TRACTABLE COMBINATIONS OF TEMPORAL CSPS

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ABSTRACT. The constraint satisfaction problem (CSP) of a first-order theory T is the computational problem of deciding whether a given conjunction of atomic formulas is satisfiable in some model of T. We study the computational complexity of CSP($T_1 \cup T_2$) where T_1 and T_2 are theories with disjoint finite relational signatures. We prove that if T_1 and T_2 are the theories of temporal structures, i.e., structures where all relations have a first-order definition in $(\mathbb{Q}, <)$, then CSP($T_1 \cup T_2$) is in P or NP-complete. To this end we prove a purely algebraic statement about the structure of the lattice of locally closed clones over the domain $\mathbb{Q}$ that contain $\mathrm{Aut}(\mathbb{Q}; <)$.

1. INTRODUCTION

Deciding the satisfiability of formulas with respect to a given theory or structure is one of the fundamental problems in theoretical computer science. One large class of problems of this kind are *Constraint Satisfaction Problems* (CSPs). For a finite relational signature τ, the CSP of a τ-theory T, written CSP(T), is the computational problem of deciding whether a given finite set S of atomic τ-formulas is satisfiable in some model of T. A general goal is to identify theories T such that CSP(T) can be solved in polynomial time.

Many theories that are relevant in program verification and automated deduction are of the form $T_1 \cup T_2$ where the signatures of T_1 and T_2 are disjoint; satisfiability problems of the form CSP($T_1 \cup T_2$) are also studied in the field of Satisfiability Modulo Theories (SMT). If we already have a decision procedure for CSP(T_1) and for CSP(T_2), then, under certain conditions, we can use these decision procedures to construct a decision procedure for CSP($T_1 \cup T_2$) in a generic way. Most results in the area of combinations of decision procedures concern decidability, rather than polynomial-time decidability; see for example [23, 38, 19, 36]. We are particularly interested in polynomial-time decidability and the borderline to NP-hardness. The seminal result in this direction is due to Greg Nelson and Derek C. Oppen, who provided a criterion assuring that satisfiability of conjunctions of atomic and negated atomic formulas can be decided in polynomial time [32, 34]. The work of Nelson and Oppen has been further developed later on (see for example [1]) and their algorithm has been implemented in many SMT solvers (see for example [27], [30]). While their result directly gives sufficient conditions for polynomial-time tractability of CSP($T_1 \cup T_2$), one of their conditions called 'convexity' can be weakened to 'independence of $\neq$' (see [20]) without changing their proof, if we only consider conjunctions of atomic formulas as input (see Section 2.3 and Section 3 for details). Interestingly, the weakened criterion also turns out to be remarkably tight; Schulz [37] as well as Bodirsky and Greiner [6] proved that in many cases not covered by the weaker criterion, CSP($T_1 \cup T_2$) is NP-hard even though both CSP(T_1) and CSP(T_2) can be solved in polynomial time. However, there are examples of theories T_1 and T_2 that do not satisfy the weakened conditions of Nelson and Oppen, but CSP($T_1 \cup T_2$) can be solved in polynomial time nevertheless (see [7]). Unfortunately,

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there is still no general theory of polynomial-time tractability for combinations of theories.

An important subclass of CSPs are *temporal CSPs*, which are CSPs for the theories of structures of the form $(\mathbb{Q}; R_1, \ldots, R_n)$ where $R_1, \ldots, R_n$ are relations defined by quantifier-free first-order formulas over $(\mathbb{Q}; <)$; we refer to such structures as *temporal structures*. A well-known example of such a structure is $(\mathbb{Q}; \mathrm{Betw})$ where $\mathrm{Betw} := \{(a,b,c) \mid a < b < c \vee c < b < a\}$. The CSP for the theory of this structure is the so-called *Betweenness problem* and is NP-complete [33]. Other well-known temporal CSPs are the Cyclic Ordering problem [22], Ord-Horn constraints [31], the network satisfaction problem for the point algebra [39], and scheduling with and/or precedence constraints [29]. It has been shown that every temporal CSP is in P or NP-complete [11]. Temporal CSPs are of particular importance for the study of polynomial-time procedures for combinations of theories, because many of the polynomial-time tractable cases do *not* satisfy the weakened conditions of Nelson and Oppen because $\neq$ is not independent in these cases. This is unlike several other classifications for CSPs where all the polynomial-time tractable cases do satisfy the weakened conditions of Nelson and Oppen [10, 13, 18, 16, 8, 26, 12] and hence CSPs for combinations of such theories can be solved in polynomial time. Some results about the complexity of CSPs for combinations of theories of temporal structures were obtained in [6], but they were restricted to temporal structures that contain the relations $<$ and $\neq$.

1.1. **Contributions.** Our main result is a complexity dichotomy for all problems of the form $\mathrm{CSP}(T_1 \cup T_2)$ where T_1 and T_2 are first-order theories of temporal structures with disjoint finite signatures. In order to phrase our results in this section, we need the concepts of *primitive positive definability* and *polymorphisms*, which are of fundamental importance in universal algebra and will be recalled in Section 2.2. The main result is the following:

Theorem 1. *Let T_1 and T_2 be the theories of temporal structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with disjoint finite signatures. Then $\mathrm{CSP}(T_1 \cup T_2)$ is polynomial-time tractable if*

(1) for both $i \in \{1,2\}$, the structure $\mathfrak{A}_i$ has a binary injective polymorphism and $\mathrm{CSP}(\mathfrak{A}_i)$ is in P, or
(2) for both $i \in \{1,2\}$, the structure $\mathfrak{A}_i$ has a constant polymorphism, or
(3) there is a temporal structure $\mathfrak{B}$ such that $\mathrm{CSP}(\mathfrak{B}) = \mathrm{CSP}(T_1 \cup T_2)$, and $\mathrm{CSP}(\mathfrak{B})$ is in P (this happens if, for some $i \in \{1,2\}$, all permutations are polymorphisms of $\mathfrak{A}_i$).

Otherwise, $\mathrm{CSP}(T_1 \cup T_2)$ is NP-complete.

The technique we use to prove NP-hardness in Theorem 1 is based on the notion of cross-prevention introduced in [6].

Definition 2. A τ-structure $\mathfrak{B}$ *can prevent crosses* if there exists a primitive positive τ-formula $\phi(x,y,u,v)$ such that

(1) $\phi(x,y,u,v) \wedge x = y \wedge u \neq v \wedge x \neq u \wedge x \neq v$ is satisfiable in $\mathfrak{B}$,
(2) $\phi(x,y,u,v) \wedge x \neq y \wedge u = v \wedge x \neq u \wedge y \neq u$ is satisfiable in $\mathfrak{B}$, and
(3) $\phi(x,y,u,v) \wedge x = y \wedge u = v$ is not satisfiable in $\mathfrak{B}$.

Any such formula ϕ will be referred to as a *cross prevention formula* of $\mathfrak{B}$.

An example of a structure that can prevent crosses is $(\mathbb{Q}; <)$; a cross prevention formula is $u < x \wedge y < v$. Another example is $(\mathbb{N}; E, N)$ where E is an equivalence relation where all classes have exactly two elements and N is the complement of E. In this structure $E(x,u) \wedge N(y,v)$ is a cross prevention formula.

Our next contribution, Theorem 3, is the complexity result underlying the hardness proof for Theorem 1 and is not limited to temporal structures. It uses the relation R^{mix}, which is of fundamental importance to this article and defined as follows:

$$R^{\mathrm{mix}} := \big\{(a, b, c) \in \mathbb{Q}^3 \mid (a = b) \vee (c < a \wedge c < b)\big\} .$$

Theorem 3. *Let $\mathfrak{A}$ be a countably infinite ω-categorical structure with finite relational signature and without algebraicity. If $\mathfrak{A}$ can prevent crosses, then the CSP of* $\mathrm{Th}(\mathbb{Q}; <, R^{\mathrm{mix}}) \cup \mathrm{Th}(\mathfrak{A})$ *is NP-hard.*

Examples for ω-categorical structures without algebraicity and with cross prevention can be found in Section 6.

Our third contribution is the algebraic cornerstone of this article, which is a result about the definability of R^{mix}. If R is a temporal relation, then $-R$ denotes the *dual of* R, which is the temporal relation $\{(a_1, \dots, a_n) \in \mathbb{Q}^n \mid (-a_1, \dots, -a_n) \in R\}$. The functions min, min, mx and ll will be explained in Section 2.5.

Theorem 4. *Let $\mathfrak{A}$ be a first-order expansion of $(\mathbb{Q}; <)$ with a finite relational signature such that* min, mi, mx, ll *or one of their duals is a polymorphism of $\mathfrak{A}$. Then the following are equivalent:*

- *$\mathfrak{A}$ does not have a binary injective polymorphism.*
- *The relation R^{mix} or its dual $-R^{\mathrm{mix}}$ has a primitive positive definition in $\mathfrak{A}$.*

Theorem 4 characterises those first-order expansions of $(\mathbb{Q}; <)$ among the polynomial-time tractable cases in the dichotomy of Bodirsky and Kára (see Theorem 14) whose first-order theory does not satisfy the weakened tractability conditions by Nelson and Oppen because $\neq$ is not independent from their theory (see Section 2.3 for the definition).

