi_{3NAE} \wedge \phi_{2SAT}$ . Here  $\phi_{3NAE}$  is a 3NAE formula, i.e., the disjunction of clauses with three literals each, where every clause NAE $(\ell_1, \ell_2, \ell_3)$  is satisfied if and only if at least one of the literals  $\{\ell_1, \ell_2, \ell_3\}$  is equal to 1 and at least one of them is equal to 0. Note that every clause NAE $(\ell_1, \ell_2, \ell_3)$  corresponds to the condition that a specific triangle in the temporal graph cannot be cyclically oriented. Furthermore  $\phi_{2SAT}$  is a 2SAT formula, i.e., the disjunction of 2CNF clauses with two literals each, where every clause  $(\ell_1 \vee \ell_2)$  is satisfied if and only if at least one of the literals  $\{\ell_1, \ell_2\}$ is equal to 1.

However, although deciding whether  $\phi_{2\text{SAT}}$  is satisfiable can be done in linear time with respect to the size of the formula [6], the problem Not-All-Equal-3-SAT is NP-complete [42]. Therefore, expressing our problem as a Boolean satisfiability problem in this sense does not help deducing a polynomial-time algorithm (unless P=NP). Nevertheless, we prove a sequence of structural properties that a temporally transitive orientation has to satisfy, which allows us to present a polynomial-time algorithm for this problem. This phenomenon of deducing a polynomial-time algorithm for an algorithmic graph problem by deciding satisfiability of a mixed Boolean formula (i.e. with both clauses of two and three literals) occurs rarely; this approach has been successfully used for the efficient recognition of simple-triangle (known also as "PI") graphs [34].

In the second part of our paper (Section 4) we consider a natural extension of the temporal orientability problem, namely the temporal transitive completion problem. In this problem we are given a temporal graph  $\mathcal{G}$  and a natural number k, and the question is whether it is possible to add at most k new edges (with the corresponding time-labels) to  $\mathcal{G}$  such that the resulting temporal graph is (strongly/strictly/strongly strictly) transitively orientable. We prove that all four versions of temporal graph  $\mathcal{G}$  is directed (i.e. if every time-labeled edge has a fixed orientation) then all versions of temporal transitive completion are solvable in polynomial time. As a corollary of our results it follows that all four versions of temporal transitive completion are solvable in polynomial time. As a corollary of our results it follows that all four versions of temporal transitive completion are fixed-parameter-tractable (FPT) with respect to the number q of unoriented time-labeled edges in  $\mathcal{G}$ .

In the third and last part of our paper (Section 5) we consider the *multilayer transitive* orientation problem. In this problem we are given an undirected temporal graph  $\mathcal{G} = (G, \lambda)$ , where G = (V, E), and we ask whether there exists an orientation F of its edges (i.e. with exactly one orientation for each edge of G) such that, for every 'time-layer"  $t \ge 1$ , the (static) oriented graph induced by the edges having time-label t is transitively oriented in F. Problem

 $<sup>^{2}</sup>$  That is, orienting an edge from u to v forces us to orient another edge from a to b.

definitions of this type are commonly referred to as multilayer problems [10], Observe that this problem trivially reduces to the static case if we assume that each edge has a single time-label, as then each layer can be treated independently of all others. However, if we allow  $\mathcal{G}$  to have multiple time-labels on every edge of G, then we show that the problem becomes NP-complete, even when every edge has at most two labels.

# 2 Preliminaries and Notation

Given a (static) undirected graph G = (V, E), an edge between two vertices  $u, v \in V$  is denoted by the unordered pair  $\{u, v\} \in E$ , and in this case the vertices u, v are said to be *adjacent*. If the graph is directed, we will use the ordered pair (u, v) (resp. (v, u)) to denote the oriented edge from u to v (resp. from v to u). For simplicity of the notation, we will usually drop the parentheses and the comma when denoting an oriented edge, i.e. we will denote (u, v) just by uv. Furthermore,  $\widehat{uv} = \{uv, vu\}$  is used to denote the set of both oriented edges uv and vu between the vertices u and v.

Let  $S \subseteq E$  be a subset of the edges of an undirected (static) graph G = (V, E), and let  $\widehat{S} = \{uv, vu : \{u, v\} \in S\}$  be the set of both possible orientations uv and vu of every edge  $\{u, v\} \in S$ . Let  $F \subseteq \widehat{S}$ . If F contains at least one of the two possible orientations uv and vu of an edge  $\{u, v\} \in S$ , then F is called an *orientation* of the edges of S. F is called a proper orientation if it contains exactly one of the orientations uv and vu of every edge  $\{u, v\} \in S$ . Note here that, in order to simplify some technical proofs, the above definition of an orientation allows F to be not proper, i.e. to contain both uv and vu for a specific edge  $\{u, v\}$ . However, whenever F is not proper, this means that F can be discarded as it cannot be used as a part of a (temporal) transitive orientation. For every orientation F denote by  $F^{-1} = \{vu : uv \in F\}$  the reversal of F. Note that  $F \cap F^{-1} = \emptyset$  if and only if F is proper.

In a temporal graph  $\mathcal{G} = (G, \lambda)$ , where G = (V, E), whenever  $\lambda(\{v, w\}) = t$  (or simply  $\lambda(v, w) = t$ ), we refer to the tuple  $(\{v, w\}, t)$  as a *time-edge* of  $\mathcal{G}$ . A triangle of  $(G, \lambda)$  on the vertices u, v, w is a synchronous triangle if  $\lambda(u, v) = \lambda(v, w) = \lambda(w, u)$ . Let G = (V, E) and let F be a proper orientation of the whole edge set E. Then  $(\mathcal{G}, F)$ , or  $(G, \lambda, F)$ , is a proper orientation of the temporal graph  $\mathcal{G}$ . An partial proper orientation of a temporal graph  $\mathcal{G} = (G, \lambda, F)$  is an orientation of a subset of E. To indicate that the edge  $\{u, v\}$  of a time-edge  $(\{u, v\}, t)$  is oriented from u to v (that is,  $uv \in F$  in a (partial) proper orientation F), we use the term ((u, v), t), or simply (uv, t). For simplicity we may refer to a (partial) proper orientation just as a (partial) orientation, whenever the term "proper" is clear from the context.

A static graph G = (V, E) is a comparability graph if there exists a proper orientation Fof E which is transitive, that is, if  $F \cap F^{-1} = \emptyset$  and  $F^2 \subseteq F$ , where  $F^2 = \{uw : uv, vw \in F$ for some vertex  $v\}$  [24]. Analogously, in a temporal graph  $\mathcal{G} = (G, \lambda)$ , where G = (V, E), we define a proper orientation F of E to be temporally transitive, if:

whenever  $(uv, t_1)$  and  $(vw, t_2)$  are oriented time-edges in  $(\mathcal{G}, F)$  such that  $t_2 \ge t_1$ , there exists an oriented time-edge  $(wu, t_3)$  in  $(\mathcal{G}, F)$ , for some  $t_3 \ge t_2$ .

In the above definition of a temporally transitive orientation, if we replace the condition " $t_3 \ge t_2$ " with " $t_3 > t_2$ ", then F is called *strongly temporally transitive*. If we instead replace the condition " $t_2 \ge t_1$ " with " $t_2 > t_1$ ", then F is called *strictly temporally transitive*. If we do both of these replacements, then F is called *strongly strictly temporally transitive*. Note that strong (strict) temporal transitivity implies (strict) temporal transitivity, while (strong) temporal transitivity implies (strong) strict temporal transitivity.

the established terminology for static graphs, we define a temporal graph  $\mathcal{G} = (G, \lambda)$ , where G = (V, E), to be a *(strongly/strictly) temporal comparability graph* if there exists a proper orientation F of E which is *(strongly/strictly) temporally transitive*.

We are now ready to formally introduce the following decision problem of recognizing whether a given temporal graph is temporally transitively orientable or not.

TEMPORAL TRANSITIVE ORIENTATION (TTO)

**Input:** A temporal graph  $\mathcal{G} = (G, \lambda)$ , where G = (V, E).

**Question:** Does  $\mathcal{G}$  admit a temporally transitive orientation F of E?

In the above problem definition of TTO, if we ask for the existence of a strictly (resp. strongly, or strongly strictly) temporally transitive orientation F, we obtain the decision problem STRICT (resp. STRONG, or STRONG STRICT) TEMPORAL TRANSITIVE ORIENTATION (TTO).

Let  $\mathcal{G} = (G, \lambda)$  be a temporal graph, where G = (V, E). Let G' = (V, E') be a graph such that  $E \subseteq E'$ , and let  $\lambda' \colon E' \to \mathbb{N}$  be a time-labeling function such that  $\lambda'(u, v) = \lambda(u, v)$  for every  $\{u, v\} \in E$ . Then the temporal graph  $\mathcal{G}' = (G', \lambda')$  is called a *temporal supergraph of*  $\mathcal{G}$ . We can now define our next problem definition regarding computing temporally orientable supergraphs of  $\mathcal{G}$ .

TEMPORAL TRANSITIVE COMPLETION (TTC)

- **Input:** A temporal graph  $\mathcal{G} = (G, \lambda)$ , where G = (V, E), a (partial) orientation F of  $\mathcal{G}$ , and an integer k.
- **Question:** Does there exist a temporal supergraph  $\mathcal{G}' = (G', \lambda')$  of  $(G, \lambda)$ , where G' = (V, E'), and a transitive orientation  $F' \supseteq F$  of  $\mathcal{G}'$  such that  $|E' \setminus E| \le k$ ?

Similarly to TTO, if we ask in the problem definition of TTC for the existence of a strictly (resp. strongly, or strongly strictly) temporally transitive orientation F', we obtain the decision problem STRICT (resp. STRONG, or STRONG STRICT) TEMPORAL TRANSITIVE COMPLETION (TTC).

Now we define our final problem which asks for an orientation F of a temporal graph  $\mathcal{G} = (G, \lambda)$  (i.e. with exactly one orientation for each edge of G) such that, for every "time-layer"  $t \geq 1$ , the (static) oriented graph defined by the edges having time-label t is transitively oriented in F. This problem does not make much sense if every edge has exactly one time-label in  $\mathcal{G}$ , as in this case it can be easily solved by just repeatedly applying any known static transitive orientation algorithm. Therefore, in the next problem definition, we assume that in the input temporal graph  $\mathcal{G} = (G, \lambda)$  every edge of G potentially has multiple time-labels, i.e. the time-labeling function is  $\lambda : E \to 2^{\mathbb{N}}$ .

MULTILAYER TRANSITIVE ORIENTATION (MTO)

**Input:** A temporal graph  $\mathcal{G} = (G, \lambda)$ , where G = (V, E) and  $\lambda : E \to 2^{\mathbb{N}}$ . **Question:** Is there an orientation F of the edges of G such that, for every  $t \ge 1$ , the (static)

# oriented graph induced by the edges having time-label t is transitively oriented?

# **3** The recognition of temporally transitively orientable graphs

In this section we investigate the computational complexity of all variants of TTO. We show that TTO as well as the two variants STRONG TTO and STRONG STRICT TTO, are solvable in polynomial time, whereas STRICT TTO turns out to be NP-complete.

	$u \overset{v}{\longleftarrow} t_{3} \overset{v}{\longrightarrow} w$			$t_1$	
	$t_1 = t_2 = t_3$	$t_1 < t_2 = t_3$	$t_1 \le t_2 < t_3$	$t_1 = t_2$	$t_1 < t_2$
TTO	non-cyclic	wu = wv	$\begin{array}{l} vw \implies uw \\ vu \implies wu \end{array}$	uv = wv	$uv \implies wv$
Strong TTO		$wu \wedge wv$	$\begin{array}{l} vw \implies uw \\ vu \implies wu \end{array}$	uv = wv	$uv \implies wv$
STRICT TTO	т	non-cyclic	$\begin{array}{l} vw \implies uw \\ vu \implies wu \end{array}$	Т	$uv \implies wv$
Str. Str. TTO	Т	$wu \wedge wv$	$\begin{array}{l} vw \implies uw \\ vu \implies wu \end{array}$	Т	$uv \implies wv$

**Table 1** Orientation conditions imposed by a triangle (left) and an induced path of length two (right) in the underlying graph G for the decision problems (STRICT/STRONG/STRONG STRICT) TTO. Here,  $\top$  means that no restriction is imposed,  $\perp$  means that the graph is not orientable, and in the case of triangles, "non-cyclic" means that all orientations except the ones that orient the triangle cyclicly are allowed.

The main idea of our approach to solve TTO and its variants is to create Boolean variables for each edge of the underlying graph G and interpret setting a variable to 1 or 0 with the two possible ways of directing the corresponding edge.

More formally, for every edge  $\{u, v\}$  we introduce a variable  $x_{uv}$  and setting this variable to 1 corresponds to the orientation uv while setting this variable to 0 corresponds to the orientation vu. Now consider the example of Figure 1(a), i.e. an induced path of length two in the underlying graph G on three vertices u, v, w, and let  $\lambda(u, v) = 1$  and  $\lambda(v, w) = 2$ . Then the orientation uv "forces" the orientation wv. Indeed, if we otherwise orient  $\{v, w\}$ as vw, then the edge  $\{u, w\}$  must exist and be oriented as uw in any temporal transitive orientation, which is a contradiction as there is no edge between u and w. We can express this "forcing" with the implication  $x_{uv} \implies x_{wv}$ . In this way we can deduce the constraints that all triangles or induced paths on three vertices impose on any (strong/strict/strong strict) temporal transitive orientation. We collect all these constraints in Table 1.

When looking at the conditions imposed on temporal transitive orientations collected in Table 1, we can observe that all conditions except "non-cyclic" are expressible in 2SAT. Since 2SAT is solvable in linear time [6], it immediately follows that the strong variants of temporal transitivity are solvable in polynomial time.

#### ▶ **Theorem 2.** STRONG TTO and STRONG STRICT TTO are solvable in polynomial time.

In the variants TTO and STRICT TTO, however, we can have triangles that impose a "non-cyclic" orientation of three edges. This can be naturally modeled by a not-all-equal (NAE) clause<sup>3</sup>. However, if we now naïvely model the conditions with a Boolean formula, we

 $<sup>^{3}</sup>$  A not all equal clause is a set of literals and it evaluates to true if and only if at least two literals in the



**Figure 1** The orientation uv forces the orientation wu and vice-versa in the examples of (a) a static graph G where  $\{u, v\}, \{v, w\} \in E(G)$  and  $\{u, w\} \notin E(G)$ , and of (b) a temporal graph  $(G, \lambda)$  where  $\lambda(u, w) = 3 < 5 = \lambda(u, v) = \lambda(v, w)$ .

obtain a formula with 2SAT clauses and 3NAE clauses. Deciding whether such a formula is satisfiable is NP-complete in general [42]. Hence, we have to investigate these two variants more thoroughly.

Indeed we obtain a surprising result: TTO is solvable in polynomial time while STRICT TTO is NP-complete. In the following subsection we provide a polynomial-time algorithm for TTO and at the end of the section we prove that STRICT TTO is NP-complete.