1.2. **Significance of the Result in Universal Algebra.** Theorem 4 is of independent interest in universal algebra; for an introduction to the universal-algebraic concepts that appear in this section we refer the reader to Section 2.2. Theorem 4 can be seen as a result about locally closed clones on a countably infinite domain B that are *highly set-transitive.* A permutation group G on a set B is said to be highly set-transitive if for all finite subsets S_1 and S_2 of B of equal size there exists a permutation in G that maps S_1 to S_2. An operation clone on a set B is said to be highly set-transitive if it contains a highly set-transitive permutation group.

It can be shown that the highly set-transitive locally closed clones are precisely the polymorphism clones of temporal structures (possibly with infinitely many relations), up to a bijection between B and $\mathbb{Q}$ [11]. These objects form a lattice: the meet of two clones is the intersection of the clones and the join can be obtained as the polymorphism clone of all relations preserved by both of the clones. Similarly, as the lattice of clones over the set $\{0, 1\}$ plays a fundamental role for studying finite algebras (it has been classified by Post [35]), the lattice of locally closed highly set-transitive clones over $\mathbb{Q}$ is of fundamental importance for the study of locally closed clones in general. This lattice is of size 2^ω even if we restrict our attention to closed clones that contain all permutations [4]. However, the lower parts of the lattice appears to be more structured and amenable to classification. We pose the following question.

Question 5. *Are there only countably many locally closed highly set-transitive clones over a fixed countably infinite set that do not contain a binary injective operation?*

Question 5 has a positive answer in the case that the clone contains all permutations of the base set [4]. Theorem 11 below shows that answering Question 5 can be split into finitely many cases, depending on whether the clone contains a constant operation, or whether it preserves one out of a finite list of temporal relations. Theorem 4 shows that in case (1) of Theorem 14, we can even focus on clones that preserve the relation R^{mix} or its dual.

1.3. **Outline of the Article.** We first recall some basic concepts from model theory in Section 2.1. Then, the classical Nelson-Oppen conditions for obtaining polynomial-time decision procedures for combined theories are presented in Section 2.3; a slight generalisation of their results can be found in Section 3. We then define the model-theoretic notion of a *generic combination* of two structures with disjoint relational signatures in Section 2.4, which plays a crucial role in our proof. The reason is that we may apply universal algebra to study the complexity of CSPs of structures but not of theories. Basic universal-algebraic concepts are introduced in Section 2.2. Our results build on the classification of the temporal CSPs that can be solved in polynomial time [11], which we present along with other known facts about temporal structures in Section 2.5.

The proof of Theorem 4 is organised as follows. The difficult direction is to find a pp-definition of R^{mix} in $\mathfrak{A}$ if $\mathfrak{A}$ is not preserved by a binary injective polymorphism. If $\mathrm{Pol}(\mathfrak{A})$ contains mi, then the proof is easier if $\leq$ is pp-definable in the structure $\mathfrak{A}$. If the relation $\leq$ is not pp-definable in $\mathfrak{A}$, then a certain operation mix is a polymorphism of $\mathfrak{A}$. We discuss mix in Section 4 and use results thereof in Section 5.1 to show the pp-definability of R^{mix} in $\mathfrak{A}$.

The case that $\mathrm{Pol}(\mathfrak{A})$ contains mx but not mi is treated in Section 5.2, and the case that $\mathrm{Pol}(\mathfrak{A})$ contains min but neither mi nor mx is treated in Section 5.3. All of these partial results are put together in Section 5.4.

Finally, Section 6 uses our definability dichotomy theorem (Theorem 4) to prove the complexity dichotomy for combinations of temporal CSPs.

2. Preliminaries

We use the notation $[k]$ for the set $\{1,\ldots,k\} \subseteq \mathbb{N}$.

2.1. **Model Theory.** A *relational signature* is a set of relation symbols, each endowed with a natural number, stating its arity. Let τ be relational signature. A *τ-structure* $\mathfrak{A}$ consists of a set A, the *domain* of $\mathfrak{A}$, and a relation $R \subseteq A^k$ for each $R \in \tau$ of arity k. We use the notation $\mathfrak{A} = (A; R_1, \ldots, R_n)$ for relational structures with finite signature.

A τ-formula is *atomic* if it is of the form $x_1 = x_2$, $\bot$ (the logical "false"), or $R(x_1,\ldots,x_n)$ for $R \in \tau$ of arity n where $x_1,\ldots,x_n$ are variables. A *literal* is either an atomic formula or a negated atomic formula. A τ-formula is *primitive positive (pp)* if it is of the form $\exists x_k, x_{k+1},\ldots,x_\ell.\, \phi(x_1,\ldots,x_\ell)$ where ϕ is a conjunction of atomic τ-formulas and $k \geq 1$ is allowed to be larger than ℓ, in which case all variables are unquantified. A τ-formula is *existential positive* if it is a disjunction of primitive positive formulas; note that every first-order formula which does not contain negation or universal quantification is equivalent to such a formula. A *τ-theory* is a set of first-order τ-sentences, i.e., τ-formulas without free variables. For a τ-strucutre $\mathfrak{A}$ the *(first-order) theory of* $\mathfrak{A}$, denoted by $\mathrm{Th}(\mathfrak{A})$, is the set of all first-order τ-sentences that hold in $\mathfrak{A}$. If T is a τ-theory and $\mathfrak{A}$ a τ-structure, then $\mathfrak{A}$ is a *model* for T, written $\mathfrak{A} \models T$, if all sentences in T hold in $\mathfrak{A}$. In particular $\mathfrak{A} \models \mathrm{Th}(\mathfrak{A})$. If $\mathfrak{A}$ is a τ-structure and $\phi(x_1,\ldots,x_n)$ is a τ-formula with free variables $x_1,\ldots,x_n$, then the *relation defined by* ϕ is the relation $\{(a_1,\ldots,a_n) \in A^n \mid \mathfrak{A} \models \phi(a_1,\ldots,a_n)\}$. We say that a relation is *primitively positively definable* (*pp-definable*) in $\mathfrak{A}$ if there is a primitive positive formula that defines R in $\mathfrak{A}$. By $\langle\mathfrak{A}\rangle_{\mathrm{pp}}$ we denote the set of all relations which are pp-definable in $\mathfrak{A}$. First-order (fo) and existential positive (ep) definability are defined analogously. Notice that a definition of a relation R via a formula ϕ in the above way also yields a bijection between coordinates of tuples of R and the free variables of ϕ. We will use this bijection implicitly whenever we say that $t \in R$ *satisfies* a formula on the free variables of ϕ. The CSP of a τ-structure $\mathfrak{A}$, written CSP($\mathfrak{A}$), is the computational problem of deciding, given a conjunction of atomic τ-formulas, whether or not the conjunction is satisfiable in $\mathfrak{A}$. In particular CSP($\mathfrak{A}$) and CSP(Th($\mathfrak{A}$)) are the same problem. Let $\mathfrak{A}$ be a relational τ-structure and $\mathfrak{B}$ a relational σ-structure with $\tau \subseteq \sigma$. If $\mathfrak{A}$ can be obtained from $\mathfrak{B}$ by deleting relations from $\mathfrak{B}$, then $\mathfrak{A}$ is called a *reduct* of $\mathfrak{B}$ and $\mathfrak{B}$ is called an *expansion* of $\mathfrak{A}$. An expansion $\mathfrak{B}$ of $\mathfrak{A}$ is called *first-order expansion* if all relations in $\mathfrak{B}$ have a first-order definition in $\mathfrak{A}$. The expansion of $\mathfrak{A}$ by a relation R is denoted by $(\mathfrak{A}; R)$. As usual, Aut($\mathfrak{A}$) denotes the set of all automorphisms of $\mathfrak{A}$, i.e., isomorphisms from $\mathfrak{A}$ to $\mathfrak{A}$. For $k \in \mathbb{N}$ and $a \in A^k$, the set $\mathrm{Aut}(\mathfrak{A})a := \{(\alpha(a_1), \dots, \alpha(a_k)) \mid \alpha \in \mathrm{Aut}(\mathfrak{A})\}$ is called the *orbit* of a.

The theory of $(\mathbb{Q}; <)$, or any first-order expansion thereof, has the remarkable property of *ω-categoricity*, that is, it has only one countable model up to isomorphism (see Proposition 3.1.1 in [2]). The class of ω-categorical relational structures can be characterised by the following theorem.

Theorem 6 (Engeler, Ryll-Nardzewski, Svenonius, see [24] p. 171)**.** *Let $\mathfrak{A}$ be a countably infinite structure with countable signature. Then, the following are equivalent:*

(1) $\mathfrak{A}$ is ω-categorical;
(2) for all $n \geq 1$ every orbit of n-tuples is first-order definable in $\mathfrak{A}$;
(3) for all $n \geq 1$ there are only finitely many orbits of n-tuples.