# 3.1 A polynomial-time algorithm for TTO

Let G = (V, E) be a static undirected graph. There are various polynomial-time algorithms for deciding whether G admits a transitive orientation F. However our results in this section are inspired by the transitive orientation algorithm described by Golumbic [24], which is based on the crucial notion of *forcing* an orientation. The notion of forcing in static graphs is illustrated in Figure 1 (a): if we orient the edge  $\{u, v\}$  as uv (i.e., from u to v) then we are forced to orient the edge  $\{v, w\}$  as wv (i.e., from w to v) in any transitive orientation Fof G. Indeed, if we otherwise orient  $\{v, w\}$  as vw (i.e. from v to w), then the edge  $\{u, w\}$ must exist and it must be oriented as uw in any transitive orientation F of G, which is a contradiction as  $\{u, w\}$  is not an edge of G. Similarly, if we orient the edge  $\{u, v\}$  as vu then we are forced to orient the edge  $\{v, w\}$  as vw. That is, in any transitive orientation F of G we have that  $uv \in F \Leftrightarrow wv \in F$ . This forcing operation can be captured by the binary forcing relation  $\Gamma$  which is defined on the edges of a static graph G as follows [24].

$$uv \ \Gamma \ u'v' \quad \text{if and only if} \quad \begin{cases} \text{either } u = u' \text{ and } \{v, v'\} \notin E \\ \text{or } v = v' \text{ and } \{u, u'\} \notin E \end{cases}$$
(1)

We now extend the definition of  $\Gamma$  in a natural way to the binary relation  $\Lambda$  on the edges of a temporal graph  $(G, \lambda)$ , see Equation (2). For this, observe from Table 1 that the only cases, where we have  $uv \in F \Leftrightarrow wv \in F$  in any temporal transitive orientation of  $(G, \lambda)$ , are when (i) the vertices u, v, w induce a path of length 2 (see Figure 1 (a)) and  $\lambda(u, v) = \lambda(v, w)$ , as well as when (ii) u, v, w induce a triangle and  $\lambda(u, w) < \lambda(u, v) = \lambda(v, w)$ . The latter situation is illustrated in the example of Figure 1 (b). The binary forcing relation  $\Lambda$  is only defined on pairs of edges  $\{u, v\}$  and  $\{u', v'\}$  where  $\lambda(u, v) = \lambda(u', v')$ , as follows.

$$uv \wedge u'v' \text{ if and only if } \lambda(u,v) = \lambda(u',v') = t \text{ and } \begin{cases} u = u' \text{ and } \{v,v'\} \notin E, \text{ or} \\ v = v' \text{ and } \{u,u'\} \notin E, \text{ or} \\ u = u' \text{ and } \lambda(v,v') < t, \text{ or} \\ v = v' \text{ and } \lambda(u,u') < t. \end{cases}$$
(2)

set evaluate to different truth values.

#### G.B. Mertzios, H. Molter, M. Renken, P.G. Spirakis and P. Zschoche

Note that, for every edge  $\{u, v\} \in E$  we have that  $uv \Lambda uv$ . The forcing relation  $\Lambda$  for temporal graphs shares some properties with the forcing relation  $\Gamma$  for static graphs. In particular, the reflexive transitive closure  $\Lambda^*$  of  $\Lambda$  is an equivalence relation, which partitions the edges of each set  $E_t = \{\{u, v\} \in E : \lambda(u, v) = t\}$  into its  $\Lambda$ -implication classes (or simply, into its implication classes). Two edges  $\{a, b\}$  and  $\{c, d\}$  are in the same  $\Lambda$ -implication class if and only  $ab \Lambda^* cd$ , i.e. there exists a sequence

$$ab = a_0b_0 \Lambda a_1b_1 \Lambda \ldots \Lambda a_kb_k = cd$$
, with  $k \ge 0$ .

Note that, for this to happen, we must have  $\lambda(a_0, b_0) = \lambda(a_1, b_1) = \ldots = \lambda(a_k, b_k) = t$  for some  $t \ge 1$ . Such a sequence is called a  $\Lambda$ -chain from ab to cd, and we say that ab (eventually)  $\Lambda$ -forces cd. Furthermore note that  $ab \Lambda^* cd$  if and only if  $ba \Lambda^* dc$ . The next observation helps the reader understand the relationship between the two forcing relations  $\Gamma$  and  $\Lambda$ .

▶ Observation 3. Let  $\{u, v\} \in E$ , where  $\lambda(u, v) = t$ , and let A be the  $\Lambda$ -implication class of uv in the temporal graph  $(G, \lambda)$ . Let G' be the static graph obtained by removing from G all edges  $\{p, q\}$ , where  $\lambda(p, q) < t$ . Then A is also the  $\Gamma$ -implication class of uv in the static graph G'.

For the next lemma, we use the notation  $\widehat{A} = \{uv, vu : uv \in A\}.$ 

▶ Lemma 4. Let A be a  $\Lambda$ -implication class of a temporal graph  $(G, \lambda)$ . Then either  $A = A^{-1} = \widehat{A}$  or  $A \cap A^{-1} = \emptyset$ .

**Proof.** Suppose that  $A \cap A^{-1} \neq \emptyset$ , and let  $uv \in A \cap A^{-1}$ , i.e.  $uv, vu \in A$ . Then, for any  $pq \in A$  we have that  $pq \Lambda^* uv$  and  $qp \Lambda^* vu$ . Since  $\Lambda^*$  is an equivalence relation and  $uv, vu \in A$ , it also follows that  $pq, qp \in A$ . Therefore also  $pq, qp \in A^{-1}$ , and thus  $A = A^{-1} = \widehat{A}$ .

▶ Lemma 5. Let F be a temporal transitive orientation of a temporal graph  $(G, \lambda)$ , and let A be a  $\Lambda$ -implication class of  $(G, \lambda)$ . Then either  $A \subseteq F$  or  $A^{-1} \subseteq F$ , and in either case  $A \cap A^{-1} = \emptyset$ .

**Proof.** We defined the binary forcing relation  $\Lambda$  to capture the fact that, for any temporal transitive orientation F of  $(G, \lambda)$ , if  $ab \Lambda cd$  and  $ab \in F$ , then also  $cd \in F$ . Applying this property repeatedly, it follows that either  $A \subseteq F$  or  $F \cap A = \emptyset$ . If  $A \subseteq F$  then  $A^{-1} \subseteq F^{-1}$ . On the other hand, if  $F \cap A = \emptyset$  then  $A \subseteq F^{-1}$ , and thus also  $A^{-1} \subseteq F$ . In either case, the fact that  $F \cap F^{-1} = \emptyset$  by the definition of a temporal transitive orientation implies that also  $A \cap A^{-1} = \emptyset$ .

Let now  $ab = a_0b_0 \Lambda a_1b_1 \Lambda \ldots \Lambda a_kb_k = cd$  be a given  $\Lambda$ -chain. Note by Equation (2) that, for every  $i = 1, \ldots, k$ , we have that either  $a_{i-1} = a_i$  or  $b_{i-1} = b_i$ . Therefore we can replace the  $\Lambda$ -implication  $a_{i-1}b_{i-1} \Lambda a_ib_i$  by the implications  $a_{i-1}b_{i-1} \Lambda a_ib_{i-1} \Lambda a_ib_i$ , since either  $a_ib_{i-1} = a_{i-1}b_{i-1}$  or  $a_ib_{i-1} = a_ib_i$ . Thus, as this addition of this middle edge is always possible in a  $\Lambda$ -implication, we can now define the notion of a canonical  $\Lambda$ -chain, which always exists.

**Definition 6.** Let  $ab \Lambda^* cd$ . Then any  $\Lambda$ -chain of the from

$$ab = a_0b_0 \Lambda a_1b_0 \Lambda a_1b_1 \Lambda \dots \Lambda a_kb_{k-1} \Lambda a_kb_k = cd$$

is a canonical  $\Lambda$ -chain.

The next lemma extends an important known property of the forcing relation  $\Gamma$  for static graphs [24, Lemma 5.3] to the temporal case.

▶ Lemma 7 (Temporal Triangle Lemma). Let  $(G, \lambda)$  be a temporal graph and with a synchronous triangle on the vertices a, b, c, where  $\lambda(a, b) = \lambda(b, c) = \lambda(c, a) = t$ . Let A, B, C be three  $\Lambda$ -implication classes of  $(G, \lambda)$ , where  $ab \in C$ ,  $bc \in A$ , and  $ca \in B$ , where  $A \neq B^{-1}$  and  $A \neq C^{-1}$ .

- 1. If some  $b'c' \in A$ , then  $ab' \in C$  and  $c'a \in B$ .
- **2.** If some  $b'c' \in A$  and  $a'b' \in C$ , then  $c'a' \in B$ .
- **3.** No edge of A touches vertex a.
- **Proof.** 1. Let  $b'c' \in A$ , and let  $bc = b_0c_0 \Lambda b_1c_0 \Lambda \ldots \Lambda b_kc_{k-1} \Lambda b_kc_k = b'c'$  be a canonical  $\Lambda$ -chain from bc to b'c'. Thus note that all edges  $b_ic_{i-1}$  and  $b_ic_i$  of this  $\Lambda$ -chain have the same time-label t in  $(G, \lambda)$ . We will prove by induction that  $ab_i \in C$  and  $c_ia \in B$ , for every  $i = 0, 1, \ldots, k$ . The induction basis follows directly by the statement of the lemma, as  $ab = ab_0 \in C$  and  $ca = c_0a \in B$ .

Assume now that  $ab_i \in C$  and  $c_i a \in B$ . If  $b_{i+1} = b_i$  then clearly  $ab_{i+1} \in C$  by the induction hypothesis. Suppose now that  $b_{i+1} \neq b_i$ . If  $\{a, b_{i+1}\} \notin E$  then  $ac_i \wedge b_{i+1}c_i$ . Then, since  $c_i a \in B$  and  $b_{i+1}c_i \in A$ , it follows that  $A = B^{-1}$ , which is a contradiction to the assumption of the lemma. Therefore  $\{a, b_{i+1}\} \in E$ . Furthermore, since  $b_i c_i \wedge b_{i+1}c_i$ , it follows that either  $\{b_i, b_{i+1}\} \notin E$  or  $\lambda(b_i, b_{i+1}) < t$ . In either case it follows that  $ab_i \wedge ab_{i+1}$ , and thus  $ab_{i+1} \in C$ .

Similarly, if  $c_{i+1} = c_i$  then  $c_{i+1}a \in B$  by the induction hypothesis. Suppose now that  $c_{i+1} \neq c_i$ . If  $\{a, c_{i+1}\} \notin E$  then  $ab_{i+1} \wedge c_{i+1}b_{i+1}$ . Then, since  $ab_{i+1} \in C$  and  $b_{i+1}c_{i+1} \in A$ , it follows that  $A = C^{-1}$ , which is a contradiction to the assumption of the lemma. Therefore  $\{a, c_{i+1}\} \in E$ . Furthermore, since  $b_{i+1}c_i \wedge b_{i+1}c_{i+1}$ , it follows that either  $\{c_i, c_{i+1}\} \notin E$  or  $\lambda(c_i, c_{i+1}) < t$ . In either case it follows that  $c_ia \wedge c_{i+1}a$ , and thus  $c_{i+1}a \in C$ . This completes the induction step.

2. Let  $b'c' \in A$  and  $a'b' \in C$ . Then part 1 of the lemma implies that  $c'a \in B$ . Now let  $ab = a_0b_0 \wedge a_1b_0 \wedge \ldots \wedge a_\ell b_{\ell-1} \wedge a_\ell b_\ell = a'b'$  be a canonical  $\Lambda$ -chain from ab to a'b'. Again, note that all edges  $a_ib_{i-1}$  and  $a_ib_i$  of this  $\Lambda$ -chain have the same time-label t in  $(G, \lambda)$ . We will prove by induction that  $c'a_i \in B$  and  $b_ic' \in A$  for every  $i = 0, 1, \ldots, k$ . First recall that  $c'a = c'a_0 \in B$ . Furthermore, by applying part 1 of the proof to the triangle with vertices  $a_0, b_0, c$  and on the edge  $c'a_0 \in B$ , it follows that  $b_0c' \in A$ . This completes the induction basis.

For the induction step, assume that  $c'a_i \in B$  and  $b_ic' \in A$ . If  $a_{i+1} = a_i$  then clearly  $c'a_{i+1} \in B$ . Suppose now that  $a_{i+1} \neq a_i$ . If  $\{a_{i+1}, c'\} \notin E$  then  $a_{i+1}b_i \wedge c'b_i$ . Then, since  $a_{i+1}b_i \in C$  and  $b_ic' \in A$ , it follows that  $A = C^{-1}$ , which is a contradiction to the assumption of the lemma. Therefore  $\{a_{i+1}, c'\} \in E$ . Now, since  $a_ib_i \wedge a_{i+1}b_i$ , it follows that either  $\{a_i, a_{i+1}\} \notin E$  or  $\lambda(a_i, a_{i+1}) < t$ . In either case it follows that  $c'a_i \wedge c'a_{i+1}$ . Therefore, since  $c'a_i \in B$ , it follows that  $c'a_{i+1} \in B$ .

If  $b_{i+1} = b_i$  then clearly  $b_{i+1}c' \in A$ . Suppose now that  $b_{i+1} \neq b_i$ . Then, since  $c'a_{i+1} \in B$ ,  $a_{i+1}b_i \in C$ , and  $b_ic' \in A$ , we can apply part 1 of the lemma to the triangle with vertices  $a_{i+1}, b_i, c'$  and on the edge  $a_{i+1}b_{i+1} \in C$ , from which it follows that  $b_ic' \in A$ . This completes the induction step, and thus  $c'a_k = c'a' \in B$ .

**3.** Suppose that  $ad \in A$  (resp.  $da \in A$ ), for some vertex d. Then, by setting b' = a and c' = d (resp. b' = d and c' = a), part 1 of the lemma implies that  $ab' = aa \in C$  (resp.  $c'a = aa \in B$ ). Thus is a contradiction, as the underlying graph G does not have the edge aa.

**Deciding temporal transitivity using Boolean satisfiability.** Starting with any undirected edge  $\{u, v\}$  of the underlying graph G, we can clearly enumerate in polynomial time the whole  $\Lambda$ -implication class A to which the oriented edge uv belongs (cf. Equation (2)). If the reversely directed edge  $vu \in A$  then Lemma 4 implies that  $A = A^{-1} = \hat{A}$ . Otherwise, if  $vu \notin A$  then  $vu \in A^{-1}$  and Lemma 4 implies that  $A \cap A^{-1} = \emptyset$ . Thus, we can also decide in polynomial time whether  $A \cap A^{-1} = \emptyset$ . If we encounter at least one  $\Lambda$ -implication class A such that  $A \cap A^{-1} \neq \emptyset$ , then it follows by Lemma 5 that  $(G, \lambda)$  is not temporally transitively orientable.

In the remainder of the section we will assume that  $A \cap A^{-1} = \emptyset$  for every  $\Lambda$ -implication class A of  $(G, \lambda)$ , which is a *necessary* condition for  $(G, \lambda)$  to be temporally transitive orientable. Moreover it follows by Lemma 5 that, if  $(G, \lambda)$  admits a temporally transitively orientation F, then either  $A \subseteq F$  or  $A^{-1} \subseteq F$ . This allows us to define a Boolean variable  $x_A$  for every  $\Lambda$ -implication class A, where  $x_A = \overline{x_{A^{-1}}}$ . Here  $x_A = 1$  (resp.  $x_{A^{-1}} = 1$ ) means that  $A \subseteq F$  (resp.  $A^{-1} \subseteq F$ ), where F is the temporally transitive orientation which we are looking for. Let  $\{A_1, A_2, \ldots, A_s\}$  be a set of  $\Lambda$ -implication classes such that  $\{\widehat{A}_1, \widehat{A}_2, \ldots, \widehat{A}_s\}$ is a partition of the edges of the underlying graph G.<sup>4</sup> Then any truth assignment  $\tau$  of the variables  $x_1, x_2, \ldots, x_s$  (where  $x_i = x_{A_i}$  for every  $i = 1, 2, \ldots, s$ ) corresponds bijectively to one possible orientation of the temporal graph  $(G, \lambda)$ , in which every  $\Lambda$ -implication class is oriented consistently.

Now we define two Boolean formulas  $\phi_{3NAE}$  and  $\phi_{2SAT}$  such that  $(G, \lambda)$  admits a temporal transitive orientation if and only if there is a truth assignment  $\tau$  of the variables  $x_1, x_2, \ldots, x_s$  such that both  $\phi_{3NAE}$  and  $\phi_{2SAT}$  are simultaneously satisfied. Here  $\phi_{3NAE}$  is a 3NAE formula, i.e., the disjunction of clauses with three literals each, where every clause NAE $(\ell_1, \ell_2, \ell_3)$  is satisfied if and only if at least one of the literals  $\{\ell_1, \ell_2, \ell_3\}$  is equal to 1 and at least one of them is equal to 0. Furthermore  $\phi_{2SAT}$  is a 2SAT formula, i.e., the disjunction of 2CNF clauses with two literals each, where every clause  $(\ell_1 \vee \ell_2)$  is satisfied if and only if at least one of the literals  $\{\ell_1, \ell_2, \ell_3\}$  is equal to 1. For simplicity of the presentation we also define a variable  $x_{uv}$  for every directed edge uv. More specifically, if  $uv \in A_i$  (resp.  $uv \in A_i^{-1}$ ) then we set  $x_{uv} = x_i$  (resp.  $x_{uv} = \overline{x_i}$ ). That is,  $x_{uv} = \overline{x_{vu}}$  for every undirected edge  $\{u, v\} \in E$ . Note that, although  $\{x_{uv}, x_{vu} : \{u, v\} \in E\}$  are defined as variables, they can equivalently be seen as *literals* in a Boolean formula over the variables  $x_1, x_2, \ldots, x_s$ . The process of building all  $\Lambda$ -implication classes and all variables  $\{x_{uv}, x_{vu} : \{u, v\} \in E\}$  is given by Algorithm 1.

**Description of the 3NAE formula**  $\phi_{3NAE}$ . Consider a synchronous triangle of  $(G, \lambda)$  on the vertices u, v, w. First observe that, if at least one of the three equalities  $x_{uv} = x_{wv}$ , or  $x_{vw} = x_{uw}$ , or  $x_{wu} = x_{vu}$  is true (or equivalently, if at least one of the pairs  $\{uv, wv\}$ ,  $\{vw, uw\}$ ,  $\{wu, vu\}$  of oriented edges belongs to the same  $\Lambda$ -implication class  $A_i$ ), then the triangle on the vertices u, v, w is never cyclically oriented in any proper orientation F. Therefore, if  $x_{uv} \neq x_{wv}, x_{vw} \neq x_{uw}$ , and  $x_{wu} \neq x_{vu}$ , then we add to  $\phi_{3NAE}$  the clause NAE $(x_{uv}, x_{vw}, x_{wu})$ . Note that the triangle on u, v, w is transitively oriented if and only if NAE $(x_{uv}, x_{vw}, x_{wu})$  is satisfiable, i.e., at least one of the variables  $\{x_{uv}, x_{vw}, x_{wu}\}$  receives the value 1 and at least one of them receives the value 0.

**Description of the 2SAT formula**  $\phi_{2SAT}$ . Consider a triangle of  $(G, \lambda)$  on the vertices u, v, w, where  $\lambda(u, v) = t_1$ ,  $\lambda(v, w) = t_2$ ,  $\lambda(w, v) = t_3$ , and  $t_1 \leq t_2 \leq t_3$ . If  $t_1 < t_2 = t_3$  then

<sup>&</sup>lt;sup>4</sup> Here we slightly abuse the notation by identifying the undirected edge  $\{u, v\}$  with the set of both its orientations  $\{uv, vu\}$ .

Algorithm 1 Building the Λ-implication classes and the edge-variables.

**Input:** A temporal graph  $(G, \lambda)$ , where G = (V, E). **Output:** The variables  $\{x_{uv}, x_{vu} : \{u, v\} \in E\}$ , or the announcement that  $(G, \lambda)$  is temporally not transitively orientable.

1:  $s \leftarrow 0$ ;  $E_0 \leftarrow E$ 2: while  $E_0 \neq \emptyset$  do  $s \leftarrow s + 1$ ; Let  $\{p, q\} \in E_0$  be arbitrary 3: Build the  $\Lambda$ -implication class  $A_s$  of the oriented edge pq (by Equation (2)) 4: if  $qp \in A_s$  then  $\{A_s \cap A_s^{-1} \neq \emptyset\}$ 5:return "NO" 6:7: else  $x_s$  is the variable corresponding to the directed edges of  $A_s$ 8: 9: for every  $uv \in A_s$  do  $x_{uv} \leftarrow x_s; x_{vu} \leftarrow \overline{x_s} \{x_{uv} \text{ and } x_{vu} \text{ become aliases of } x_s \text{ and } \overline{x_s} \}$ 10:11:  $E_0 \leftarrow E_0 \setminus A_s$ 12: **return**  $\Lambda$ -implication classes  $\{A_1, A_2, \ldots, A_s\}$  and variables  $\{x_{uv}, x_{vu} : \{u, v\} \in E\}$ 

we add to  $\phi_{2\text{SAT}}$  the clauses  $(x_{uw} \lor x_{wv}) \land (x_{vw} \lor x_{wu})$ ; note that these clauses are equivalent to  $x_{wu} = x_{wv}$ . If  $t_1 \leq t_2 < t_3$  then we add to  $\phi_{2\text{SAT}}$  the clauses  $(x_{wv} \lor x_{uw}) \land (x_{uv} \lor x_{wu})$ ; note that these clauses are equivalent to  $(x_{vw} \Rightarrow x_{uw}) \land (x_{vu} \Rightarrow x_{wu})$ . Now consider a path of length 2 that is induced by the vertices u, v, w, where  $\lambda(u, v) = t_1, \lambda(v, w) = t_2$ , and  $t_1 \leq t_2$ . If  $t_1 = t_2$  then we add to  $\phi_{2\text{SAT}}$  the clauses  $(x_{vu} \lor x_{wv}) \land (x_{vw} \lor x_{uv})$ ; note that these clauses are equivalent to  $(x_{uv} = x_{wv})$ . Finally, if  $t_1 < t_2$  then we add to  $\phi_{2\text{SAT}}$  the clause  $(x_{vu} \lor x_{wv})$ ; note that this clause is equivalent to  $(x_{uv} \Rightarrow x_{wv})$ .

By slightly abusing the notation, we say that  $\phi_{3NAE} \wedge \phi_{2SAT}$  is *satisfiable* if and only if there exists a truth assignment  $\tau$  which simultaneously satisfies both  $\phi_{3NAE}$  and  $\phi_{2SAT}$ . Given the above definitions of  $\phi_{3NAE}$  and  $\phi_{2SAT}$ , it is easy to check that their clauses model all conditions of the oriented edges imposed by the row of "TTO" in Table 1. Therefore we obtain that the temporal graph  $(G, \lambda)$  is transitively orientable if and only if  $\phi_{3NAE} \wedge \phi_{2SAT}$ is satisfiable.

Although deciding whether  $\phi_{2\text{SAT}}$  is satisfiable can be done in linear time with respect to the size of the formula [6], the problem Not-All-Equal-3-SAT is NP-complete [42]. We overcome this problem and present a polynomial-time algorithm for deciding whether  $\phi_{3\text{NAE}} \wedge \phi_{2\text{SAT}}$  is satisfiable as follows. In the *initialization phase*, we exhaustively check which truth values are *forced* in  $\phi_{3\text{NAE}} \wedge \phi_{2\text{SAT}}$  by using INITIAL-FORCING (see Algorithm 2) as a subroutine. During the execution of INITIAL-FORCING, we either replace the formulas  $\phi_{3\text{NAE}}$  and  $\phi_{2\text{SAT}}$  by the equivalent formulas  $\phi_{3\text{NAE}}^{(0)}$  and  $\phi_{2\text{SAT}}^{(0)}$ , respectively, or we reach a contradiction by showing that  $\phi_{3\text{NAE}} \wedge \phi_{2\text{SAT}}$  is unsatisfiable. Clearly, we have the following.

▶ **Observation 8.** The temporal graph  $(G, \lambda)$  is transitively orientable if and only if  $\phi_{3NAE}^{(0)} \land \phi_{2SAT}^{(0)}$  is satisfiable.

The main phase of the algorithm starts once the formulas  $\phi_{3\text{NAE}}^{(0)}$  and  $\phi_{2\text{SAT}}^{(0)}$  have been computed. During this phase, we iteratively modify the formulas such that, at the end of iteration j we have the formulas  $\phi_{3\text{NAE}}^{(j)}$  and  $\phi_{2\text{SAT}}^{(j)}$ . As we prove,  $\phi_{3\text{NAE}}^{(j-1)} \wedge \phi_{2\text{SAT}}^{(j-1)}$  is satisfiable if and only if  $\phi_{3\text{NAE}}^{(j)} \wedge \phi_{2\text{SAT}}^{(j)}$  is satisfiable. Note that, during the execution of the algorithm, we can both add clauses to and remove clauses from  $\phi_{2\text{SAT}}^{(j)}$ . On the other hand, we can only

#### G.B. Mertzios, H. Molter, M. Renken, P.G. Spirakis and P. Zschoche

remove clauses from  $\phi_{3\text{NAE}}^{(j)}$ .

For the remainder of the section we write  $x_{ab} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}} x_{uv}$  (resp.  $x_{ab} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}^{(j)}} x_{uv}$ ) if the truth assignment  $x_{ab} = 1$  forces the truth assignment  $x_{uv} = 1$  from the clauses of  $\phi_{2SAT}$  (resp. of  $\phi_{2SAT}^{(j)}$  at the iteration j of the algorithm); in this case we say that  $x_{ab}$  implies  $x_{uv}$  in  $\phi_{2SAT}$  (resp. in  $\phi_{2SAT}^{(j)}$ ).

We next introduce the notion of *uncorrelated* triangles, which lets us formulate some important properties of the implications in  $\phi_{2\text{SAT}}$  and  $\phi_{2\text{SAT}}^{(0)}$ .

▶ **Definition 9.** Let u, v, w induce a synchronous triangle in  $(G, \lambda)$ , where each of the variables of the set  $\{x_{uv}, x_{vu}, x_{vw}, x_{wv}, x_{uw}\}$  belongs to a different  $\Lambda$ -implication class. If none of the variables of the set  $\{x_{uv}, x_{vu}, x_{vu}, x_{vw}, x_{wv}, x_{wv}, x_{uw}\}$  implies any other variable of the same set in the formula  $\phi_{2SAT}$  (resp. in the formula  $\phi_{2SAT}^{(0)}$ ), then the triangle of u, v, w is  $\phi_{2SAT}$ -uncorrelated (resp.  $\phi_{2SAT}^{(0)}$ -uncorrelated).

Now we present two crucial technical lemmas which prove some crucial structural properties of the 2SAT formulas  $\phi_{2SAT}$  and  $\phi_{2SAT}^{(0)}$ . These structural properties will allow us to prove the correctness of our main algorithm in this section (Algorithm 4). In a nutshell, these two lemmas guarantee that, whenever we have specific implications in  $\phi_{2SAT}$  (resp. in  $\phi_{2SAT}^{(0)}$ ), then we also have some specific *other* implications in the same formula.

▶ Lemma 10. Let u, v, w induce a synchronous and  $\phi_{2SAT}$ -uncorrelated triangle in  $(G, \lambda)$ , and let  $\{a, b\} \in E$  be an edge of G such that  $\{a, b\} \cap \{u, v, w\} \leq 1$ . If  $x_{ab} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}} x_{uv}$ , then  $x_{ab}$  also implies at least one of the four variables in the set  $\{x_{vw}, x_{wv}, x_{uw}, x_{wu}\}$  in  $\phi_{2SAT}$ .

**Proof.** Let t be the common time-label of all the edges in the synchronous triangle of the vertices u, v, w. That is,  $\lambda(u, v) = \lambda(v, w) = \lambda(w, u) = t$ . Denote by A, B, and C the A-implication classes in which the directed edges uv, vw, and wu belong, respectively. Let  $x_{ab} = x_{a_0b_0} \Rightarrow_{\phi_{2SAT}} x_{a_1b_1} \Rightarrow_{\phi_{2SAT}} \dots \Rightarrow_{\phi_{2SAT}} x_{a_{k-1}b_{k-1}} \Rightarrow_{\phi_{2SAT}} x_{a_kb_k} = x_{uv}$  be a  $\phi_{2SAT}$ -implication chain from  $x_{ab}$  to  $x_{uv}$ . Note that, without loss of generality, for each variable  $x_{a_ib_i}$  in this chain, the directed edge  $a_ib_i$  is a representative of a different  $\Lambda$ -implication class than all other directed edges in the chain (otherwise we can just shorten the  $\phi_{2SAT}$ -implication chain from  $x_{ab}$  to  $x_{uv}$ ). Furthermore, since  $x_{a_kb_k} = x_{uv}$ , note that  $a_kb_k$  and uv are both representatives of the same  $\Lambda$ -implication class A. Therefore Lemma 7 (the temporal triangle lemma) implies that  $wa_k \in C$  and  $b_k w \in B$ . Therefore we can assume without loss of generality that  $u = a_k$  and  $v = b_k$ . Moreover, let  $A' \notin \{A, A^{-1}, B, B^{-1}, C, C^{-1}\}$  be the  $\Lambda$ -implication class in which the directed edge  $a_{k-1}b_{k-1}$  belongs. Since  $x_{a_k-1}b_{k-1} \Rightarrow_{\phi_{2SAT}} x_{a_kb_k}$ , note that without loss of generality we can choose the directed edge  $a_{k-1}b_{k-1} = a_k$  or  $b_{k-1} = b_k$ . We now distinguish these two cases.

**Case 1:**  $u = a_k = a_{k-1}$  and  $v = b_k \neq b_{k-1}$ . Then, since  $x_{a_{k-1}b_{k-1}} = x_{a_kb_{k-1}} \Rightarrow_{\phi_{2SAT}} x_{a_kb_k} = x_{uv}$  and  $\lambda(a_k, b_k) = t$ , it follows that  $\lambda(u, b_{k-1}) \geq t+1$ . Suppose that  $\{w, b_{k-1}\} \notin E$ . Then  $x_{ub_{k-1}} \Rightarrow_{\phi_{2SAT}} x_{uw}$ , which proves the lemma. Now suppose that  $\{w, b_{k-1}\} \in E$ . If  $\lambda(w, b_{k-1}) \leq \lambda(u, b_{k-1}) - 1$  then  $x_{ub_{k-1}} \Rightarrow_{\phi_{2SAT}} x_{uw}$ , which proves the lemma. Suppose that  $\{w, b_{k-1}\} \in E$ . If  $\lambda(w, b_{k-1}) \geq \lambda(u, b_{k-1}) + 1$ . Then  $x_{ub_{k-1}} \Rightarrow_{\phi_{2SAT}} x_{uw}$ , which proves the lemma. Suppose that  $\lambda(w, b_{k-1}) \geq \lambda(u, b_{k-1}) + 1$ . Then  $x_{ub_{k-1}} \Rightarrow_{\phi_{2SAT}} x_{wu}$ , i.e.  $x_{ub_{k-1}} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}} x_{wu}$ , which again proves the lemma. Suppose finally that  $\lambda(w, b_{k-1}) = \lambda(u, b_{k-1})$ . Then, since  $\lambda(u, w) = t < \lambda(w, b_{k-1}) = \lambda(u, b_{k-1})$ , it follows that  $wb_{k-1} \Lambda ub_{k-1}$ . If  $\{v, b_{k-1}\} \notin E$  then  $x_{ub_{k-1}} = x_{wb_{k-1}} \Rightarrow_{\phi_{2SAT}} x_{wv}$ , which proves the lemma. Now let  $\{v, b_{k-1}\} \in E$ . If  $\lambda(v, b_{k-1}) \leq \lambda(w, b_{k-1}) - 1$  then  $x_{ub_{k-1}} = x_{wb_{k-1}} \Rightarrow_{\phi_{2SAT}} x_{wv}$ , which proves the lemma. If  $\lambda(v, b_{k-1}) \geq \lambda(w, b_{k-1}) + 1$  then  $x_{ub_{k-1}} = x_{wb_{k-1}} \Rightarrow_{\phi_{2SAT}} x_{vb_{k-1}} \Rightarrow_{\phi_{2SAT}} x_{wv}$ , which

proves the lemma. If  $\lambda(v, b_{k-1}) = \lambda(w, b_{k-1})$  then  $ub_{k-1} \Lambda vb_{b-1}$ , and thus  $x_{ub_{k-1}} = x_{a_{k-1}b_{k-1}} \not\Rightarrow_{\phi_{2SAT}} x_{a_kb_k} = x_{uv}$ , which is a contradiction.