2.2. **Universal Algebra.** A operation $f\colon A^m \to A$ *preserves* a relation $R \subseteq A^n$ if for all $t_1, \dots, t_m \in R$ we have $f(t_1, \dots, t_m) \in R$ where f is applied componentwise. For instance, the projection of arity n to the i-th coordinate, denoted by π_i^n, preserves every relation over A. For a set S of relations over A we define Pol(S) as the set of all operations on A that preserve all relations in S. We define Pol($\mathfrak{A}$) as Pol(S) where S is the set of all relations of $\mathfrak{A}$. Unary polymorphisms are also called *endomorphisms* of $\mathfrak{A}$; the set of all endomorphisms is denoted by End($\mathfrak{A}$).

For a set S of functions on a set A we define Inv(S) ('invariants of S') as the set of all finitary relations over A which are preserved by all functions in S.

Theorem 7 (Theorem 4 in [14])**.** *Let $\mathfrak{A}$ be a countable ω-categorical relational structure. Then a relation R over A is invariant under the polymorphisms of $\mathfrak{A}$ if and only if R has a pp-definition in $\mathfrak{A}$, i.e.,* $\mathrm{Inv}(\mathrm{Pol}(\mathfrak{A})) = \langle\mathfrak{A}\rangle_{\mathrm{pp}}$.

As a consequence of Theorem 7, we may go back and forth between the (non)-existence of certain polymorphisms and the (non)-pp-definability of certain relations. Furthermore, Theorem 7 implies that the set of polymorphisms of an ω-categorical relational structure $\mathfrak{A}$ fully captures the complexity of CSP($\mathfrak{A}$).

One of the central notions of universal algebra is that of a clone. A set of operations on a common domain is a *clone* if it contains all projections and is closed under composition of functions. Thus, if we fix the domain, an arbitrary intersection of clones is again a clone. Therefore, given a set of operations F over a common domain, there is a unique minimal clone $\langle F\rangle$ containing F, which we call the clone *generated* by F. For a clone $\mathcal{F}$ on domain A we will also need the *local closure* of $\mathcal{F}$, denoted by $\overline{\mathcal{F}}$, which is the smallest clone which contains $\mathcal{F}$ and for any $n \in \mathbb{N}$ and $g\colon A^n \to A$ the following holds: If for all finite $S \subseteq A$ there exists

$f_S \in \mathcal{F}$ such that $f_S|_{S^n} = g|_{S^n}$ then $g \in \overline{\mathcal{F}}$. If $\mathcal{F} = \overline{\mathcal{F}}$, then $\mathcal{F}$ is *locally closed*. It is easy to show that $\mathrm{Pol}(\mathfrak{A})$ is always a locally closed clone for any relational structure $\mathfrak{A}$.

2.3. **The Conditions of Nelson and Oppen.** In this section we recall the classical conditions of Nelson and Oppen on theories T_1 and T_2 with disjoint signatures that guarantee the polynomial-time tractability of $\mathrm{CSP}(T_1 \cup T_2)$. Their condition can be found in [32, 34] and [1] and are the following:

- Both theories T_1 and T_2 must be *stably infinite*, i.e., whenever a finite set of literals S is satisfiable in a model of the theory, then there is also an infinite model of the theory where S is satisfiable.
- Both theories must be *convex*, i.e., if we choose a finite set of literals S such that for all $i \in [n]$ there exist a model of the theory where $S \cup \{x_i \neq y_i\}$ is satisfiable, then there exists a model of the theory where $S \cup \{x_1 \neq y_1, \ldots, x_n \neq y_n\}$ is satisfiable.
- For $i = 1$ and $i = 2$ there exist polynomial-time decision procedures to decide whether a finite set of τ_i-literals is satisfiable in some model of T_i.

The theorem of Nelson and Oppen states that if T_1 and T_2 satisfy these three conditions, then there exists a polynomial-time procedure that decides whether a given set of literals over the signatures of T_1 and T_2 is satisfiable in a model of $T_1 \cup T_2$. Note that this decision problem is in general not equal to $\mathrm{CSP}(T_1 \cup T_2)$, as S is restricted to atomic formulas in the latter. Nelson and Oppen always allow relations of the form $x \neq y$ in the input, which we would like to avoid, because there are tractable first-order expansions of $(\mathbb{Q}; <)$ where adding $\neq$ to the signature makes the CSP hard, as the following examples shows.

Example 8. Let $\mathfrak{A}$ be the temporal structure $(\mathbb{Q}; <, R^{\min}_{\leq})$ where $R^{\min}_{\leq}$ is the relation defined by $\phi(x, y, z) :=x \geq y \vee x \geq z$. Then $\mathrm{CSP}(\mathfrak{A})$ is in P by Theorem 14 below because $\mathfrak{A}$ is preserved by min. But $\mathrm{CSP}(\mathfrak{A}; \neq)$ is NP-hard by Theorem 14 because $(\mathfrak{A}; \neq)$ is neither preserved by a constant operation, mi, mx, min, nor by their duals.

An analysis of the correctness proof of the algorithm of Nelson and Oppen yields that the set of literals in the definition of convexity can be exchanged by a set of atomic formulas if the input of the decision problem is restricted to a set of atomic formulas, i.e., we only require that $\neq$ *is independent from* T_1 *and* T_2 (see Definition 18). Independence of $\neq$, stably infinite theories, tractable CSPs and the presence of $\neq$ in the signature of T_1 and T_2 is what we refer to as the *weakened* conditions of Nelson and Open.

Furthermore, Nelson and Oppen did not require that the signature is purely relational. However, this difference is rather a formal one, because a function can be represented by its graph and nested functions can be unnested in polynomial time by introducing new existentially quantified variables for nested terms. In Section 3 we will prove a tractability criterion which is slightly stronger than the criterion of Nelson and Oppen with weakened conditions.

2.4. **Generic Combinations.** In the context of combining decision procedures for CSPs, the notion of generic combinations has been introduced in [6]. However, others have studied such structures before (for instance in [21, 25, 17, 28]).

Definition 9. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be countably infinite ω-categorical structures with disjoint relational signatures τ_1 and τ_2. A countable model $\mathfrak{A}$ of $\mathrm{Th}(\mathfrak{A}_1) \cup \mathrm{Th}(\mathfrak{A}_2)$ is called a *generic combination of* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ if for any $k \in \mathbb{N}$ and any $a, b \in A^k$

with pairwise distinct coordinates

$$\mathrm{Aut}(\mathfrak{A}^{\tau_1})a \cap \mathrm{Aut}(\mathfrak{A}^{\tau_2})b \neq \emptyset \quad \text{and}$$
$$\mathrm{Aut}(\mathfrak{A}^{\tau_1})a \cap \mathrm{Aut}(\mathfrak{A}^{\tau_2})a = \mathrm{Aut}(\mathfrak{A})a.$$

All generic combinations of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic (Lemma 2.8 in [6]), so we will speak of *the* generic combination of two structures, and denote it by $\mathfrak{A}_1 * \mathfrak{A}_2$.

By definition, the τ_i reduct of $\mathfrak{A} := \mathfrak{A}_1 * \mathfrak{A}_2$ is a model of $\mathrm{Th}(\mathfrak{A}_i)$, which is ω-categorical, and therefore, $\mathfrak{A}^{\tau_i} \cong \mathfrak{A}_i$ for $i = 1$ and $i = 2$. Hence, we may assume without loss of generality that $\mathfrak{A}_1$, $\mathfrak{A}_2$, and $\mathfrak{A}$ have the same domain. It is an easy observation that an instance $\phi_1 \wedge \phi_2$ of $\mathrm{CSP}(T_1 \cup T_2)$, where ϕ_i is a τ_i-formula, is satisfiable if and only if for $i = 1$ and $i = 2$ there exist models $\mathfrak{A}_i$ of T_i with $|A_1| = |A_2|$ such that ϕ_i is satisfiable in $\mathfrak{A}_i$ and the satisfying assignments of ϕ_1 and ϕ_2 identify exactly the same variables. Therefore, the fact that $\mathrm{CSP}(\mathfrak{A}) = \mathrm{CSP}(\mathrm{Th}(\mathfrak{A}_1) \cup \mathrm{Th}(\mathfrak{A}_2))$ easily follows from $\mathrm{Aut}(\mathfrak{A}^{\tau_1})a \cap \mathrm{Aut}(\mathfrak{A}^{\tau_2})b \neq \emptyset$ and ω-categoricity of $\mathfrak{A}_1$ and $\mathfrak{A}_2$.

A structure $\mathfrak{A}$ has *no algebraicity* if every set defined by a first-order formula over $\mathfrak{A}$ with parameters from A is either contained in the set of parameters or infinite. The following proposition characterises when generic combinations of ω-categorical structures exist.

Proposition 10 (Proposition 1.1 in [6]). *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be countably infinite ω-categorical structures with disjoint relational signatures. Then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have a generic combination if and only if either both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ do not have algebraicity or one of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ does have algebraicity and the other structure is preserved by all permutations.*

2.5. **Temporal Structures.** A relation with a first-order definition over $(\mathbb{Q}; <)$ is called *temporal*. An example of a temporal relation is the relation Betw from the introduction. A *temporal structure* is a relational structure $\mathfrak{A}$ with domain $\mathbb{Q}$ all of whose relations are temporal. The structure $(\mathbb{Q}; <)$ is homogeneous, i.e., every order-preserving map between two finite subsets of $\mathbb{Q}$ can be extended to an automorphism of $(\mathbb{Q}; <)$. Therefore, the orbit of a tuple in $\mathfrak{A}$ is determined by identifications and the ordering among the coordinates. It follows from Theorem 6 that all temporal structures are ω-categorical.

2.5.1. *Polymorphisms of Temporal Structures.* One of the fundamental results in the proof of the complexity dichotomy for temporal CSPs, Theorem 11 below, also plays an important role for combinations of temporal CSPs. To understand Theorem 11 and for later use, we define the relations Cycl, Betw, and Sep:

$$\begin{aligned}
\mathrm{Betw} &:= \{(x, y, z) \in \mathbb{Q}^3 \mid (x < y < z) \vee (z < y < x)\}\\
\mathrm{Cycl} &:= \{(x, y, z) \in \mathbb{Q}^3 \mid (x < y < z) \vee (y < z < x) \vee (z < x < y)\}\\
\mathrm{Sep} &:= \{(x, y, u, v) \in \mathbb{Q}^3 \mid (x < u < y < v) \vee (y < u < x < v) \vee\\
&\qquad (x < v < y < u) \vee (y < v < x < u)\}
\end{aligned}$$

Theorem 11 (Bodirsky and Kára [11], Theorem 20). *Let $\mathfrak{A}$ be a temporal structure. Then at least one of the following cases applies.*

- *$\mathfrak{A}$ has a constant endomorphism;*
- *One of the relations $<$, Cycl, Betw, or Sep has a pp-definition in $\mathfrak{A}$.*
- *$\mathfrak{A}$ is preserved by all permutations of $\mathbb{Q}$.*

We introduce several notions that are needed to describe the polynomial-time tractable temporal CSPs from [11]. However, as opposed to [11] we flip the roles of 0 and 1 in the following definition because in this way the resulting systems of equations are homogeneous (see Proposition 16 (4) below; we follow [15]).

Definition 12. For a tuple $t \in \mathbb{Q}^n$ we define the *min-indicator function* $\chi\colon \mathbb{Q}^n \to \{0,1\}^n$ by $\chi(t)[i] \coloneqq 1$ if and only if $t[i] \leq t[j]$ for all $1 \leq j \leq n$. The tuple $\chi(t) \in \{0,1\}^n$ is called the *min-tuple* of $t \in \mathbb{Q}^n$. For an n-ary relation R we define

$$\chi(R) \coloneqq \{\chi(t) \mid t \in R\} \text{ and } \chi_0(R) \coloneqq \chi(R) \cup \{(\underbrace{0,\ldots,0}_{n \text{ zeros}})\}.$$

Let min denote the binary minimum operation on $\mathbb{Q}$. For any fixed endomorphisms α,β,γ of $(\mathbb{Q};<)$ which satisfy $\alpha(a) < \beta(a) < \gamma(a) < \alpha(a+\epsilon)$ for every $a \in \mathbb{Q}$ and every $\epsilon \in \mathbb{Q}$ with $\epsilon > 0$, the binary operation mi on $\mathbb{Q}$ is defined by

$$\mathrm{mi}(x,y) \coloneqq \begin{cases} \alpha(x) & \text{if } x = y, \\ \beta(y) & \text{if } x > y, \\ \gamma(x) & \text{if } x < y. \end{cases}$$

For possibly different α,β satisfying the same conditions, mx is the binary operation on $\mathbb{Q}$ defined by

$$\mathrm{mx}(x,y) \coloneqq \begin{cases} \alpha(\min(x,y)) & \text{if } x \neq y, \\ \beta(x) & \text{if } x = y. \end{cases}$$

Theorem 13 (Lemma 4.1 and Theorem 5.2 in [15]). *We have* $\overline{\langle \{\mathrm{mx}\} \cup \mathrm{Aut}(\mathbb{Q};<)\rangle} = \mathrm{Pol}(\mathbb{Q};X)$ *where*

$$X \coloneqq \{(x,y,z) \in \mathbb{Q}^3 \mid x = y < z \vee x = z < y \vee y = z < x\}.$$

Moreover, every temporal structure $\mathfrak{B}$ *preserved by* mx *either admits a pp-definition of* X *or is preserved by a constant operation or by* min.

Let ll be an arbitrary binary operation on $\mathbb{Q}$ such that $\mathrm{ll}(a,b) < \mathrm{ll}(a',b')$ if

- $a \leq 0$ and $a < a'$, or
- $a \leq 0$ and $a = a'$ and $b < b'$, or
- $a, a' > 0$ and $b < b'$, or
- $a > 0$ and $b = b'$ and $a < a'$.

Let lex: $\mathbb{Q}^2 \to \mathbb{Q}$ be an arbitrary operation that induces the lexicographic order on $\mathbb{Q}^2$ (just like ll if the first argument is not positive). Let pp: $\mathbb{Q}^2 \to \mathbb{Q}$ be an arbitrary operation such that $\mathrm{pp}(a,b) \leq \mathrm{pp}(a',b')$ if and only if either

- $a \leq 0$ and $a \leq a'$, or
- $0 < a$, $0 < a'$ and $b \leq b'$ holds.

The *dual* of an operation $f\colon \mathbb{Q}^n \to \mathbb{Q}$ is defined by $(x_1,\ldots,x_n) \mapsto -f(-x_1,\ldots,-x_n)$. Hence, for any temporal relation R and any operation f on $\mathbb{Q}$, R is preserved by f if and only if the dual of R is preserved by the dual of f.

Notice that the functions mi, mx, pp, ll, their duals, and lex are not uniquely specified by their definitions. They rather specify a unique weak linear order on $\mathbb{Q}^2$. By Observation 10.2.3 in [2], any two functions in $\mathrm{Pol}(\mathbb{Q};<)$ which generate the same weak linear order on $\mathbb{Q}^2$ are equivalent with respect to containment in sub-clones of $\mathrm{Pol}(\mathbb{Q};<)$. Hence, we may assume the following additional properties for convenience:

- $\mathrm{mx}(0,0) = 1$ and $\mathrm{mx}(1,0) = 0$,
- $\mathrm{mi}(0,0) = 0, \mathrm{mi}(1,0) = 1, \mathrm{mi}(0,1) = 2, \mathrm{mi}(1,1) = 3$,
- $\mathrm{ll}(0,0) = 0$, $\mathrm{ll}(1,0) = 1$, $\mathrm{ll}(2,0) = 2$, $\mathrm{ll}(3,0) = 3$ and $\mathrm{ll}(1,1) = 4$.

The polymorphisms we presented are connected by the following inclusions. For $m \in \{\text{min}, \text{mi}, \text{mx}\}$ and $l \in \{\text{ll}, \text{dual-ll}\}$ we have

$$\overline{\langle \text{pp}, \text{Aut}(\mathbb{Q})\rangle} \subseteq \overline{\langle m, \text{Aut}(\mathbb{Q})\rangle},$$
$$\overline{\langle \text{dual-pp}, \text{Aut}(\mathbb{Q})\rangle} \subseteq \overline{\langle \text{dual-}m, \text{Aut}(\mathbb{Q})\rangle},$$
$$\overline{\langle \text{lex}, \text{Aut}(\mathbb{Q})\rangle} \subseteq \overline{\langle l, \text{Aut}(\mathbb{Q})\rangle}.$$

2.5.2. *Complexity of Temporal CSPs.* We can now state the complexity dichotomy for temporal CSPs.

Theorem 14 (Theorem 50 in [11])**.** *Let $\mathfrak{A}$ be a temporal structure with finite signature. Then one of the following applies:*

(1) *$\mathfrak{A}$ is preserved by* min*,* mi*,* mx*,* ll*, the dual of one of these operations, or by a constant operation. In this case* $\text{CSP}(\mathfrak{A})$*, is in P.*
(2) $\text{CSP}(\mathfrak{A})$ *is NP-complete.*

In our proofs, we also need some intermediate results from [11]. In particular, we use the ternary temporal relation introduced in Definition 3 in [11]:

$$T_3 := \{(x, y, z) \mid x = y < z \vee x = z < y\}$$

T_3 is preserved by pp, but by none of the polymorphisms listed in item (1) of Theorem 14 and therefore $\text{CSP}(\mathbb{Q}; T_3)$ is NP-complete.

Lemma 15 (Lemma 36 in [11])**.** *Let $\mathfrak{A}$ be a first-order expansion of $(\mathbb{Q}; <)$ preserved by* pp*. Then either T_3 has a pp-definition in $\mathfrak{A}$, or $\mathfrak{A}$ is preserved by* mi*,* mx*, or* min*.*

2.5.3. *Known Syntactic Descriptions of Temporal Relations.* We also need syntactic descriptions for temporal relations preserved by the operations introduced in the previous sections.