**Case 2:**  $u = a_k \neq a_{k-1}$  and  $v = b_k = b_{k-1}$ . Then, since  $x_{a_{k-1}b_{k-1}} = x_{a_{k-1}b_k} \Rightarrow_{\phi_{2SAT}} x_{a_kb_k} = x_{uv}$  and  $\lambda(a_k, b_k) = t$ , it follows that  $\lambda(v, a_{k-1}) \leq t-1$ . Suppose that  $\{w, a_{k-1}\} \notin E$ . Then  $x_{a_{k-1}v} \Rightarrow_{\phi_{2SAT}} x_{wv}$ , which proves the lemma. Now suppose that  $\{w, a_{k-1}\} \in E$ . If  $\lambda(w, a_{k-1}) \leq t-1$  then  $x_{a_{k-1}v} \Rightarrow_{\phi_{2SAT}} x_{wv}$ , which proves the lemma. Suppose that  $\lambda(w, a_{k-1}) = t$ . Then, since  $\lambda(v, a_{k-1}) \leq t-1$ , it follows that  $vw \land a_{t-1}w$ . If  $\{u, a_{k-1}\} \notin E$ then also  $a_{t-1}w \land uw$ , and thus  $x_{wv} = x_{wu}$ , which is a contradiction to the assumption that the triangle of u, v, w is uncorrelated. Thus  $\{u, a_{k-1}\} \in E$ . If  $\lambda(u, a_{k-1}) \leq t-1$  then again  $a_{t-1}w \land uw$ , which is a contradiction. On the other hand, if  $\lambda(u, a_{k-1}) \geq t$  then  $x_{a_{k-1}v} = x_{a_{k-1}b_{k-1}} \Rightarrow_{\phi_{2SAT}} x_{a_kb_k} = x_{uv}$ , which is a contradiction.

Finally suppose that  $\lambda(w, a_{k-1}) \ge t+1$ . Then, since  $\lambda(v, w) = t$  and  $\lambda(v, a_{k-1}) \le t-1$ , it follows that  $x_{vw} \Rightarrow_{\phi_{2SAT}} x_{a_{k-1}w} \Rightarrow_{\phi_{2SAT}} x_{a_{k-1}v}$ . However, since  $x_{a_{k-1}v} = x_{a_{k-1}b_k} \Rightarrow_{\phi_{2SAT}} x_{a_{k}b_k} = x_{uv}$ , it follows that  $x_{vw} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}} x_{uv}$ , which is a contradiction to the assumption that the triangle of u, v, w is uncorrelated.

▶ Lemma 11. Let u, v, w induce a synchronous and  $\phi_{2SAT}^{(0)}$ -uncorrelated triangle in  $(G, \lambda)$ , and let  $\{a, b\} \in E$  be an edge of G such that  $\{a, b\} \cap \{u, v, w\} \leq 1$ . If  $x_{ab} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}^{(0)}} x_{uv}$ , then  $x_{ab}$  also implies at least one of the four variables in the set  $\{x_{vw}, x_{wv}, x_{uw}, x_{wu}\}$  in  $\phi_{2SAT}^{(0)}$ . **Proof.** Assume we have  $\{a, b\} \cap \{u, v, w\} \leq 1$  and  $x_{ab} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}^{(0)}} x_{uv}$ . Then we make a case distinction on the last implication in the implication chain  $x_{ab} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}^{(0)}} x_{uv}$ .

- 1. The last implication is an implication from  $\phi_{2\text{SAT}}$ , i.e.,  $x_{ab} \stackrel{*}{\Rightarrow}_{\phi_{2\text{SAT}}^{(0)}} x_{pq} \stackrel{*}{\Rightarrow}_{\phi_{2\text{SAT}}} x_{uv}$ . If  $\{p,q\} \subseteq \{u,v,w\}$  then we are done, since we can assume that  $\{p,q\} \neq \{u,v\}$  because no such implications are contained in  $\phi_{2\text{SAT}}$ . Otherwise Lemma 10 implies that  $x_{pq}$  also implies at least one of the four variables in the set  $\{x_{vw}, x_{wv}, x_{uw}, x_{wu}\}$  in  $\phi_{2\text{SAT}}$ . If follows that  $x_{ab}$  also implies at least one of the four variables in the set  $\{x_{vw}, x_{wv}, x_{uw}, x_{uw}, x_{uw}, x_{uw}, x_{wu}\}$  in  $\phi_{2\text{SAT}}^{(0)}$ .
- 2. The last implication is *not* an implication from  $\phi_{2\text{SAT}}$ , i.e.,  $x_{ab} \stackrel{*}{\Rightarrow}_{\phi_{2\text{SAT}}^{(0)}} x_{pq} \Rightarrow_{\phi_{1\text{NIT}}} x_{uv}$ , there the implication  $x_{pq} \Rightarrow_{\phi_{1\text{NIT}}} x_{uv}$  was added to  $\phi_{2\text{SAT}}^{(0)}$  by INITIAL-FORCING. If  $x_{pq} \Rightarrow_{\phi_{1\text{NIT}}} x_{uv}$  was added in Line 7 or Line 10 of INITIAL-FORCING, then we have that  $\{p,q\} \subseteq \{u,v,w\}$  and  $\{p,q\} \neq \{u,v\}$ , hence the u,v,w is not a  $\phi_{2\text{SAT}}^{(0)}$ -uncorrelated triangle, a contradiction. If  $x_{pq} \Rightarrow_{\phi_{1\text{NIT}}} x_{uv}$  was added in Line 14 of INITIAL-FORCING, then we have that  $x_{pq} \Rightarrow_{\phi_{1\text{NIT}}} x_{uw}$ , hence we are done.

**The algorithm.** We are now ready to present our polynomial-time algorithm (Algorithm 4) for deciding whether a given temporal graph  $(G, \lambda)$  is temporally transitively orientable. The main idea of our algorithm is as follows. First, the algorithm computes all  $\Lambda$ -implication classes  $A_1, \ldots, A_s$  by calling Algorithm 1 as a subroutine. If there exists at least one  $\Lambda$ -implication class  $A_i$  where  $uv, vu \in A_i$  for some edge  $\{u, v\} \in E$ , then we announce that  $(G, \lambda)$  is a *no*-instance, due to Lemma 5. Otherwise we associate to each  $\Lambda$ -implication class  $A_i$  a variable  $x_i$ , and we build the 3NAE formula  $\phi_{3NAE}$  and the 2SAT formula  $\phi_{2SAT}$ , as described in Section 3.1.

In the *initialization phase* of Algorithm 4 we build the formulas  $\phi_{3\text{NAE}}^{(0)}$  and  $\phi_{2\text{SAT}}^{(0)}$  by checking which truth values are being *forced* in  $\phi_{3\text{NAE}} \wedge \phi_{2\text{SAT}}$ . Furthermore, we if  $x \Rightarrow_{\phi_{2\text{SAT}}^{(0)}} a$  and  $x \Rightarrow_{\phi_{2\text{SAT}}^{(0)}} b$ , and we have that  $\text{NAE}(a, b, c) \in \phi_{3\text{NAE}}^{(0)}$ , then we add  $x \Rightarrow_{\phi_{2\text{SAT}}^{(0)}} \overline{c}$  to  $\phi_{2\text{SAT}}^{(0)}$ ,

since clearly, if x = 1 then a = b = 1 and we have to set c = 0 to satisfy the NAE clause NAE(a, b, c). This is done by calling algorithm INITIAL-FORCING (see Algorithm 2) as a subroutine. The main idea of INITIAL-FORCING is that, whenever setting  $x_i = 1$  (resp.  $x_i = 0$ ) forces  $\phi_{3NAE} \wedge \phi_{2SAT}$  to become unsatisfiable, we choose to set  $x_i = 0$  (resp.  $x_i = 1$ ).

The main phase of the algorithm starts once the formulas  $\phi_{3NAE}^{(0)}$  and  $\phi_{2SAT}^{(0)}$  have been computed. Then we iteratively try assigning to each variable  $x_i$  the truth value 1 or 0. Once we have set  $x_i = 1$  (resp.  $x_i = 0$ ) during the iteration  $j \ge 1$  of the algorithm, we call algorithm BOOLEAN-FORCING (see Algorithm 3) as a subroutine to check which implications this value of  $x_i$  has on the current formulas  $\phi_{3NAE}^{(j-1)}$  and  $\phi_{2SAT}^{(j-1)}$  and which other truth values of variables are forced. The correctness of BOOLEAN-FORCING can be easily verified by checking all subcases of BOOLEAN-FORCING. During the execution of BOOLEAN-FORCING, we either replace the current formulas by  $\phi_{3NAE}^{(j)}$  and  $\phi_{2SAT}^{(j)}$ , or we reach a contradiction by showing that, setting  $x_i = 1$  (resp.  $x_i = 0$ ) makes  $\phi_{3NAE}^{(j-1)} \wedge \phi_{2SAT}^{(j-1)}$  unsatisfiable. If each of the truth assignments { $x_i = 1, x_i = 0$ } leads to such a contradiction, we return that  $(G, \lambda)$  is a *no*-instance. Otherwise, if at least one of the truth assignments { $x_i = 1, x_i = 0$ } does not lead to such a contradiction, we follow this truth assignment and proceed with the next variable. We prove that, if we manage to sequentially assign to each variable a truth value without reaching a contradiction, the resulting truth assignment of  $x_1, \ldots, x_s$  naturally defines a temporal transitive orientation F of  $(G, \lambda)$ .

We now prove formally that Algorithm 4 is correct. More specifically, we show that if Algorithm 4 gets a *yes*-instance as input then it outputs a temporally transitive orientation, while if it gets a *no*-instance as input then it outputs "NO". We start by proving in the following auxiliary lemma that, if the algorithm returns "NO" at the *j*th iteration, then the formula  $\phi_{3NAE}^{(j-1)} \wedge \phi_{2SAT}^{(j-1)}$  of the previous iteration is not satisfiable.

▶ Lemma 12. For every iteration j of Algorithm 4, if Algorithm 4 returns "NO" in Line 16, then  $\phi_{3NAE}^{(j-1)} \wedge \phi_{2SAT}^{(j-1)}$  is not satisfiable.

**Proof.** Assume otherwise that  $\phi_{3\text{NAE}}^{(j-1)} \wedge \phi_{2\text{SAT}}^{(j-1)}$  is satisfiable, and let  $x_i$  be the variable of  $\phi_{3\text{NAE}}^{(j-1)} \wedge \phi_{2\text{SAT}}^{(j-1)}$  that is considered by the algorithm at iteration j. Let  $\tau$  be a satisfying truth assignment of  $\phi_{3\text{NAE}}^{(j-1)} \wedge \phi_{2\text{SAT}}^{(j-1)}$ . If  $x_i = 1$  (resp.  $x_i = 0$ ) in  $\tau$  then the algorithm will proceed by computing the next formula  $\phi_{3\text{NAE}}^{(j)} \wedge \phi_{2\text{SAT}}^{(j)}$  in Line 11 (resp. in Line 14) and thus it will not return "NO" in Line 16, which is a contradiction.

The next crucial observation follows immediately by the construction of  $\phi_{3\text{NAE}}$  in Section 3.1, and by the fact that, at every iteration j, Algorithm 4 can only remove clauses from  $\phi_{3\text{NAE}}^{(j-1)}$ .

▶ **Observation 13.** If Algorithm 3 removes a clause from  $\phi_{3NAE}^{(j-1)}$ , then this clause is satisfied for all satisfying assignments of  $\phi_{2SAT}^{(j)}$ .

Next, we prove a crucial insight about the Boolean forcing steps in the main phase of Algorithm 4. This will allow us to assume that whenever a clause is added to the 2SAT part of the formula in the main phase of the algorithm, then we have set one variable of a NAE clause to some truth value.

▶ Lemma 14. Whenever BOOLEAN-FORCING (Algorithm 3) is called in an iteration  $j \ge 1$  (i.e. in the main phase) of Algorithm 4, Lines 11 and 12 of BOOLEAN-FORCING are not executed if this call of BOOLEAN-FORCING does not output "NO".

# **Algorithm 2** INITIAL-FORCING

**Input:** A 2-SAT formula  $\phi_{2SAT}$  and a 3-NAE formula  $\phi_{3NAE}$ **Output:** A 2-SAT formula  $\phi_{2SAT}^{(0)}$  and a 3-NAE formula  $\phi_{3NAE}^{(0)}$  such that  $\phi_{2SAT}^{(0)} \wedge \phi_{3NAE}^{(0)}$ is satisfiable if and only if  $\phi_{2SAT} \wedge \phi_{3NAE}$  is satisfiable, or the announcement that  $\phi_{2SAT} \wedge \phi_{3NAE}$  is not satisfiable. 1:  $\phi_{3\text{NAE}}^{(0)} \leftarrow \phi_{3\text{NAE}}; \quad \phi_{2\text{SAT}}^{(0)} \leftarrow \phi_{2\text{SAT}} \text{ {initialization}}$ 2: for every variable  $x_i$  appearing in  $\phi_{3NAE}^{(0)} \wedge \phi_{2SAT}^{(0)}$  do if BOOLEAN-FORCING  $\left(\phi_{3\text{NAE}}^{(0)}, \phi_{2\text{SAT}}^{(0)}, x_i, 1\right) = \text{"NO" then}$ 3: if BOOLEAN-FORCING  $\left(\phi_{3\text{NAE}}^{(0)}, \phi_{2\text{SAT}}^{(0)}, x_i, 0\right) = \text{"NO" then}$ 4: **return** "NO" {both  $x_i = 1$  and  $x_i = 0$  invalidate the formulas} 5: else 6:  $\left(\phi_{3\text{NAE}}^{(0)}, \phi_{2\text{SAT}}^{(0)}\right) \leftarrow \text{Boolean-Forcing}\left(\phi_{3\text{NAE}}^{(0)}, \phi_{2\text{SAT}}^{(0)}, x_i, 0\right)$ 7: else 8: if BOOLEAN-FORCING  $\left(\phi_{3\text{NAE}}^{(0)}, \phi_{2\text{SAT}}^{(0)}, x_i, 0\right) = \text{"NO" then}$ 9:  $\left(\phi_{3\text{NAE}}^{(0)}, \phi_{2\text{SAT}}^{(0)}\right) \leftarrow \text{Boolean-Forcing}\left(\phi_{3\text{NAE}}^{(0)}, \phi_{2\text{SAT}}^{(0)}, x_i, 1\right)$ 10: 11: for every clause NAE $(x_{uv}, x_{vw}, x_{wu})$  of  $\phi_{3NAE}^{(0)}$  do for every variable  $x_{ab}$  do 12:if  $x_{ab} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}^{(0)}} x_{uv}$  and  $x_{ab} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}^{(0)}} x_{vw}$  then {add  $(x_{ab} \Rightarrow x_{uw})$  to  $\phi_{2SAT}^{(0)}$ } 13:  $\phi_{2\text{SAT}}^{(0)} \leftarrow \phi_{2\text{SAT}}^{(0)} \land (x_{ba} \lor x_{uw})$ 14: 15: Repeat lines 2 and 11 until no changes occur on  $\phi_{2\text{SAT}}^{(0)}$  and  $\phi_{3\text{NAE}}^{(0)}$ 16: **return**  $\left(\phi_{3\text{NAE}}^{(0)}, \phi_{2\text{SAT}}^{(0)}\right)$ 

**Proof.** Assume for contradiction that Lines 11 and 12 of Algorithm 3 are executed in iteration j of Algorithm 4 and Algorithm 3 does not output "NO". Let j be the first iteration where this happens. This means that there is a clause NAE $(x_{uv}, x_{vw}, x_{wu})$  of  $\phi'_3$  and an implication  $x_{uv} \stackrel{*}{\Rightarrow} \phi'_2 x_{vw}$  during the execution of Algorithm 3. Let NAE $(x_{uv}, x_{vw}, x_{wu})$  and  $x_{uv} \stackrel{*}{\Rightarrow} \phi'_2 x_{vw}$  appear in the first execution of Lines 11 and 12 of Algorithm 3.