Proposition 16 (Theorems 4, 5, and 6 in [5], Proposition 10.4.7 and Theorem 10.5.18 in [2] and observation above Theorem 42 in [11])**.** *A temporal relation R is preserved by*

(1) pp *if and only if R can be defined by a conjunction of formulas of the form*
$$x_1 \circ_2 x_2 \vee \cdots \vee x_1 \circ_n x_n \text{ where } \circ_i \in \{\neq, \geq\};$$
(2) min *if and only if R can be defined by a conjunction of formulas of the form*
$$x_1 \circ_2 x_2 \vee \cdots \vee x_1 \circ_n x_n \text{ where } \circ_i \in \{>, \geq\}.$$
(3) mi *if and only if R can be defined by a conjunction of formulas of the form*
$$x_1 \circ_2 x_2 \vee \cdots \vee x_1 \circ_n x_n \text{ where } \circ_i \in \{\neq, >, \geq\} \text{ with at most one } \circ_i \text{ equal to } \geq .$$
(4) mx *if and only if R can be defined by a conjunction of formulas $\phi(x_1, \ldots, x_n)$ for which there exists a homogeneous system $Ax = 0$ of linear equations over* GF_2 *such that for every $t \in \mathbb{Q}^n$*
$$t \text{ satisfies } \phi \text{ if and only if } A\chi(t) = 0.$$
In this case, there exists a homogeneous system $Ax = 0$ of linear equations over GF_2 *with solution space $\chi_0(R)$.*
(5) ll *if and only if R can be defined by a conjunction of formulas of the form*
$$(x_1 > x_2 \vee \cdots \vee x_1 > x_m) \vee (x_1 = \cdots = x_m) \vee \bigvee_{m<2i<n} x_{2i} \neq x_{2i+1},$$
where the clause $x_1 = \cdots = x_m$ may be omitted.

Note that the relation R^{mix} can equivalently be written as

$$R^{\mathrm{mix}} = \big\{(a,b,c) \in \mathbb{Q}^3 \mid (a \geq b \vee a > c) \wedge (b \geq a \vee b > c)\big\}.$$

In this representation, it is clear that R^{mix} is preserved by pp and therefore by min, mi and mx.

Every temporal relation can be defined by a quantifier-free $\{<\}$-formula $\phi(x_1,\ldots,x_n)$ and one may assume that ϕ is written in *conjunctive normal form (CNF)*

$$\bigwedge_{\ell=1}^{k} \bigvee_{i \in I_\ell} \phi_{\ell,i}$$

where $\phi_{\ell,i}$ is an atomic $\{<\}$-formula. We say that ϕ is in *reduced CNF* if we cannot remove any disjunct $\phi_{\ell,i}$ from ϕ without altering the defined relation. If ϕ is in reduced CNF, then for any $\ell \in [k]$ and $i \in I_\ell$ there exists $t \in R$ that satisfies $\phi_{\ell,i}$ and does not satisfy any other disjunct $\phi_{\ell,j}$ for $j \in I_\ell \setminus \{i\}$. We use the symbols $\leq, \neq, \geq, >$ as the usual shortcuts, for $x < y \vee x = y$, etc. Clearly, every formula is equivalent to a formula in reduced CNF. Remarkably, the syntactic form in (2) is preserved by removing literals; hence, in (2) we may assume without loss of generality that the definition of R is additionally reduced.

2.6. **Known Relational Generating Sets.** Many important temporal structures $\mathfrak{A}$ can also be described elegantly and concisely by specifying a finite set of temporal relations such that the temporal relations of $\mathfrak{A}$ are precisely those that have a primitive positive definition in $\mathfrak{A}$. Note that such a finite set might not exist even if $\mathfrak{A}$ contains all relations that are pp-definable in $\mathfrak{A}$. We need such a result for the temporal structure that contains all temporal relations preserved by pp.

Proposition 17 (Theorem 12.7.4 in [3])**.** *A temporal relation is preserved by* pp *if and only if it has a primitive positive definition in* $(\mathbb{Q}; \neq, R^{\mathrm{min}}_{\leq}, S^{\mathrm{mi}})$ *where*

$$R^{\mathrm{min}}_{\leq} := \big\{(x,y,z) \in \mathbb{Q}^3 \mid x \geq y \vee x \geq z\big\} \qquad \textit{and}$$
$$S^{\mathrm{mi}} := \big\{(x,y,z) \in \mathbb{Q}^3 \mid x \neq y \vee x \geq z\big\}.$$

3. Polynomial-Time Tractable Combinations

The following definition already appeared in [3] and [20] and is closely related to the convexity condition of Nelson and Oppen. The key difference to convexity of T is that we consider conjunctions of atomic formulas instead of conjunctions of literals.

Definition 18. Let T be a τ-theory. We say that $\neq$ *is independent from* T if for any conjunction of atomic τ-formulas ϕ the formula $\phi \wedge \bigwedge_{i=1}^{k} x_i \neq y_i$ is satisfiable in some model of T whenever for all $i \in [k]$ the formula $\phi \wedge x_i \neq y_i$ is satisfiable is some model of T.

The following is easy to see (see, e.g., [3]).

Proposition 19. *For every structure* $\mathfrak{A}$ *with a binary injective polymorphism,* $\neq$ *is independent from* $\mathrm{Th}(\mathfrak{A})$*.*

Nelson and Oppen require that both theories are stably infinite. We will make a weaker assumption captured by the following notion.

Definition 20. Let T_1 and T_2 be theories with signatures τ_1 and τ_2, respectively. We say that T_1 and T_2 are *cardinality compatible* if for all for $i \in [2]$ and all conjunctions $\phi_i(x_1,\ldots,x_n)$ of atomic τ_i-formulas, such that $\{\exists x_1,\ldots,x_n.\phi_i\} \cup T_i$ has a model, there are models of $\{\exists x_1,\ldots,x_n.\phi_1\} \cup T_1$ and $\{\exists x_1,\ldots,x_n.\phi_2\} \cup T_2$ of equal cardinality.

Clearly, if T_1 and T_2 are stably infinite, then they are also cardinality compatible. Contrary to stably infinite theories where we require that we can choose the cardinality of the models to be countably infinite, the definition of cardinality compatibility also allows theories with finite models only. We also allow theories where some formulas are only satisfiable in finite models while others have infinite models, as the following example shows.

Example 21. Let T be the theory $\{\forall x, y\,(\neg Q(x) \vee x = y)\}$ whose signature only contains the unary relation symbol Q. There is an infinite model for T where Q is empty. However, if ϕ is the formula $Q(x)$, then all models for $T \cup \{\phi\}$ have exactly one element and this element is contained in Q. Hence, T is not stably infinite, but cardinality compatible with itself.

The sufficient conditions for polynomial-time tractability of $\mathrm{CSP}(T_1 \cup T_2)$ given in the following theorem are slightly weaker than those by Nelson and Oppen.

Theorem 22. *Let T_1 and T_2 be cardinality compatible theories with finite, disjoint relational signatures and polynomial-time tractable CSPs. If $\neq$ is independent from both T_1 and T_2 and $\neq$ has an ep-definition in both T_1 and T_2, then* $\mathrm{CSP}(T_1 \cup T_2)$ *is polynomial-time tractable.*

Proof. Let τ_1 and τ_2 be the signatures of T_1 and T_2, respectively. Let S be a set of atomic $\tau_1 \cup \tau_2$-formulas with free variables among $x_1, \ldots, x_n$. Then we may partition S into S_1 and S_2 such that S_i is a set of τ_i-formulas and $S = S_1 \cup S_2$. Without loss of generality, we may assume that all variables occur in both S_1 and S_2 (this can also be attained by introduction of dummy constraints like $x = x$). Let $\phi_i(x, y)$ be an existential positive definition of $x \neq y$ in T_i for $i \in \{1, 2\}$. For $i = 1$ and $i = 2$ and for each tuple of variables (x_k, x_l) and each disjunct $D(\cdot\,, \cdot)$ in ϕ_i we test whether $S_i \cup \{D(x_k, x_l)\}$ is satisfiable is some model of T_i. If, for a fixed tuple (x_k, x_l), the answer is *'unsatisfiable'* for all disjuncts of ϕ_i, then we replace all occurrences of x_l in S_1 and in S_2 by x_k. We iterate this procedure until no more replacements are made. If S_1 or S_2 is unsatisfiable in all models of T_1, T_2 respectively thereafter, we return *'unsatisfiable'*. Otherwise, we return *'satisfiable'*.