We first partition the implication chain  $x_{uv} \stackrel{*}{\Rightarrow} \phi'_2 x_{vw}$  into "old" and "new" implications, where "old" implications are contained in  $\phi^{(0)}_{2\text{SAT}}$  and all other implications (that were added in previous iterations) are considered "new". If there are several NAE clauses and implication chains that fulfill the condition in Line 9 of Algorithm 3, we assume that  $x_{uv} \stackrel{*}{\Rightarrow} \phi'_2 x_{vw}$  is one that contains a minimum number of "new" implications. Observe that since we assume  $x_{uv} \stackrel{*}{\Rightarrow} \phi'_2 x_{vw}$  is a condition for the first execution of Lines 11 and 12 of Algorithm 3, it follows that all "new" implications in  $x_{uv} \stackrel{*}{\Rightarrow} \phi'_2 x_{vw}$  were added in Line 17 or Line 19 of Algorithm 3 in previous iterations.

Note that by definition of  $\phi_{2\text{SAT}}^{(0)}$ , we know that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$  contains at least one "new" implication. Furthermore, we can observe that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$  contains at least two

**Algorithm 3** BOOLEAN-FORCING

**Input:** A 2-SAT formula  $\phi_2$ , a 3-NAE formula  $\phi_3$ , and a variable  $x_i$  of  $\phi_2 \wedge \phi_3$ , and a truth value VALUE  $\in \{0, 1\}$ 

**Output:** A 2-SAT formula  $\phi'_2$  and a 3-NAE formula  $\phi'_3$ , obtained from  $\phi_2$  and  $\phi_3$  by setting  $x_i = \text{VALUE}$ , or the announcement that  $x_i = \text{VALUE}$  does not satisfy  $\phi_2 \wedge \phi_3$ .

1:  $\phi_2' \leftarrow \phi_2$ ;  $\phi_3' \leftarrow \phi_3$ 

2: while  $\phi'_2$  has a clause  $(x_{uv} \lor x_{pq})$  and  $x_{uv} = 1$  do

3: Remove the clause  $(x_{uv} \lor x_{pq})$  from  $\phi'_2$ 

4: while  $\phi'_2$  has a clause  $(x_{uv} \lor x_{pq})$  and  $x_{uv} = 0$  do

5: **if**  $x_{pq} = 0$  **then return** "NO"

6: Remove the clause  $(x_{uv} \lor x_{pq})$  from  $\phi'_2$ ;  $x_{pq} \leftarrow 1$ 

7: for every variable  $x_{uv}$  that does not yet have a truth value do

8: **if**  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi_2''} x_{vu}$ , where  $\phi_2'' = \phi_2' \setminus \phi_2$  **then**  $x_{uv} \leftarrow 0$ 

9: for every clause NAE $(x_{uv}, x_{vw}, x_{wu})$  of  $\phi'_3$  do {synchronous triangle on vertices u, v, w}

10: **if**  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$  **then** {add  $(x_{uv} \Rightarrow x_{uw}) \land (x_{uw} \Rightarrow x_{vw})$  to  $\phi'_2$ }

- 11:  $\phi'_2 \leftarrow \phi'_2 \land (x_{vu} \lor x_{uw}) \land (x_{wu} \lor x_{vw})$
- 12: Remove the clause NAE $(x_{uv}, x_{vw}, x_{wu})$  from  $\phi'_3$
- 13: **if**  $x_{uv}$  already got the value 1 or 0 **then**
- 14: Remove the clause NAE $(x_{uv}, x_{vw}, x_{wu})$  from  $\phi'_3$

15: **if**  $x_{vw}$  and  $x_{wu}$  do not have yet a truth value **then** 

16: **if**  $x_{uv} = 1$  **then** {add  $(x_{vw} \Rightarrow x_{uw})$  to  $\phi'_2$ }

17:  $\phi'_2 \leftarrow \phi'_2 \land (x_{wv} \lor x_{uw})$ 

18: **else** { $x_{uv} = 0$ ; in this case add ( $x_{uw} \Rightarrow x_{vw}$ ) to  $\phi'_2$ }

19:  $\phi'_2 \leftarrow \phi'_2 \land (x_{wu} \lor x_{vw})$ 

```
20: if x_{vw} = x_{uv} and x_{wu} does not have yet a truth value then
```

```
21: \qquad x_{wu} \leftarrow 1 - x_{uv}
```

```
22: if x_{vw} = x_{wu} = x_{uv} then return "NO"
```

23: Repeat lines 2, 4, 7, and 9 until no changes occur on  $\phi_2'$  and  $\phi_3'$ 

24: if both  $x_{uv} = 0$  and  $x_{uv} = 1$  for some variable  $x_{uv}$  then return "NO"

25: return  $(\phi'_2, \phi'_3)$ 

implications overall.

We first consider the case that  $x_{uv} \stackrel{*}{\Rightarrow} \phi'_2 x_{vw}$  contains at least one "old" implication. We assume w.l.o.g. that  $x_{uv} \stackrel{*}{\Rightarrow} \phi'_2 x_{vw}$  contains an "old" implication that is directly followed by a "new" implication (if this is not the case, then we can consider the contraposition of the implication chain).

Note that since the "new" implication was added in Line 17 or Line 19 of Algorithm 3, we can assume w.l.o.g that the "new" implication is  $x_{ab} \Rightarrow_{BF} x_{cb}$  and that  $x_{ca} = 1$  for some synchronous triangle on the vertices a, b, c, that is, we have  $NAE(x_{ab}, x_{bc}, x_{ca}) \in \phi_{3NAE}^{(0)}$  (this is the Line 17 case, Line 19 works analogously). Let  $x_{pq} \Rightarrow_{\phi_{2SAT}}^{(0)} x_{ab}$  be the "old" implication.

**Algorithm 4** Temporal transitive orientation. **Input:** A temporal graph  $(G, \lambda)$ , where G = (V, E). **Output:** A temporal transitive orientation F of  $(G, \lambda)$ , or the announcement that  $(G, \lambda)$  is temporally not transitively orientable. 1: Execute Algorithm 1 to build the  $\Lambda$ -implication classes  $\{A_1, A_2, \ldots, A_s\}$  and the Boolean variables  $\{x_{uv}, x_{vu} : \{u, v\} \in E\}$ 2: if Algorithm 1 returns "NO" then return "NO" 3: Build the 3NAE formula  $\phi_{3NAE}$  and the 2SAT formula  $\phi_{2SAT}$ 4: **if** INITIAL-FORCING  $(\phi_{3NAE}, \phi_{2SAT}) \neq$  "NO" **then** {Initialization phase}  $\left(\phi_{3\text{NAE}}^{(0)}, \phi_{2\text{SAT}}^{(0)}\right) \leftarrow \text{INITIAL-FORCING}\left(\phi_{3\text{NAE}}, \phi_{2\text{SAT}}\right)$ 5:6: else { $\phi_{3NAE} \land \phi_{2SAT}$  leads to a contradiction} return "NO" 7: 8:  $j \leftarrow 1$ ;  $F \leftarrow \emptyset$  {Main phase} 9: while a variable  $x_i$  appearing in  $\phi_{3NAE}^{(j-1)} \wedge \phi_{2SAT}^{(j-1)}$  did not yet receive a truth value do if BOOLEAN-FORCING  $\left(\phi_{3\text{NAE}}^{(j-1)}, \phi_{2\text{SAT}}^{(j-1)}, x_i, 1\right) \neq \text{``NO''}$  then 10:  $\left(\phi_{3\text{NAE}}^{(j)}, \phi_{2\text{SAT}}^{(j)}\right) \leftarrow \text{Boolean-Forcing}\left(\phi_{3\text{NAE}}^{(j-1)}, \phi_{2\text{SAT}}^{(j-1)}, x_i, 1\right)$ 11: else { $x_i = 1$  leads to a contradiction} 12:if BOOLEAN-FORCING  $\left(\phi_{3\text{NAE}}^{(j-1)}, \phi_{2\text{SAT}}^{(j-1)}, x_i, 0\right) \neq \text{``NO''}$  then 13: $\left(\phi_{3\text{NAE}}^{(j)}, \phi_{2\text{SAT}}^{(j)}\right) \leftarrow \text{Boolean-Forcing}\left(\phi_{3\text{NAE}}^{(j-1)}, \phi_{2\text{SAT}}^{(j-1)}, x_i, 0\right)$ 14:else 15:return "NO" 16: $j \leftarrow j + 1$ 17:18: for i = 1 to s do if  $x_i$  did not yet receive a truth value then  $x_i \leftarrow 1$ 19:if  $x_i = 1$  then  $F \leftarrow F \cup A_i$  else  $F \leftarrow F \cup \overline{A_i}$ 20:21: return the temporal transitive orientation F of  $(G, \lambda)$ 

Then we have that  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{ab} \Rightarrow_{BF} x_{cb}$  is contained in  $x_{uv} \Rightarrow_{\phi'_2} x_{vw}$ . Furthermore, by definition of  $\phi_{2SAT}^{(0)}$ , we have that  $|\{p,q\} \cap \{a,b,c\}| \leq 1$ , hence we can apply Lemma 11 and obtain one of the following four scenarios:

1.  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{cb}$ : In this case we can replace  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{ab} \Rightarrow_{BF} x_{cb}$  with  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{cb}$  in the implication chain  $x_{uv} \stackrel{\circ}{\Rightarrow}_{\phi'_2} x_{vw}$  to obtain an implication chain from  $x_{uv}$  to  $x_{vw}$  with strictly fewer "new" implications, a contradiction.

2.  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{bc}$ :

Now we have that  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{ab}$  and  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{bc}$ . Then by definition of  $\phi_{2SAT}^{(0)}$  we also have that  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{ac}$  and hence  $x_{ca} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{qp}$ . Recall that we know that  $x_{ca} = 1$ . It follows that  $x_{pq} = 0$  in iteration j, a contradiction to the assumption that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$ 

exists.

 x<sub>pq</sub>⇒<sub>φ<sup>(0)</sup><sub>2SAT</sub></sub> x<sub>ca</sub>: Now we have that x<sub>pq</sub>⇒<sub>φ<sup>(0)</sup><sub>2SAT</sub></sub> x<sub>ab</sub> and x<sub>pq</sub>⇒<sub>φ<sup>(0)</sup><sub>2SAT</sub></sub> x<sub>ca</sub>. Then by definition of φ<sup>(0)</sup><sub>2SAT</sub> we also have that x<sub>pq</sub>⇒<sub>φ<sup>(0)</sup><sub>2SAT</sub></sub> x<sub>cb</sub>. From here it is the same as case 1.
x<sub>pq</sub>⇒<sub>φ<sup>(0)</sup><sub>2SAT</sub></sub> x<sub>ac</sub>: Same as case 2.

Hence, we have a contradiction in every case and can conclude that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$  does not contain any "old" implications.

Now consider the case that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$  only contains "new" implications. We first assume that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$  contains exactly two "new" implications. Then the first implication is either  $x_{uv} \Rightarrow_{BF} x_{wv}$  or  $x_{uv} \Rightarrow_{BF} x_{uw}$ . Note that we cannot add the implication  $x_{wv} \Rightarrow x_{vw}$  in BOOLEAN-FORCING, hence the first implication has to be  $x_{uv} \Rightarrow_{BF} x_{uw}$  which implies that  $x_{vw} = 1$ , which is a contradiction to the existence of the implication chain  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$ .

From now on we assume that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$  contains strictly more than two implications. Consider two consecutive BF implications in  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$ . Denote these two implications by  $x_{ab} \Rightarrow_{BF} x_{cd}$  and  $x_{cd} \Rightarrow_{BF} x_{fg}$ . By the BOOLEAN-FORCING algorithm, we have that either b = d or a = c, and in both cases the edges ab and cd belong to a synchronous triangle. Suppose that b = d (the case a = c can be treated analogously), i.e., we have the implication  $x_{ab} \Rightarrow_{BF} x_{cb}$ . Let a, b, c be the vertices of the synchronous triangle for this implication. Similarly, for the implication  $x_{cd} = x_{cb} \Rightarrow_{BF} x_{fg}$  we know that either b = gor c = f. Suppose that b = g (the case c = f can be treated analogously), i.e., we have the implication  $x_{cb} \Rightarrow_{BF} x_{fb}$ . Let f', c', b' be the vertices of the synchronous triangle for this implication; that is, the edges cb and c'b' are both representatives of the  $\Lambda$ -implication class of the variable  $x_{cb}$ . Therefore Lemma 7 (the temporal triangle lemma) implies that ab'(resp. ac') exists in the graph and belongs to the same  $\Lambda$ -implication class of ab (resp. ac). Therefore we can assume without loss of generality that b = b' and c = c'. Summarizing, we have a synchronous triangle on the vertices a, b, c and another synchronous triangle b, c, f'. For convenience of the presentation, in the remainder of the proof we rename f' to d; that is, the two synchronous triangles from the two consecutive BF implications are on the vertices a, b, c and b, c, d, respectively. Note that both of these triangles must be also synchronous to each other, i.e., all their edges have the same time label t, see Figure 2 (a).

We now have the following cases for the two consecutive implications:

(A)  $x_{ab} \Rightarrow_{BF} x_{cb} \Rightarrow_{BF} x_{cd}$  is contained in  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$  and  $x_{ca} = 1$  and  $x_{bd} = 1$  (Figure 2 (b)).

(B)  $x_{ca} \Rightarrow_{BF} x_{cb} \Rightarrow_{BF} x_{cd}$  is contained in  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$  and  $x_{ab} = 1$  and  $x_{bd} = 1$  (Figure 2 (c)). All other cases are symmetric to one of the two cases above. We now make a case-distinction on the (possibly missing) edge  $\{a, d\}$  (dash-dotted green line in Figure 2). Cf. Table 1 for the following cases.

- 1.  $\{a, d\}$  is a non-edge or  $\lambda(a, d) < t$ :
  - (A) In this case  $\phi_{2\text{SAT}}^{(0)}$  by definition then contains  $x_{bd} \Rightarrow_{\phi_{2\text{SAT}}^{(0)}} x_{ba}$ . Hence, we have that  $x_{ab} = 0$ , a contradiction to the assumption that  $x_{ab} \Rightarrow_{\text{BF}} x_{cb} \Rightarrow_{\text{BF}} x_{cd}$  is contained in  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$ .
  - (B) Contradiction since  $\phi_{2\text{SAT}}^{(0)}$  by definition then contains  $x_{ab} \Rightarrow_{\phi_{\alpha\beta,T}^{(0)}} x_{db}$ .
- **2.**  $\lambda(a,d) > t$ :
  - (A) In this case  $\phi_{2\text{SAT}}^{(0)}$  by definition then contains  $x_{ca} \Rightarrow_{\phi_{2\text{SAT}}^{(0)}} x_{da}$  and  $x_{bd} \Rightarrow_{\phi_{2\text{SAT}}^{(0)}} x_{ad}$ , a contradiction.
  - (B) In this case we know that  $x_{ad} = 1$ , since by definition  $\phi_{2\text{SAT}}^{(0)}$  then contains  $x_{bd} \Rightarrow_{\phi_{2\text{SAT}}^{(0)}} x_{ad}$ . Furthermore,  $\phi_{2\text{SAT}}^{(0)}$  by definition then contains  $x_{ad} \Rightarrow_{\phi_{2\text{SAT}}^{(0)}} x_{ac}$  and



**Figure 2** Illustration of the scenario of two consecutive "new" BF implications (a) appearing in the proof of Lemma 14. The green dash-dotted line indicates that edge  $\{a, d\}$  may exist with some time label or not. The proof makes a case distinction here. Subfigure (b) illustrates the case that  $x_{ab} \Rightarrow_{BF} x_{cb} \Rightarrow_{BF} x_{cd}$  and  $x_{ca} = x_{bd} = 1$ , indicated by the red arrows. Subfigure (c) illustrates the case that  $x_{ca} \Rightarrow_{BF} x_{cb} \Rightarrow_{BF} x_{cd}$  and  $x_{ab} = x_{bd} = 1$ , indicated by the red arrows.

hence we have  $x_{ca} = 0$ , a contradiction to the assumption that  $x_{ca} \Rightarrow_{BF} x_{cb} \Rightarrow_{BF} x_{cd}$ is contained in  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_{a}} x_{vw}$ .

**3.**  $\lambda(a, d) = t$ :

Note that the above two cases do not apply, we can assume that all pairs of consecutive implication appearing in  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$  fall into this case. In particular, also the first one. Hence, we have that  $x_{uv} \Rightarrow_{BF} x_{pv} \stackrel{*}{\Rightarrow}_{BF} x_{vw}$  or  $x_{uv} \Rightarrow_{BF} x_{up} \stackrel{*}{\Rightarrow}_{BF} x_{vw}$ .

Assume that  $x_{uv} \Rightarrow_{BF} x_{pv} \stackrel{*}{\Rightarrow}_{BF} x_{vw}$ . Then in particular, using Lemma 7 (the temporal triangle lemma) similarly as described above, we get that vertices p, v, w induce a synchronous triangle and  $NAE(x_{pv}, x_{vw}, x_{wp}) \in \phi_{3NAE}^{(0)}$ . Hence,  $x_{pv} \stackrel{*}{\Rightarrow}_{BF} x_{vw}$  is an implication chain that fulfills the condition in Line 9 but contains less "new" implication than  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$ , a contradiction.

Now assume that  $x_{uv} \Rightarrow_{BF} x_{up} \stackrel{*}{\Rightarrow}_{BF} x_{vw}$ . Then we have that  $x_{vp} = 1$ , otherwise the implication  $x_{uv} \Rightarrow_{BF} x_{up}$  would not have been added by Algorithm 3. In this case we also consider the second implication in the chain. There are two cases:

- =  $x_{uv} \Rightarrow_{BF} x_{up} \Rightarrow_{BF} x_{uq} \Rightarrow_{BF} x_{vw}$  and  $x_{pb} = 1$ . Since we have both  $x_{vp} = 1$  and  $x_{pq} = 1$ , we have that Algorithm 3 also sets  $x_{vq} = 1$ . It follows that we have that  $x_{uv} \Rightarrow_{BF} x_{uq}$ and hence  $x_{uv} \Rightarrow_{BF} x_{uq} \Rightarrow_{BF} x_{vw}$ , an implication chain that fulfills the condition in Line 9 but contains less "new" implication than  $x_{uv} \Rightarrow_{\phi'_2} x_{vw}$ , a contradiction.
- =  $x_{uv} \Rightarrow_{\mathrm{BF}} x_{up} \Rightarrow_{\mathrm{BF}} x_{qp} \stackrel{*}{\Rightarrow}_{\mathrm{BF}} x_{vw}$  and  $x_{qu} = 1$ . In this case we also have  $x_{uv} \Rightarrow_{\mathrm{BF}} x_{bv}$  and  $x_{qv} \Rightarrow_{\mathrm{BF}} x_{qp}$ . Hence, we have an alternative implication chain  $x_{uv} \Rightarrow_{\mathrm{BF}} x_{qv} \Rightarrow_{\mathrm{BF}} x_{qp} \Rightarrow_{\mathrm{BF}} x_{vw}$  that fulfills the condition in Line 9 of the same length. Now if  $\{q, w\}$  is a nonedge,  $\lambda(q, w) < t$ , or  $\lambda(q, w) > t$ , then one of the previous cases applies to the new implication chain and we get a contradiction. Hence, assume that  $\lambda(q, w) = t$ . Then (using Lemma 7) we have that vertices q, v, w induce a synchronous triangle and NAE $(x_{qv}, x_{vw}, x_{wq}) \in \phi_{3\mathrm{NAE}}^{(0)}$ . It follows that  $x_{qv} \Rightarrow_{\mathrm{BF}} x_{qp} \Rightarrow_{\mathrm{BF}} x_{vw}$  is an implication chain that fulfills the condition in Line 9 but contains less "new" implication than  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi'_2} x_{vw}$ , a contradiction.

This finished the proof.

◀

We next show that for all iterations the 2SAT part of the formula does not contain an implication chain from a variable to its negation or vice versa.

▶ Lemma 15. For every iteration j of Algorithm 4 we have that if  $\phi_{3NAE}^{(j-1)} \wedge \phi_{2SAT}^{(j-1)}$  is satisfiable and there is no  $x_{uv}$  in  $\phi_{2SAT}^{(j-1)}$  such that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}^{(j-1)}} x_{vu}$ , then there is no  $x_{uv}$  in  $\phi_{2SAT}^{(j)}$  such that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi^{(j)}_{acar}} x_{vu}$ .

**Proof.** By Lemma 12 we have that if  $\phi_{3\text{NAE}}^{(j-1)} \wedge \phi_{2\text{SAT}}^{(j-1)}$  is satisfiable, then  $\phi_{2\text{SAT}}^{(j)}$  is well-defined. Assume for contradiction that there is no  $x_{uv}$  in  $\phi_{2\text{SAT}}^{(j-1)}$  such that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi_{2\text{SAT}}^{(j-1)}} x_{vu}$  but we have a  $x_{uv}$  in  $\phi_{2\text{SAT}}^{(j)}$  such that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi_{2\text{SAT}}^{(j)}} x_{vu}$ .

Then we can partition the implication chain  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}^{(j)}} x_{vu}$  into "old" parts, that are also present in  $\phi_{2SAT}^{(0)}$  and "new" implications, that were added by BOOLEAN-FORCING during some iteration  $j' \leq j$ .

Note that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi_{0}^{(j)}} x_{vu}$  contains at least one "new" implication. Consider an "old" implication in the implication chain followed by a "new" implication (if there is none, then there is one in the contraposition of the implication chain). By Lemma 14 the "new" implication was added by Algorithm 3 in Line 17 or Line 19. We can assume w.l.o.g that the "new" implication is  $x_{ab} \Rightarrow_{BF} x_{cb}$  and that  $x_{ca} = 1$  for some synchronous triangle on the vertices a, b, c, that is, we have  $NAE(x_{ab}, x_{bc}, x_{ca}) \in \phi_{3NAE}^{(0)}$  (this is the Line 17 case, Line 19 works analogously). Let  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{ab}$  be the "old" implication. Then we have that  $x_{pq} \Rightarrow_{\phi_{2\text{SAT}}^{(0)}} x_{ab} \Rightarrow_{\text{BF}} x_{cb}$  is contained in  $x_{uv} \Rightarrow_{\phi_{2\text{SAT}}^{(j)}} x_{vw}$ . Furthermore, by definition of  $\phi_{2\text{SAT}}^{(0)}$ , we have that  $|\{p,q\} \cap \{a,b,c\}| \leq 1$ , hence we can apply Lemma 11 and obtain one of the following four scenarios:

1.  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{cb}$ :

In this case we can replace  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{ab} \Rightarrow_{BF} x_{cb}$  with  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{cb}$  in the implication chain  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}^{(j)}} x_{vw}$  to obtain an implication chain from  $x_{uv}$  to  $x_{vw}$  with strictly fewer "new" implications, a contradiction.

**2.**  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{bc}$ :

Now we have that  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{ab}$  and  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{bc}$ . Then by definition of  $\phi_{2SAT}^{(0)}$  we also have that  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{ac}$  and hence  $x_{ca} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{qp}$ . Recall that we know that  $x_{ca} = 1$ . It follows that  $x_{pq} = 0$  in iteration j, a contradiction to the assumption that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi_{\alpha\alpha}} x_{vw}$ exists.

3.  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{ca}$ :

Now we have that  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{ab}$  and  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{ca}$ . Then by definition of  $\phi_{2SAT}^{(0)}$  we also have that  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{cb}$ . From here it is the same as case 1. 4.  $x_{pq} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{ac}$ : Same as case 2.

Now we are ready to show that if Algorithm 4 gets a yes-instance as input, it will compute a valid orientation.

▶ Lemma 16. For every iteration j of Algorithm 4 we have that if  $\phi_{3NAE}^{(j-1)} \land \phi_{2SAT}^{(j-1)}$  is satisfiable and there is no  $x_{uv}$  in  $\phi_{2SAT}^{(j-1)}$  such that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}^{(j-1)}} x_{vu}$ , then  $\phi_{3NAE}^{(j)} \land \phi_{2SAT}^{(j)}$  is satisfiable and there is no  $x_{uv}$  in  $\phi_{2SAT}^{(j)}$  such that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}}^{(j)} x_{vu}$ .

**Proof.** By Lemma 12 we have that if  $\phi_{3NAE}^{(j-1)} \wedge \phi_{2SAT}^{(j-1)}$  is satisfiable, then  $\phi_{3NAE}^{(j)} \wedge \phi_{2SAT}^{(j)}$  is well-defined.

Note that if  $\phi_{3\text{NAE}}^{(j-1)} = \emptyset$ , this also implies that  $\phi_{3\text{NAE}}^{(j)} = \emptyset$ , and then  $\phi_{3\text{NAE}}^{(j)} \wedge \phi_{2\text{SAT}}^{(j)}$  is satisfiable and there is no  $x_{uv}$  in  $\phi_{2\text{SAT}}^{(j)}$  such that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi_{2\text{SAT}}^{(j)}} x_{vu}$  by Lemma 15.



**Figure 3** Illustration of the variable gadget used in the reduction in the proof of Theorem 18.

From now on we assume that  $\phi_{3NAE}^{(j-1)} \neq \emptyset$ . We now argue that whenever  $\phi_{3NAE}^{(j-1)} \neq \phi_{3NAE}^{(j)}$ , we have removed some clauses from  $\phi_{3NAE}^{(j-1)}$  in Line 12 or in Line 14 of Algorithm 3. By Observation 13 the removed clauses are satisfied for all satisfying assignments of  $\phi_{2SAT}^{(j)}$  and by Lemma 15 we know that  $\phi_{2SAT}^{(j)}$  is satisfiable and there is no  $x_{uv}$  in  $\phi_{2SAT}^{(j)}$  such that  $x_{uv} \stackrel{*}{\Rightarrow}_{\phi_{2SAT}^{(j)}} x_{vu}$ . It follows that  $\phi_{3NAE}^{(j)} \wedge \phi_{2SAT}^{(j)}$  is also satisfiable.

Now we have all ingredients to show that we can solve TTO in polynomial time.

▶ Theorem 17. Algorithm 4 solves TTO in polynomial time.

**Proof.** If Algorithm 4 outputs an orientation, then clearly  $\phi_{3NAE}^{(0)}$  and  $\phi_{2SAT}^{(0)}$  is satisfiable and by Observation 8 we know that this implies that the input instance is a *yes*-instance.

If the input instance is a *yes*-instance, then by Observation 8 we have that  $\phi_{3NAE}^{(0)}$  and  $\phi_{2SAT}^{(0)}$  is satisfiable. Furthermore, by construction  $\phi_{2SAT}^{(0)}$  does not contain any  $x_{uv}$  such that  $x_{uv} \Rightarrow_{\phi_{2SAT}^{(0)}} x_{vu}$ . Lemma 16 now implies that Algorithm 4 finds a satisfying assignment and outputs an orientation.

Lastly, it is easy to check that Algorithm 4 runs in polynomial time.

# 3.2 Strict TTO is NP-Complete

In this section we show that in contrast to the other variants, STRICT TTO is NP-complete.

▶ **Theorem 18.** STRICT TTO is NP-complete even if the temporal input graph has only four different time labels.

**Proof.** We present a polynomial time reduction from (3,4)-SAT [46] where, given a CNF formula  $\phi$  where each clause contains exactly three literals and each variably appears in exactly four clauses, we are asked whether  $\phi$  is satisfiable or not. Given a formula  $\phi$ , we construct a temporal graph  $\mathcal{G}$  as follows.

Variable gadget. For each variable x that appears in  $\phi$ , we add eight vertices  $a_x, a'_x, b_x, b'_x, c_x, c'_x, d_x, d'_x$  to  $\mathcal{G}$ . We connect these vertices as depicted in Figure 3, that is, we add the following time edges to  $\mathcal{G}$ :  $(\{a_x, a'_x\}, 1), (\{a'_x, b_x\}, 2), (\{b_x, b'_x\}, 1), (\{b'_x, c_x\}, 2), (\{c_x, c'_x\}, 1), (\{c'_x, d_x\}, 2), (\{d_x, d'_x\}, 1), (\{d'_x, a_x\}, 2).$ 

Clause gadget. For each clause c of  $\phi$ , we add six vertices  $u_c, u'_c, v_c, v'_c, w_c, w'_c$  to  $\mathcal{G}$ . We connect these vertices as depicted in Figure 4, that is, we add the following time edges to  $\mathcal{G}$ :  $(\{u_c, u'_c\}, 2), (v_c, v'_c\}, 1), (\{w_c, w'_c\}, 2), (\{u_c, v_c\}, 2), (\{v_c, w_c\}, 3), (\{w_c, u_c\}, 3), (\{v_c, w'_c\}, 3), (\{w_c, v'_c\}, 3).$ 

Connecting variable gadgets and clause gadgets. Let variable x appear for the *i*th time in clause c and let x appear in the *j*th literal of c. The four vertex pairs  $(a_x, a'_x), (b_x, b'_x), (c_x, c'_x), (d_x, d'_x)$  from the variable gadget of x correspond to the first, second, third, and fourth appearance of x, respectively. The three vertices  $u'_c, v'_c, w'_c$  correspond to the first, second, and third literal



**Figure 4** Illustration of the clause gadget used in the reduction in the proof of Theorem 18 and three ways how to orient the edges in it.

of c, respectively. Let i = 1 and j = 1. If x appears non-negated, then we add the time edge  $(\{a_x, u_c'\}, 4)$ . Otherwise, if x appears negated, we add the time edge  $(\{a'_x, u'_c\}, 4)$ . For all other values of i and j we add time edges analogously.

This finishes the reduction. It can clearly be performed in polynomial time.

(⇒): a satisfying assignment Assume that we have for  $\phi$ , then we can orient G as follows. Then if a variable xis set  $\mathrm{to}$ true, we the edges of the corresponding variable orient gadgets asfollows:  $(a_x, a'_x), (b_x, a'_x), (b_x, b'_x), (c_x, b'_x), (c_x, c'_x), (d_x, c'_x), (d_x, d'_x), (a_x, d'_x).$  Otherwise, if x is set to false, we orient as follows:  $(a'_x, a_x), (a'_x, b_x), (b'_x, b_x), (b'_x, c_x), (c'_x, c_x), (c'_x, d_x), (d'_x, d_x), (d'_x, a_x).$ It is easy so see that both orientations are transitive.

Now consider a clause in  $\phi$  with literals u, v, w corresponding to vertices  $u'_c, v'_c, w'_c$  of the clause gadget, respectively. We have that at least one of the three literals satisfies the clause. If it is u, then we orient the edges in the clause gadgets as illustrated in Figure 4 (a). It is easy so see that this orientation is transitive. Furthermore, we orient the three edges connecting the clause gadgets to variable gadget. Assume, we have edges  $\{u'_c, x\}, \{v'_c, y\}, \{w'_c, z\}$ . Then we orient as follows:  $(x, u'_c), (v'_c, y), (w'_c, z)$ , that is, we orient the edge connecting the literal that satisfies the clause towards the clause gadget. Note that the variable gadgets have time labels 1 and 2 so we can always orient the connecting edges except  $(x, u'_c)$ . This edge is oriented from the variable gadget. We do this with all connecting edges except  $(x, u'_c)$ . This edge is oriented from the variable gadget towards the clause gadget towards the clause gadget, however it also corresponds to a literal that satisfies the clause. Then by construction, the edges incident to x in the variable gadget are oriented away from x, hence our orientation is transitive.

Otherwise and if v satisfies the clause, then we orient the edges in the clause gadgets as illustrated in Figure 4 (b). Otherwise (in this case w has to satisfy the clause), we orient the edges in the clause gadgets as illustrated in Figure 4 (c). It is easy so see that each of these orientation is transitive. In both cases we orient the edges connecting the clause gadgets to the variable gadgets analogously to the first case discussed above. By analogous arguments we get that the resulting orientation is transitive.