To prove that this algorithm is correct, notice that if $S_i \cup \{D(x_k, x_l)\}$ is unsatisfiable for all disjuncts D of ϕ_i, then clearly $S_i \cup \{x_k \neq x_l\}$ is not satisfiable. Moreover, if S_1 or S_2 is unsatisfiable, then their union is unsatisfiable as well. Hence, the substitutions done by the algorithm do not change the satisfiability of $S_1 \cup S_2$ in models of $T_1 \cup T_2$. Let us therefore assume that after the substitution process both S_1 and S_2 are satisfiable in some model of T_1 and T_2 respectively. Without loss of generality we may assume that the variables $x_1, \ldots, x_m$ remain in S_1 and in S_2. Furthermore, we know that $S_i \cup \{x_k \neq x_l\}$ is satisfiable for all $k \neq l$ with $k, l \leq m$ and both $i \in \{1, 2\}$. Therefore, $S_i \cup \bigcup_{k \neq l} \{x_k \neq x_l\}$ is satisfiable in some model of T_i, because $\neq$ is independent from T_i, i.e., there exists $\mathfrak{M}_i \models T_i$ and an injective assignment $s_i \colon \{x_1, \ldots, x_m\} \to \mathfrak{M}_i$ such that $\mathfrak{M}_i \models \bigwedge_{\sigma \in S_i} \sigma(s_i)$, where $\sigma(s_i)$ denotes $\sigma(s_i(y_1), \ldots, s_i(y_k))$ and $y_1, \ldots, y_k$ denote the variables in σ. By the cardinality compatibility of T_1 and T_2, we may assume that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have the same cardinality. Therefore, there exists a bijection $f \colon M_1 \to M_2$ between their domains such that $f(s_1(x_k)) = s_2(x_k)$ for all $k \in [m]$. With this bijection we define a $\tau_1 \cup \tau_2$ structure $\mathfrak{M}$ which is a model of $T_1 \cup T_2$ via $R^{\mathfrak{M}} := R^{\mathfrak{M}_1}$ for $R \in \tau_1$, and $a \in R^{\mathfrak{M}}$ if and only if $f(a) \in R^{\mathfrak{M}_2}$ for $R \in \tau_2$. This is well-defined, because the signatures of T_1 and T_2 are disjoint and because s_1 and s_2 are both injective. It is easy to verify that $\mathfrak{M}$ is a model of $T_1 \cup T_2$ and $\mathfrak{M} \models \bigwedge_{\sigma \in S_1} \sigma(s_1) \wedge \bigwedge_{\sigma \in S_2} \sigma(s_1)$ and hence, the original instance is satisfiable.

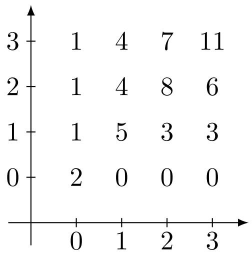

FIGURE 1. The image of mix on $\{0,1,2,3\}^2$.

The number of calls to the decision procedures for T_1 and T_2 is bounded by the number of pairs (x_k, x_l) multiplied by the maximal number of rounds of substitutions and the number of disjuncts in ϕ_1 and ϕ_2. Hence, the runtime of the algorithm is in $O(n^3)$. $\square$

Notice that the tractability result by Nelson and Oppen can be obtained as a special case of Theorem 22 when we consider theories which are stably infinite and where the set of atomic formulas is closed under negation. The following example shows that our condition covers strictly more cases already for combinations of temporal CSPs.

Example 23. For $i = 1$ and $i = 2$, let $(\mathbb{Q}; <_i, \leq_i)$ be a structure where $<_i$ denotes the usual strict linear order on the rational numbers, and $\leq_i$ denotes the corresponding weak linear order. Let $T_i := \mathrm{Th}(\mathbb{Q}; <_i, \leq_i)$. Note that the relation $\neq$ does not have a pp-definition in $(\mathbb{Q}; <_i, \leq_i)$; however, it has the existential positive definition $x <_1 y \vee y <_1 x$. It is well-known that $\mathrm{CSP}(\mathbb{Q}; <_i, \leq_i)$ can be solved in polynomial time [39] and that $\neq$ is independent from T_i [20]. Then T_1 and T_2 satisfy the conditions from Theorem 22 but do not satisfy the conditions of Nelson and Oppen.

4. The Operation mix

A certain temporal structure plays an important role in our proof; it contains the set of all temporal relations preserved by an operation, which we call mix, and which is similar to the polymorphisms mi and mx. We also present an equivalent description of these relations in terms of syntactically restricted quantifier-free $\{<\}$-formulas (Theorem 29).

Definition 24. Let α, β, γ be endomorphisms of $(\mathbb{Q}; <)$ such that $\gamma(a) < \alpha(a) < \beta(a) < \gamma(a + \epsilon)$ for every $a, \epsilon \in \mathbb{Q}$ with $\epsilon > 0$. Then mix is the binary operation on $\mathbb{Q}$ defined by

$$\mathrm{mix}(x, y) := \begin{cases} \alpha(x) & \text{if } x < y, \\ \beta(x) & \text{if } x = y, \\ \gamma(y) & \text{if } x > y. \end{cases}$$

Analogously to the definitions of mi and mx we may without loss of generality fix some concrete values of α, β, and γ. It is convenient to choose $\gamma \colon a \mapsto 3a, \alpha \colon a \mapsto 3a + 1, \beta \colon a \mapsto 3a + 2$ for $a \in \mathbb{Z}^+$. Figure 1 shows some values for mix.

Lemma 25. *The locally closed clone generated by* mix *and* $\mathrm{Aut}(\mathbb{Q}; <)$ *contains* mi.

Proof. It is easy to check that $f(x, y) := \mathrm{mix}(\mathrm{mix}(x, y), 3y)$ induces the same linear order as $\mathrm{mi}(x, y)$ on $(\mathbb{Z}^+)^2$. Hence, for any finite set $S \subseteq \mathbb{Q}$ there exist $\alpha, \beta, \gamma \in \mathrm{Aut}(\mathbb{Q}; <)$ such that $\alpha f(\beta(x), \gamma(y))|_{S^2} = \mathrm{mi}(x, y)|_{S^2}$. Then, by definition, $\mathrm{mi} \in \overline{\langle \mathrm{mix}, \mathrm{Aut}(\mathbb{Q}; <) \rangle}$. $\square$

The relation R^{mix} has the generalisation R_n^{mix} of arity $n \geq 3$ defined as follows.

$$(1) \qquad R_n^{\mathrm{mix}} := \big\{(a_1, \dots, a_n) \in \mathbb{Q}^n \mid \min(a_3, \dots, a_n) \geq \min(a_1, a_2) \Rightarrow a_1 = a_2\big\}$$

Note that $R_n^{\mathrm{mix}}(x_1, \dots, x_n)$ has the following definition in CNF

$$\phi_n^{\mathrm{mix}}(x_1, \dots, x_n) := \big(x_1 \geq x_2 \vee \bigvee_{i \in \{3,\dots,n\}} x_1 > x_i) \wedge (x_2 \geq x_1 \vee \bigvee_{i \in \{3,\dots,n\}} x_2 > x_i\big\}$$

which is both of the form described in item (2) and of the form described in (3) in Proposition 16. Hence, R_n^{mix} is preserved by min and by mi. Also note that $R^{\mathrm{mix}} = R_3^{\mathrm{mix}}$ and that $R^{\mathrm{mix}}(a, b, c)$ is equivalent to $R^{\mathrm{mi}}(a, b, c) \wedge R^{\mathrm{mi}}(b, a, c)$ where

$$R^{\mathrm{mi}} := \big\{(a, b, c) \in \mathbb{Q}^3 \mid a \geq b \vee a > c\big\}.$$

The relation R_n^{mix} is also preserved by mx; we first prove this for R_3^{mix}.

Lemma 26. *The relation R^{mix} is pp-definable in $(\mathbb{Q}; X)$ and hence is preserved by* mx*.*

Proof. It is easy to check that $\exists h\big(X(z, z, h) \wedge X(x, y, h)\big)$ pp-defines $R^{\mathrm{mix}}(x, y, z)$. The second part of the statement then follows from Theorem 13 and Theorem 7. ☐

Lemma 27. *For every $n \geq 3$, the relation R_n^{mix} has a pp-definition in $(\mathbb{Q}; <, R^{\mathrm{mix}})$.*

Proof. A primitive positive definition of R_n^{mix} can be obtained inductively by the observation that $R_n^{\mathrm{mix}}(x_1, \dots, x_n)$ is equivalent to the following formula.

$$(2) \qquad \exists h \left(R_{n-1}^{\mathrm{mix}}(x_1, h, x_3, \dots, x_{n-1}) \wedge R^{\mathrm{mix}}(h, x_2, x_n)\right)$$

Every tuple $t \in R_n^{\mathrm{mix}}$ satisfies (2): if t satisfies $x_1 = x_2$ or if t satisfies $x_n < \min(x_1, x_2)$, choose $h = x_1$; if t satisfies $x_i < \min(x_1, x_2)$ for some $i \in \{3, \dots, n-1\}$, choose $h = x_2$. Conversely, suppose that $t \in \mathbb{Q}^n$ satisfies (2). If t satisfies $x_1 = h$, then t satisfies $x_1 = x_2 = h$ or $x_n < x_1 \wedge x_n < x_2$ and therefore R_n^{mix}. The case that t satisfies $x_2 = h$ is analogous. If t satisfies $x_n < h \wedge x_n < x_2$ and $x_i < x_1 \wedge x_i < h$ for some $i \in \{3, \dots, n\}$, then it also satisfies $\min(x_i, x_n) < \min(x_1, x_2)$ and hence t satisfies R_n^{mix}. ☐