( $\Leftarrow$ ): Note that all variable gadgets are cycles of length eight with edges having labels alternating between 1 and 2 and hence the edges have to also be ori-

Consider the variable gadget corresponding to x. ented alternately. We inter- $(a_x, a'_x), (b_x, a'_x), (b_x, b'_x), (c_x, b'_x), (c_x, c'_x), (d_x, c'_x), (d_x, d'_x), (a_x, d'_x)$ the orientation pret interpret x $\operatorname{to}$ true and we the as setting orientation  $(a'_x, a_x), (a'_x, b_x), (b'_x, b_x), (b'_x, c_x), (c'_x, c_x), (c'_x, d_x), (d'_x, d_x), (d'_x, a_x)$  as setting x to true. We claim that this yields a satisfying assignment for  $\phi$ .

Assume for contradiction that there is a clause c in  $\phi$  that is not satisfied by this assignment. Then by construction of the connection of variable gadgets and clause gadgets, the connecting edges have to be oriented towards the variable gadget in order to keep the variable gadget transitive. Let the three connecting edges be  $\{u'_c, x\}, \{v'_c, y\}, \{w'_c, z\}$  and their orientation  $(u'_c, x), (v'_c, y), (w'_c, z)$ . Then we have that  $(u'_c, x)$  forces  $(u'_c, u_c)$  which in turn forces  $(w_c, u_c)$ . We have that  $(v'_c, y)$  forces  $(v'_c, v_c)$  which in turn forces  $(v_c, u_c)$ . Furthermore, we now have that  $(w_c, u_c)$  and  $(v_c, u_c)$  force  $(w_c, v_c)$ . Lastly, we have that  $(w'_c, z)$  forces  $(w'_c, w_c)$  which in turn forces  $(w_c, v_c)$  which in turn forces  $(w_c, v_c)$  a contradiction to the fact that we forced  $(w_c, v_c)$  previously.

# 4 Temporal Transitive Completion

We now study the computational complexity of TEMPORAL TRANSITIVE COMPLETION (TTC). In the static case, the so-called *minimum comparability completion* problem, i.e. adding the smallest number of edges to a static graph to turn it into a comparability graph, is known to be NP-hard [25]. Note that minimum comparability completion on static graphs is a special case of TTC and thus it follows that TTC is NP-hard too. Our other variants, however, do not generalize static comparability completion in such a straightforward way. Note that for STRICT TTC we have that the corresponding recognition problem STRICT TTO is NP-complete (Theorem 18), hence it follows directly that STRICT TTC is NP-hard, giving the result that all four variants of TTC are NP-hard. Furthermore, we present a polynomial-time algorithm for all four problem variants for the case that all edges of underlying graph are oriented, see Theorem 20. This allows directly to derive an FPT algorithm for the number of unoriented edges as a parameter.

▶ Theorem 19. All four variants of TTC are NP-hard.

**Proof.** We give a reduction from the NP-hard MAX-2-SAT problem [23].

Max-2-Sat

**Input:** A boolean formula  $\phi$  in implicative normal form<sup>5</sup> and an integer k.

**Question:** Is there an assignment of the variables which satisfies at least k clauses in  $\phi$ ?

We only describe the reduction from MAX-2-SAT to TTC. However, the same construction can be used to show NP-hardness of the other variants.

Let  $(\phi, k)$  be an instance of MAX-2-SAT with m clauses. We construct a temporal graph  $\mathcal{G}$  as follows. For each variable x of  $\phi$  we add two vertices denoted  $v_x$  and  $v_{\overline{x}}$ , connected by an edge with label 1. Furthermore, for each  $1 \leq i \leq m - k + 1$  we add two vertices  $v_x^i$  and  $v_{\overline{x}}^i$  connected by an edge with label 1. We then connect  $v_x^i$  with  $v_{\overline{x}}$  and  $v_{\overline{x}}^i$  with  $v_x$  using two edges labeled 4. Thus  $v_x, v_{\overline{x}}, v_x^i, v_{\overline{x}}^i$  is a 4-cycle whose edges alternating between 1 and 4. Afterwards, for each clause  $(a \Rightarrow b)$  of  $\phi$  with a, b being literals, we add a new vertex  $w_{a,b}$ .

<sup>&</sup>lt;sup>5</sup> i.e. a conjunction of clauses of the form  $(a \Rightarrow b)$  where a, b are literals.



**Figure 5** Temporal graph constructed from the formula  $(x \Rightarrow \overline{y}) \land (\overline{x} \Rightarrow z) \land (\overline{y} \Rightarrow \overline{z})$  for k = 1 with orientation corresponding to the assignment x = true, y = false, z = true. Since this assignment does not satisfy the third clause, the dashed blue edge is required to make the graph temporally transitive.

Then we connect  $w_{a,b}$  to  $v_a$  by an edge labeled 2 and to  $v_b$  by an edge labeled 3. Consider Figure 5 for an illustration. Observe that  $\mathcal{G}$  can be computed in polynomial time.

We claim that  $(\mathcal{G} = (G, \lambda), \emptyset, m - k)$  is a yes-instance of TTC if and only if  $\phi$  has a truth assignment satisfying k clauses.

For the proof, begin by observing that  $\mathcal{G}$  does not contain any triangle. Thus an orientation of  $\mathcal{G}$  is (weakly) (strict) transitive if and only if it does not have any oriented temporal 2-path, i.e. a temporal path of two edges with both edges being directed forward. We call a vertex v of  $\mathcal{G}$  happy about some orientation if v is not the center vertex of an oriented temporal 2-path. Thus an orientation of  $\mathcal{G}$  is transitive if and only if all vertices are happy.

( $\Leftarrow$ ): Let  $\alpha$  be a truth assignment to the variables (and thus literals) of  $\phi$  satisfying k clauses of  $\phi$ . For each literal a with  $\alpha(a) = \text{true}$ , orient all edges such that they point away from  $v_a$  and  $v_a^i$ ,  $1 \leq i \leq m - k + 1$ . For each literal a with  $\alpha(a) = \text{false}$ , orient all edges such that they point towards  $v_a$  and  $v_a^i$ ,  $1 \leq i \leq m - k + 1$ . Note that this makes all vertices  $v_a$ and  $v_a^i$  happy. Now observe that a vertex  $w_{a,b}$  is happy unless its edge with  $v_a$  is oriented towards  $w_{a,b}$  and its edge with  $v_b$  is oriented towards  $v_b$ . In other words,  $w_{a,b}$  is happy if and only if  $\alpha$  satisfies the clause ( $a \Rightarrow b$ ). Thus there are at most m - k unhappy vertices. For each unhappy vertex  $w_{a,b}$ , we add a new oriented edge from  $v_a$  to  $v_b$  with label 5. Note that this does not make  $v_a$  or  $v_b$  unhappy as all adjacent edges are directed away from  $v_a$ and towards  $v_b$ . The resulting temporal graph is transitively oriented.

 $(\Rightarrow)$ : Now let a transitive orientation F' of  $\mathcal{G}' = (G', \lambda')$  be given, where  $\mathcal{G}'$  is obtained from  $\mathcal{G}$  by adding at most m - k time edges. Clearly we may also interpret F' as an orientation induced of  $\mathcal{G}$ . Set  $\alpha(x) =$ true if and only if the edge between  $v_x$  and  $v_{\overline{x}}$  is oriented towards  $v_{\overline{x}}$ . We claim that this assignment  $\alpha$  satisfies at least k clauses of  $\phi$ .

First observe that for each variable x and  $1 \le i \le m - k + 1$ , F' is a transitive orientation of the 4-cycle  $v_x, v_{\overline{x}}, v_x^i, v_{\overline{x}}^i$  if and only if the edges are oriented alternatingly. Thus, for each variable, at least one of these k + 1 4-cycles is oriented alternatingly. In particular, for



**Figure 6** Example of a tail-heavy path.

every literal a with  $\alpha(a) = \text{true}$ , there is an edge with label 4 that is oriented away from  $v_a$ . Conversely, if  $\alpha(b) = \text{false}$ , then there is an edge with label 1 oriented towards  $v_b$  (this is simply the edge from  $v_{\overline{b}}$ ).

This implies that every edge with label 2 or 3 oriented from some vertex  $w_{c,d}$  (where either a = c or a = d) towards  $v_a$  with  $\alpha(a) = \texttt{true}$  requires  $E(G') \setminus E(G)$  to contain an edge from  $w_{c,d}$  to some  $v_{\overline{a}}^i$ . Analogously every edge with label 2 or 3 oriented from  $v_a$  with  $\alpha(a) = \texttt{false}$  to some  $w_{c,d}$  requires  $E(G') \setminus E(G)$  to contain an edge from  $v_{\overline{a}}$  to  $w_{c,d}$ .

Now consider the alternative orientation F'' obtained from  $\alpha$  as detailed in the converse orientation of the proof. For each edge between  $v_a$  and  $w_{c,d}$  where F' and F'' disagree, F''might potentially require  $E(G') \setminus E(G)$  to contain the edge  $v_c v_d$  (labeled 5, say), but in turn saves the need for some edge  $w_{c,d}v_{\overline{a}}^i$  or  $v_{\overline{a}}w_{c,d}$ , respectively. Thus, overall, F'' requires at most as many edge additions as F', which are at most m - k. As we have already seen in the converse direction, F'' requires exactly one edge to be added for every clause of  $\phi$  which is not satisfied. Thus,  $\alpha$  satisfies at least k clauses of  $\phi$ .

We now show that TTC can be solved in polynomial time, if all edges are already oriented, as the next theorem states. While we only discuss the algorithm for TTC the algorithm only needs marginal changes to work for all other variants.

▶ **Theorem 20.** An instance  $(\mathcal{G}, F, k)$  of TTC where  $\mathcal{G} = (G, \lambda)$  and G = (V, E), can be solved in  $O(m^2)$  time if F is an orientation of E, where m = |E|.

The actual proof of Theorem 20 is deferred to the end of this section. The key idea for the proof is based on the following definition. Assume a temporal graph  $\mathcal{G}$  and an orientation F of  $\mathcal{G}$  to be given. Let G' = (V, F) be the underlying graph of  $\mathcal{G}$  with its edges directed according to F. We call a (directed) path P in G' tail-heavy if the time-label of its last edge is largest among all edges of P, and we define t(P) to be the time-label of that last edge of P. For all  $u, v \in V$ , denote by  $T_{u,v}$  the maximum value t(P) over all tail-heavy (u, v)-paths P of length at least 2 in G'; if such a path does not exist then  $T_{u,v} = \bot$ . If the temporal graph  $\mathcal{G}$  with orientation F can be completed to be transitive, then adding the time edges of the set

 $X(\mathcal{G}, F) \coloneqq \{(uv, T_{u,v}) \mid T_{u,v} \neq \bot\},\$ 

which are not already present in  $\mathcal{G}$  is an optimal way to do so. Consider Figure 6 for an example.

▶ Lemma 21. The set  $X(\mathcal{G}, F)$  can be computed in  $O(m^2)$  time, where  $\mathcal{G}$  is a temporal graph with m time-edges and F an orientation of  $\mathcal{G}$ .

**Proof.** For each edge vw, we can take G' (defined above), remove w and all arcs whose label is larger than  $\lambda(v, w)$ , and do a depth-first-search from v to find all vertices u which can reach v in the resulting graph. Each of these then has  $T_{u,w} \geq \lambda(v, w)$ . By doing this for every edge vw, we obtain  $T_{u,w}$  for every vertex pair u, w. The overall running time is clearly  $O(m^2)$ .

#### G.B. Mertzios, H. Molter, M. Renken, P.G. Spirakis and P. Zschoche

Until the end of this section we are only considering the instance  $(\mathcal{G}, F, k)$  of TTC, where  $\mathcal{G} = (G, \lambda), G = (V, E)$ , and F is an orientation of  $\mathcal{G}$ . Hence, we can say a set X of oriented time-edges is a solution to I if  $X' := \{\{u, v\} \mid (uv, t) \in X\}$  is disjoint from E, satisfies  $|X| = |X'| \leq k$ , and  $F' := F \cup \{uv \mid (uv, t) \in X\}$  is a transitive orientation of the temporal graph  $\mathcal{G} + X := ((V, E \cup X'), \lambda')$ , where  $\lambda'(e) := \lambda(e)$  if  $e \in E$  and  $\lambda'(u, v) := t$  if X contains (uv, t) or (vu, t).

The algorithm we use to show Theorem 20 will use  $X(\mathcal{G}, F)$  to construct a solution (if there is any) of a given instance  $(\mathcal{G}, F, k)$  of TTC where F is a orientation of E. To prove the correctness of this approach, we make use of the following.

▶ Lemma 22. Let  $I = (\mathcal{G} = (G, \lambda), F, k)$  be an instance of TTC, where G = (V, E) and F is an orientation of E and X an solution for I. Then, for any  $(vu, T_{v,u}) \in X(\mathcal{G}, F)$  there is a (vu, t) in  $\mathcal{G} + X$  with  $t \ge T_{v,u}$ .

**Proof.** Let  $(v_0v_\ell, T_{v_0,v_\ell}) \in X(\mathcal{G}, F)$ , and G' = (V, F). Hence, there is a tail-heavy  $(v_0, v_\ell)$ -path P in G' of length  $\ell \geq 2$ . If  $\ell = 2$ , then clearly  $\mathcal{G} + X$  must contain the time edge  $(v_1v_\ell, t)$  such that  $t \geq T_{v_1,v_\ell}$ . Now let  $\ell > 2$  and  $V(P) := \{v_i \mid i \in \{0, 1, \ldots, \ell\}\}$  and  $E(P) = \{v_{i-1}v_i \mid i \in [\ell]\}$ . Since there is a tail-heavy  $(v_{\ell-2}, v_\ell)$ -path in G' of length 2,  $\mathcal{G} + X$  must contain a time-edge  $(v_{\ell-2}v_\ell, t)$  with  $t \geq T_{v_0,v_\ell}$ . Therefore, the (directed) underlying graph of  $\mathcal{G} + X$  contains a tail-heavy  $(v_0, v_\ell)$ -path of length  $\ell - 1$ . By induction,  $\mathcal{G} + X$  must contain the time edge  $(v_1v_\ell, t')$  such that  $t' \geq t \geq T_{v_0,v_\ell}$ .

Form Lemma 22, it follows that we can use  $X(\mathcal{G}, F)$  to identify *no*-instances in some cases.

▶ Corollary 23. Let  $I = (\mathcal{G} = (G, \lambda), F, k)$  be an instance of TTC, where G = (V, E) and F is an orientation of E. Then, I is a no-instance, if for some  $v, u \in V$ 

- 1. there are time-edges  $(vu, t) \in X(\mathcal{G}, F)$  and  $(uv, t') \in X(\mathcal{G}, F)$ ,
- **2.** there is an edge  $uv \in F$  such that  $(vu, T_{v,u}) \in X(\mathcal{G}, F)$ , or
- **3.** there is an edge  $vu \in F$  such that  $(vu, T_{v,u}) \in X(\mathcal{G}, F)$  with  $\lambda(v, u) < T_{v,u}$ .

We are now ready to prove Theorem 20.

**Proof of Theorem 20.** Let  $I = (\mathcal{G} = (G, \lambda), F, k)$  be an instance of TTC, where F is a orientation of E. First we compute  $X(\mathcal{G}, F)$  in polynomial time, see Lemma 21. Let  $Y = \{(vu, t) \in X(\mathcal{G}, F) \mid \{v, u\} \notin E\}$  and report that I is a *no*-instance if |Y| > k or one of the conditions of Corollary 23 holds true. Otherwise report that I is a *yes*-instance. This gives an overall running time of  $O(m^2)$ .

Clearly, if one of the conditions of Corollary 23 holds true, then I is a *no*-instance. Moreover, by Lemma 22 any solution contains at least |Y| time edges. Thus, if |Y| > k, then I is a *no*-instance.

If we report that I is a *yes*-instance, then we claim that Y is a solution for I. Let  $F' \supseteq F$  be a orientation of  $\mathcal{G} + Y$ . Assume towards a contradiction that F' is not transitive. Then, there is a temporal path  $((vu, t_1), (uw, t_2))$  in  $\mathcal{G} + Y$  such that there is no time-edge (uw, t) in  $\mathcal{G} + Y$ , with  $t \ge t_2$ . By definition of  $X(\mathcal{G}, F)$ , the directed graph G' = (V, F) contains a tail-heavy (v, u)-path  $P_1$  with  $t_1 = t(P_1)$  and a tail-heavy (u, w)-path  $P_2$  with  $t_2 = t(P_2) \ge t_1$ . By concatenation of  $P_1$  and  $P_2$ , we obtain that the G' contains a (v, w)-path P' of length at least two such that  $t_2 = t(P')$ . Thus,  $t_2 \le T_{v,w}$  and  $(vw, T_{v,w}) \in X(\mathcal{G})$ —a contradiction.