Lemma 28. *For every $n \geq 3$, the operation* mix *preserves R_n^{mix}.*

Proof. To prove that mix preserves R_n^{mix} it suffices prove that mix preserves R^{mix} due to Lemma 27 and Theorem 7. Suppose for contradiction that there are $t_1, t_2 \in R^{\mathrm{mix}}$ such that $t_3 := \mathrm{mix}(t_1, t_2) \notin R^{\mathrm{mix}}$. Then t_3 must satisfy $(x < y \wedge x \leq z) \vee (y < x \wedge y \leq z)$. Without loss of generality we may assume that t_3 satisfies the first disjunct. As $t_3[x]$ is minimal in t_3, the coordinate x must be minimal in either t_1 or t_2. Assume the coordinate x is minimal in t_1; the case with t_2 can be proven analogously. Then t_1 satisfies $x = y$ because $t_1 \in R^{\mathrm{mix}}$. If t_2 satisfies $x = y$ then t_3 satisfies $x = y$, contrary to our assumptions. This implies that $t_2 \in R^{\mathrm{mix}}$ satisfies $z < \min(x, y)$. If $t_2[z] < t_1[x]$ then $\min(t_1[z], t_2[z]) = t_2[z] < \min(t_1[x], t_2[x])$, and hence $t_3[z] < t_3[x]$, a contradiction. Therefore, $\min(t_2[x], t_2[y]) > t_1[x] = t_1[y]$ and hence $t_3[x] = t_3[y]$, a contradiction. ☐

Theorem 29. *A temporal relation is preserved by* mix *if and only if it has a definition by a conjunction of clauses of the form*

$$(3) \qquad \bigvee_{i=1}^{n} x \neq z_i \vee \bigvee_{i=1}^{m} x > y_i \qquad \textit{for } n, m \in \mathbb{N}$$

$$(4) \qquad \textit{and} \quad \phi_n^{\mathrm{mix}}(x_1, x_2, x_3, \dots, x_n) \qquad \textit{for } n \geq 3.$$

	x	y	$y_{1\leq i\leq m}$
$t_3' \coloneqq \alpha(t_3)$	>0	0	≥ 0
$t_1' \coloneqq \beta(t_1)$	0	0	≥ 0
$\mathrm{mix}(t_3', t_1')$	0	>0	≥ 0

TABLE 1. Calculation for the proof of Lemma 29 in case $n = 0$.

Proof. Let R be a temporal relation preserved by mix. Due to Lemma 25, the relation R is also preserved by mi. By Proposition 16 case (3) the relation R can be defined by a conjunction ϕ of clauses of the form

$$x \geq y \vee \bigvee_{i=1}^{n} x \neq z_i \vee \bigvee_{i=1}^{m} x > y_i \qquad \text{for } n, m \in \mathbb{N} \tag{5}$$

where the literal $x \geq y$ can be omitted. Let U_ϕ be the set of clauses in ϕ which do have a literal of the form $x \geq y$ and which cannot be paired with another clause such that their conjunction is of the form ϕ_k^{mix} for some k. Without loss of generality, we may assume that ϕ is chosen such that $|U_\phi|$ is minimal and such that no literal of the form $x \neq z_j$ can be replaced by $x > z_j$ without altering the relation defined by ϕ. If U_ϕ is empty, then we are done. Suppose towards a contradiction that U_ϕ contains a clause $C \coloneqq \big(x \geq y \vee \bigvee_1^n x \neq z_i \vee \bigvee_1^m x > y_i\big)$. Consider the new formulas $\phi_1, \ldots, \phi_{n+3}$ obtained from ϕ by replacing C by, respectively,

$$x > y \vee \bigvee_{1}^{n} x \neq z_i \vee \bigvee_{1}^{m} x > y_i, \tag{6}$$

$$x \geq y \vee x > z_1 \vee \bigvee_{2}^{n} x \neq z_i \vee \bigvee_{1}^{m} x > y_i, \tag{7}$$

$$\phi_{2+n+m}^{\mathrm{mix}}(x, y, z_1, \ldots, z_n, y_1, \ldots, y_m), \tag{8}$$

$$\text{or} \quad \phi_{2+n+m}^{\mathrm{mix}}(z_i, y, z_1, \ldots, z_{i-1}, x, z_{i+1}, \ldots, z_n, y_1, \ldots, y_m) \quad \text{for some } i \in [n]. \tag{9}$$

Note that ϕ_j implies ϕ for each $j \in [n+3]$. Also note that if ϕ is equivalent to ϕ_j we found a contradiction to our choice of ϕ because either $|U_{\phi_j}| < |U_\phi|$ or we can replace a literal of the form $x \neq z_j$. This implies the existence of tuples $t_1, \ldots, t_{n+3} \in R$ that do not satisfy $\phi_1, \ldots, \phi_{n+3}$, respectively. We start the analysis of these tuples with the special case $n = 0$. In this case we get

- a tuple $t_1 \in R$ that does not satisfy Clause (6), i.e., t_1 satisfies $x = y \wedge \bigwedge_{i=1}^m x \leq y_i$;
- a tuple $t_3 \in R$ that does not satisfy Clause (8), i.e., t_3 satisfies $x > y \wedge \bigwedge_{i=1}^m y \leq y_i$.

But then there exist $\alpha, \beta \in \mathrm{Aut}(\mathbb{Q}; <)$ such that $t \coloneqq \mathrm{mix}(\alpha(t_3), \beta(t_1))$ does not satisfy C (see Table 1). Therefore, t does not satisfy ϕ, contradicting the assumption that R is preserved by mix. If $n \geq 1$ the tuples are as follows:

- t_1 does not satisfy Clause (6), i.e., t_1 satisfies $x = y \wedge \bigwedge_{i=1}^n x = z_i \wedge \bigwedge_{i=1}^m x \leq y_i$;
- t_2 does not satisfy Clause (7), i.e., t_2 satisfies $x < y \wedge x < z_1 \wedge \bigwedge_{i=2}^n x = z_i \wedge \bigwedge_{i=1}^m x \leq y_i$;
- t_4 does not satisfy Clause (9) for $i = 1$, i.e., t_4 satisfies

$$z_1 \neq y \wedge (x \geq z_1 \vee x \geq y) \wedge \bigwedge_{j=2}^{n} (z_j \geq z_1 \vee z_j \geq y) \wedge \bigwedge_{j=1}^{m} (y_j \geq z_1 \vee y_j \geq y).$$

	x	y	z_1	$z_{2\leq i\leq n}$	$y_{1\leq i\leq m}$
$t_2' \coloneqq \alpha_2(t_2)$	0	>0	>0	0	≥ 0
$t_1' \coloneqq \alpha_1(t_1)$	0	0	0	0	≥ 0
$t_{yz_1} \coloneqq \text{mix}(t_2', t_1')$	>0	0	0	>0	≥ 0
$t'_{4,xz_1} \coloneqq \alpha_3(t_{4,xz_1})$	0	>0	0	≥ 0	≥ 0
t_{yz_1}	>0	0	0	>0	≥ 0
$t''_{4,y} \coloneqq \text{mix}(t'_{4,xz_1}, t_{yz_1})$	>0	0	>0	>0	≥ 0
$t'_{4,xy} \coloneqq \alpha_4(t_{4,xy})$	0	0	>0	≥ 0	≥ 0
t_1'	0	0	0	0	≥ 0
$t''_{4,z_1} \coloneqq \text{mix}(t'_{4,xy}, t_1')$	>0	>0	0	≥ 0	≥ 0
$t'_{4,z_1} \coloneqq \alpha_5(t_{4,z_1})$	>0	>0	0	≥ 0	≥ 0
t_{yz_1}	>0	0	0	>0	≥ 0
$t'''_{4,y} \coloneqq \text{mix}(t'_{4,z_1}, t_{yz_1})$	>0	0	>0	>0	≥ 0
$t'_{4,y} \coloneqq \alpha_6(t_{4,y})$	>0	0	>0	>0	≥ 0
t_1'	0	0	0	0	≥ 0
$t^* \coloneqq \text{mix}(t'_{4,y}, t_1')$	0	>0	0	0	≥ 0

TABLE 2. Calculation for the proof of Lemma 29 in case $n \geq 1$.

One of the following cases must apply:

(1) R contains t_{4,z_1} satisfying $\psi_{z_1} \coloneqq y > z_1 \wedge x > z_1 \wedge \bigwedge_{j=2}^n z_j \geq z_1 \wedge \bigwedge_{j=1}^m y_j \geq z_1$;
(2) R contains t_{4,xz_1} satisfying $\psi_{xz_1} \coloneqq y > z_1 \wedge x = z_1 \wedge \bigwedge_{j=2}^n z_j \geq z_1 \wedge \bigwedge_{j=1}^m y_j \geq z_1$;
(3) R contains $t_{4,y}$ satisfying $\psi_y \coloneqq z_1 > y \wedge x > y \wedge \bigwedge_{j=2}^n z_j \geq y \wedge \bigwedge_{j=1}^m y_j \geq y$;
(4) R contains $t_{4,xy}$ satisfying $\psi_{xy} \coloneqq z_1 > y \wedge x = y \wedge \bigwedge_{j=2}^n z_j \geq y \wedge \bigwedge_{j=1}^m y_j \geq y$.