Using Theorem 20 we can now prove that TTC is fixed-parameter tractable (FPT) with respect to the number of unoriented edges in the input temporal graph  $\mathcal{G}$ .



**Figure 7** Temporal graph constructed from the formula  $NAE(x_1, x_2, x_2) \land NAE(x_1, x_2, x_3)$  and orientation corresponding to setting  $x_1 = false$ ,  $x_2 = true$ , and  $x_3 = false$ . Each attachment vertex is at the clockwise end of its edge.

▶ Corollary 24. Let  $I = (\mathcal{G} = (G, \lambda), F, k)$  be an instance of TTC, where G = (V, E). Then I can be solved in  $O(2^q \cdot m^2)$ , where q = |E| - |F| and m the number of time edges.

**Proof.** Note that there are  $2^q$  ways to orient the q unoriented edges. For each of these  $2^q$  orientations of these q edges, we obtain a fully oriented temporal graph. Then we can solve TTC on each of these fully oriented graphs in  $O(m^2)$  time by Theorem 20. Summarizing, we can solve TTC on I in  $2^q \cdot m^2$  rime.

# 5 Deciding Multilayer Transitive Orientation

In this section we prove that MULTILAYER TRANSITIVE ORIENTATION (MTO) is NPcomplete, even if every edge of the given temporal graph has at most two labels. Recall that this problem asks for an orientation F of a temporal graph  $\mathcal{G} = (G, \lambda)$  (i.e. with exactly one orientation for each edge of G) such that, for every "time-layer"  $t \geq 1$ , the (static) oriented graph defined by the edges having time-label t is transitively oriented in F. As we discussed in Section 2, this problem makes more sense when every edge of G potentially has multiple time-labels, therefore we assume here that the time-labeling function is  $\lambda : E \to 2^{\mathbb{N}}$ .

▶ **Theorem 25.** *MTO is NP-complete, even on temporal graphs with at most two labels per edge.* 

**Proof.** We give a reduction from monotone NOT-ALL-EQUAL-3SAT, which is known to be NP-hard [42]. So let  $\phi = \bigwedge_{i=1}^{m} \text{NAE}(y_{i,1}, y_{i,2}, y_{i,3})$  be a monotone NOT-ALL-EQUAL-3SAT instance and  $X := \{x_1, \ldots, x_n\} := \bigcup_{i=1}^{m} \{y_{i,1}, y_{i,2}, y_{i,3}\}$  be the set of variables.

Start with an empty temporal graph  $\mathcal{G}$ . For every clause NAE $(y_{i,1}, y_{i,2}, y_{i,3})$ , add to  $\mathcal{G}$  a triangle on three new vertices and label its edges  $a_{i,1}, a_{i,2}, a_{i,3}$ . Give all these edges label n+1. For each of these edges, select one of its endpoints to be its *attachment vertex* in such a way that no two edges share an attachment vertex. Next, for each  $1 \leq i \leq n$ , add a new vertex  $v_i$ . Let  $A_i := \{a_{i,j} \mid y_{i,j} = x_i\}$ . Add the label *i* to every edge in  $A_i$  and connect its attachment vertex to  $v_i$  with an edge labeled *i*. See also Figure 7.

We claim that  $\mathcal{G}$  is a *yes*-instance of MTO if and only if  $\phi$  is satisfiable.

( $\Leftarrow$ ): Let  $\alpha : X \to \{ \texttt{true}, \texttt{false} \}$  be an assignment satisfying  $\omega$ . For every  $x_i \in X$ , orient all edges adjacent to  $v_i$  away from  $v_i$  if  $\alpha(x_i) = \texttt{true}$  and towards  $v_i$  otherwise. Then, orient every edge  $a_{i,j}$  towards its attachment vertex if  $\alpha(y_{i,j}) = \texttt{true}$  and away from it otherwise.

#### G.B. Mertzios, H. Molter, M. Renken, P.G. Spirakis and P. Zschoche

Note that in the layers 1 through n every vertex either has all adjacent edges oriented towards it or away from it. Thus these layers are clearly transitive. It remains to consider layer n + 1 which consists of a disjoint union of triangles. Each such triangle  $a_{i,1}, a_{i,2}, a_{i,3}$ is oriented non-transitively (i.e. cyclically) if and only if  $\alpha(y_{i,1}) = \alpha(y_{i,2}) = \alpha(y_{i,3})$ , which never happens if  $\alpha$  satisfies  $\phi$ .

 $(\Rightarrow)$ : Let  $\omega$  be an orientation of the underlying edges of  $\mathcal{G}$  such that every layer is transitive. Since they all share the same label *i*, the edges adjacent to  $v_i$  must be all oriented towards or all oriented away from  $v_i$ . We set  $\alpha(x_i) = \texttt{false}$  in the former and  $\alpha(x_i) = \texttt{true}$  in the latter case. This in turn forces each edge  $a_{i,j}$  to be oriented towards its attachment vertex if and only if  $\alpha(a_{i,j}) = \texttt{true}$ . Therefore, every clause  $\mathsf{NAE}(y_{i,1}, y_{i,2}, y_{i,3})$  is satisfied, since the three edges  $a_{i,1}, a_{i,2}, a_{i,3}$  form a triangle in layer n+1 and can thus not be oriented cyclically (i.e. all towards or all away from their respective attachment vertices).

— References

- 1 Eleni C. Akrida, Leszek Gasieniec, George B. Mertzios, and Paul G. Spirakis. Ephemeral networks with random availability of links: The case of fast networks. *Journal of Parallel and Distributed Computing*, 87:109–120, 2016.
- 2 Eleni C. Akrida, Leszek Gasieniec, George B. Mertzios, and Paul G. Spirakis. The complexity of optimal design of temporally connected graphs. *Theory of Computing Systems*, 61(3):907–944, 2017.
- 3 Eleni C. Akrida, George B. Mertzios, Sotiris E. Nikoletseas, Christoforos L. Raptopoulos, Paul G. Spirakis, and Viktor Zamaraev. How fast can we reach a target vertex in stochastic temporal graphs? *Journal of Computer and System Sciences*, 114:65–83, 2020. An extended abstract appeared at ICALP 2019.
- 4 Eleni C. Akrida, George B. Mertzios, Paul G. Spirakis, and Viktor Zamaraev. Temporal vertex cover with a sliding time window. *Journal of Computer and System Sciences*, 107:108–123, 2020. An extended abstract appeared at ICALP 2018.
- 5 Josh Alman and Virginia Vassilevska Williams. A refined laser method and faster matrix multiplication. In Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 522–539, 2021.
- 6 Bengt Aspvall, Michael F. Plass, and Robert Endre Tarjan. A linear-time algorithm for testing the truth of certain quantified boolean formulas. *Information Processing Letters*, 8(3):121–123, 1979.
- 7 Kyriakos Axiotis and Dimitris Fotakis. On the size and the approximability of minimum temporally connected subgraphs. In Proceedings of the 43rd International Colloquium on Automata, Languages, and Programming, (ICALP), pages 149:1–149:14, 2016.
- 8 Matthias Bentert, Anne-Sophie Himmel, Hendrik Molter, Marco Morik, Rolf Niedermeier, and René Saitenmacher. Listing all maximal k-plexes in temporal graphs. ACM Journal of Experimental Algorithmics, 24(1):13:1–13:27, 2019.
- 9 Matthias Bentert, Anne-Sophie Himmel, André Nichterlein, and Rolf Niedermeier. Efficient computation of optimal temporal walks under waiting-time constraints. Applied Network Science, 5(1):73, 2020.
- 10 Robert Bredereck, Christian Komusiewicz, Stefan Kratsch, Hendrik Molter, Rolf Niedermeier, and Manuel Sorge. Assessing the computational complexity of multilayer subgraph detection. *Network Science*, 7(2):215–241, 2019.
- 11 Binh-Minh Bui-Xuan, Afonso Ferreira, and Aubin Jarry. Computing shortest, fastest, and foremost journeys in dynamic networks. *International Journal of Foundations of Computer Science*, 14(02):267–285, 2003.

- 12 Sebastian Buß, Hendrik Molter, Rolf Niedermeier, and Maciej Rymar. Algorithmic aspects of temporal betweenness. In Proceedings of the 26th ACM SIGKDD Conference on Knowledge Discovery and Data Mining (KDD), pages 2084–2092. ACM, 2020.
- 13 Arnaud Casteigts and Paola Flocchini. Deterministic Algorithms in Dynamic Networks: Formal Models and Metrics. Technical report, Defence R&D Canada, April 2013. URL: https://hal.archives-ouvertes.fr/hal-00865762.
- 14 Arnaud Casteigts and Paola Flocchini. Deterministic Algorithms in Dynamic Networks: Problems, Analysis, and Algorithmic Tools. Technical report, Defence R&D Canada, April 2013. URL: https://hal.archives-ouvertes.fr/hal-00865764.
- 15 Arnaud Casteigts, Paola Flocchini, Walter Quattrociocchi, and Nicola Santoro. Time-varying graphs and dynamic networks. *International Journal of Parallel, Emergent and Distributed Systems*, 27(5):387–408, 2012.
- 16 Arnaud Casteigts, Anne-Sophie Himmel, Hendrik Molter, and Philipp Zschoche. Finding temporal paths under waiting time constraints. In *31st International Symposium on Algorithms and Computation (ISAAC)*, pages 30:1–30:18, 2020.
- 17 Arnaud Casteigts, Joseph G. Peters, and Jason Schoeters. Temporal cliques admit sparse spanners. In Proceedings of the 46th International Colloquium on Automata, Languages, and Programming (ICALP), volume 132, pages 134:1–134:14, 2019.
- 18 Jiehua Chen, Hendrik Molter, Manuel Sorge, and Ondřej Suchý. Cluster editing in multi-layer and temporal graphs. In Proceedings of the 29th International Symposium on Algorithms and Computation (ISAAC), pages 24:1–24:13, 2018.
- **19** J. Enright, K. Meeks, G.B. Mertzios, and V. Zamaraev. Deleting edges to restrict the size of an epidemic in temporal networks. *Journal of Computer and System Sciences*. To appear.
- 20 Jessica Enright, Kitty Meeks, and Fiona Skerman. Assigning times to minimise reachability in temporal graphs. Journal of Computer and System Sciences, 115:169–186, 2021.
- 21 Thomas Erlebach, Michael Hoffmann, and Frank Kammer. On temporal graph exploration. In Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming (ICALP), pages 444–455, 2015.
- 22 Till Fluschnik, Hendrik Molter, Rolf Niedermeier, Malte Renken, and Philipp Zschoche. Temporal graph classes: A view through temporal separators. *Theoretical Computer Science*, 806:197–218, 2020.
- 23 M.R. Garey, D.S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. Theoretical Computer Science, 1(3):237–267, 1976.
- 24 Martin Charles Golumbic. *Algorithmic graph theory and perfect graphs*. Elsevier, 2nd edition, 2004.
- 25 S Louis Hakimi, Edward F Schmeichel, and Neal E Young. Orienting graphs to optimize reachability. *Information Processing Letters*, 63(5):229–235, 1997.
- 26 Anne-Sophie Himmel, Hendrik Molter, Rolf Niedermeier, and Manuel Sorge. Adapting the Bron-Kerbosch algorithm for enumerating maximal cliques in temporal graphs. Social Network Analysis and Mining, 7(1):35:1–35:16, 2017.
- 27 Petter Holme and Jari Saramäki. Temporal network theory, volume 2. Springer, 2019.
- 28 David Kempe, Jon M. Kleinberg, and Amit Kumar. Connectivity and inference problems for temporal networks. *Journal of Computer and System Sciences*, 64(4):820–842, 2002.
- **29** Hyoungshick Kim and Ross Anderson. Temporal node centrality in complex networks. *Physical Review E*, 85(2):026107, 2012.
- 30 Ross M. McConnell and Jeremy P. Spinrad. Linear-time modular decomposition and efficient transitive orientation of comparability graphs. In *Proceedings of the 5th Annual ACM-SIAM* Symposium on Discrete Algorithms (SODA), pages 536–545, 1994.
- 31 Ross M. McConnell and Jeremy P. Spinrad. Linear-time transitive orientation. In Proceedings of the 8th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 19–25, 1997.
- 32 Ross M. McConnell and Jeremy P. Spinrad. Modular decomposition and transitive orientation. Discrete Mathematics, 201(1-3):189–241, 1999.

#### G.B. Mertzios, H. Molter, M. Renken, P.G. Spirakis and P. Zschoche

- 33 David B McDonald and Daizaburo Shizuka. Comparative transitive and temporal orderliness in dominance networks. *Behavioral Ecology*, 24(2):511–520, 2013.
- 34 George B. Mertzios. The recognition of simple-triangle graphs and of linear-interval orders is polynomial. *SIAM Journal on Discrete Mathematics*, 29(3):1150–1185, 2015.
- 35 George B. Mertzios, Othon Michail, Ioannis Chatzigiannakis, and Paul G. Spirakis. Temporal network optimization subject to connectivity constraints. In *Proceedings of the 40th International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 657–668, 2013.
- 36 George B Mertzios, Hendrik Molter, Rolf Niedermeier, Viktor Zamaraev, and Philipp Zschoche. Computing maximum matchings in temporal graphs. In Proceedings of the 37th International Symposium on Theoretical Aspects of Computer Science (STACS), volume 154, pages 27:1– 27:14, 2020.
- 37 George B Mertzios, Hendrik Molter, and Viktor Zamaraev. Sliding window temporal graph coloring. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)*, volume 33, pages 7667–7674, 2019.
- 38 Othon Michail and Paul G. Spirakis. Elements of the theory of dynamic networks. Communications of the ACM, 61(2):72–72, January 2018.
- **39** Robert Moskovitch and Yuval Shahar. Medical temporal-knowledge discovery via temporal abstraction. In *Proceedings of the AMIA Annual Symposium*, page 452, 2009.
- 40 Robert Moskovitch and Yuval Shahar. Fast time intervals mining using the transitivity of temporal relations. *Knowledge and Information Systems*, 42(1):21–48, 2015.
- 41 V. Nicosia, J. Tang, C. Mascolo, M. Musolesi, G. Russo, and V. Latora. Graph metrics for temporal networks. In *Temporal Networks*. Springer, 2013.
- 42 Thomas J. Schaefer. The complexity of satisfiability problems. In Proceedings of the 10th Annual ACM Symposium on Theory of Computing (STOC), pages 216–226, 1978.
- **43** Jeremy P. Spinrad. On comparability and permutation graphs. *SIAM Journal on Computing*, 14(3):658–670, 1985.
- 44 Jeremy P. Spinrad. *Efficient graph representations*, volume 19 of *Fields Institute Monographs*. American Mathematical Society, 2003.
- **45** Xavier Tannier and Philippe Muller. Evaluating temporal graphs built from texts via transitive reduction. *Journal of Artificial Intelligence Research (JAIR)*, 40:375–413, 2011.
- 46 Craig A Tovey. A simplified NP-complete satisfiability problem. Discrete Applied Mathematics, 8(1):85–89, 1984.
- 47 Tiphaine Viard, Matthieu Latapy, and Clémence Magnien. Computing maximal cliques in link streams. *Theoretical Computer Science*, 609:245–252, 2016.
- 48 Huanhuan Wu, James Cheng, Yiping Ke, Silu Huang, Yuzhen Huang, and Hejun Wu. Efficient algorithms for temporal path computation. *IEEE Transactions on Knowledge and Data Engineering*, 28(11):2927–2942, 2016.
- 49 Philipp Zschoche, Till Fluschnik, Hendrik Molter, and Rolf Niedermeier. The complexity of finding separators in temporal graphs. *Journal of Computer and System Sciences*, 107:72–92, 2020.