Using suitable automorphisms $\alpha_1, \ldots, \alpha_6 \in \text{Aut}(\mathbb{Q}; <)$, we deduce the following (see Table 2):

- in case (1) there is also $t'''_{4,y} \in R$ satisfying ψ_y, so we are also in case (3);
- in case (2) there is also $t''_{4,y} \in R$ satisfying ψ_y, so we also in case (3);
- in case (3) the tuple $t^* \coloneqq \text{mix}(t'_{4,y}, t'_1) \in R$ does not satisfy C, a contradiction.
- in case (4) there is also $t''_{4,z_1} \in R$ satisfying ψ_{z_1}, so we are also in case (3).

Hence, in each case we reached a contradiction, which shows that the assumption that U_ϕ is non-empty must be false.

It remains to show that conjunctions of clauses of the form (3) and (4) are preserved by mix. It suffices to verify that every relation defined by a single clause of this form is preserved by mix. For the clauses of the form (4) we have already shown this in Lemma 28. Let S be the relation defined by $\bigvee_{i=1}^n x \neq z_i \vee \bigvee_{i=1}^m x > y_i$. Suppose for contradiction that there exist $t_1, t_2 \in S$ such that $t_3 \coloneqq \text{mix}(t_1, t_2) \notin S$. Then t_3 must satisfy $x = z_1 = \cdots = z_n \wedge \bigwedge_{i=1}^m x \leq y_i$. Therefore, either t_1 or t_2 must satisfy $C \coloneqq x = z_1 = \cdots = z_n > y_j$ for some j, because mix is only constant on a set of pairs if one coordinate is constant and the other coordinate is bigger or equal to the first one. Without loss of generality we may assume that t_1 satisfies C with $j = 1$.

If t_2 satisfies C with some j_2 then $\min(t_1[x], t_2[x]) > \min(t_1[y_1], t_2[y_{j_2}])$ and therefore $t_3[x] > \min(t_3[y_1], t_3[y_{j_2}])$, contradicting $t_3 \notin R$. If t_2 satisfies $x \neq z_j$

for some j, then $t_3[x] = t_3[z_j]$ implies that $\min(t_2[x], t_2[z_j]) > t_1[x]$. But then $t_3[x] > t_3[y_1]$, contradicting $t_3 \notin S$. □

5. Primitive Positive Definability of the Relation R^{mix}

In this section we prove the following theorem.

Theorem 30. *Let $\mathfrak{A}$ be a first-order expansion of $(\mathbb{Q}; <)$ that is preserved by* pp*. Then R^{mix} has a pp-definition in $\mathfrak{A}$ if and only if $\mathfrak{A}$ is not preserved by* ll.

The proof of this results is organised as follows. If the relation T_3 is pp-definable in $\mathfrak{A}$, then so is R^{mix} (Proposition 40). Otherwise, Lemma 15 implies that $\mathfrak{A}$ is preserved by mi, mx, or min. It therefore suffices to treat first-order expansions $\mathfrak{A}$ of $(\mathbb{Q}; <)$ that are

- preserved by mi (Section 5.1),
- preserved by mx but not by mi (Section 5.2), and finally
- preserved by min but not by mi and not by mx (Section 5.3).

5.1. Temporal Structures Preserved by mi.

In this section we prove Theorem 30 for first-order expansions $\mathfrak{A}$ of $(\mathbb{Q}; <)$ that are preserved by mi (Proposition 35). For this purpose, it turns out to be highly useful to distinguish whether the relation $\leq$ has a pp-definition in $\mathfrak{A}$ or not. If yes, then the statement can be shown directly (Proposition 31). Otherwise, $\mathfrak{A}$ is preserved by the operation mix from Section 4 (Proposition 33). Then the syntactic normal form for temporal relations preserved by mix from Section 4 can be used to show the statement.

Proposition 31. *Let $\mathfrak{A}$ be a first-order expansion of $(\mathbb{Q}; \leq)$ which is preserved by* mi *but not by* ll*. Then R^{mi} and hence R^{mix} has a pp-definition in $\mathfrak{A}$.*

Proof. Let R be a relation of $\mathfrak{A}$ which is not preserved by ll. As R is preserved by mi, Proposition 16 (3) implies that R can be defined by a conjunction ϕ of clauses of the form

$$x \geq y \vee \bigvee_{i=1}^{m} x > y_i \vee \bigvee_{i=1}^{n} x \neq z_i.$$

We may assume that the literals $x > y_1, \dots, x > y_i$ cannot be removed from such clauses without changing the relation defined by the formula. As R is not preserved by ll, Proposition 16 (5) implies that ϕ must contain a conjunct C of the form $x \geq y \vee \bigvee_{i=1}^{m} x > y_i \vee \bigvee_{i=1}^{n} x \neq z_i$ where $m \geq 1$. Assume for contradiction that $\phi \wedge x = y$ implies $x = y_1 = \dots = y_m \vee \bigvee_{i=1}^{m} x > y_i \vee \bigvee_{i=1}^{n} x \neq z_i$. Then we can replace C by $x > y \vee \bigvee_{i=1}^{m} x > y_i \vee x = y = y_1 = \dots = y_m \vee \bigvee_{i=1}^{n} x \neq z_i$. However, if this is possible for all C with $m \geq 1$, then R is preserved by ll, contradiction. So we may suppose that there exists a tuple $t_1 \in R$ and $j \in [m]$ such that

$$t_1 \text{ satisfies} \quad x = y \wedge x < y_j \wedge \bigwedge_{i \neq j} x \leq y_i \wedge \bigwedge_{i=1}^{n} x = z_i.$$

For the sake of notation, we assume that $j = 1$. As the literal $x > y_1$ can not be removed from C without changing the relation defined by ϕ, there is a tuple $t_2 \in R$ such that

$$t_2 \text{ satisfies} \quad y > x \wedge x > y_j \wedge \bigwedge_{i \neq j} x \leq y_i \wedge \bigwedge_{i=1}^{n} x = z_i.$$

We may assume that $x, y, y_1, \dots, y_m, z_1, \dots, z_n$ refer to the first $2+m+n$ coordinates of R, in that order. Choose $k \in \mathbb{N}$ such that $2 + m + n + k$ is the arity of R and let

$u_1, \ldots, u_k, y', z$ be fresh variables. The following is a pp-definition of R^{mi} in $\mathfrak{A}$:

$$\psi(x,y',z) \coloneqq \exists y, y_1, y_2, \ldots, y_m, z_1, \ldots, z_n, u_1, \ldots, u_k \Big(y' \leq y \wedge z \leq y_1$$
$$\wedge\; R(x,y,y_1,\ldots,y_m,z_1,\ldots,z_n,u_1,\ldots,u_k) \wedge \bigwedge_{i=2}^{m} x \leq y_i \wedge \bigwedge_{i=1}^{n} x = z_i \Big)$$

To see this, first note that the quantifier-free part of ψ implies that $x \geq y \vee x > y_1$, and hence that $x \geq y' \vee x > z$.

Conversely, choose $(a,b,c) \in R^{\mathrm{mi}}$. If $a \geq b$ then choose $\alpha \in \mathrm{Aut}(\mathbb{Q};<)$ such that $\alpha(t_1[x]) = a$ and $\alpha(t_1[y_1]) \geq c$ and set $y' = b$ and $z = c$. This is possible because $t_1[y_1] > t_1[x]$. Then $\alpha(t_1)$ provides values for $y, y_1, \ldots, u_k$ which satisfy all conjuncts of ψ: the conjunct $R(x,y,y_1,\ldots)$ is satisfied because $\alpha(t_1) \in R$, and for the other conjuncts this is immediate. Hence, $\psi(a,b,c)$ holds. If $a > c$ then choose $\alpha \in \mathrm{Aut}(\mathbb{Q};<)$ such that $\alpha(t_2[x]) = a$, $\alpha(t_2(y)) \geq b$ and $\alpha(t_2[y_1]) = c$, $y' = b$ and $z = c$. This is possible because $t_2[y] > t_2[x] > t_2[y_1]$. Then $\alpha(t_2)$ provides values for $y, y_1, \ldots, u_k$ which satisfy all conjuncts of ψ: the conjunct $R(x,y,y_1,\ldots)$ is satisfied because $\alpha(t_2) \in R$ and for the other conjuncts this is immediate. □

Lemma 32. *Let $\mathfrak{A}$ be a first-order expansion of $(\mathbb{Q};<)$ which is preserved by* mi *and where $\leq$ is not pp-definable. Then $\mathfrak{A}$ has a binary polymorphism f such that for all positive $a_1, a_2, b_1, b_2 \in \mathbb{Q}$*

(10) $$2 = f(0,0) > f(0,b_1) = 1 = f(0,b_2) > f(a_1,0) = f(a_2,0) = 0.$$

Proof. By Theorem 7 there exists a polymorphism of $\mathfrak{A}$ that does not preserve $\leq$. There is also a binary polymorphism g with this property, by Lemma 10 in [11]. We can without loss of generality assume that there exist $p_1, p_2, q \in \mathbb{Q}$ such that $p_1 < p_2$ and $g(p_1,q) > g(p_2,q)$. Define $g' \coloneqq \gamma g(\alpha,\beta)$ with $\alpha,\beta,\gamma \in \mathrm{Aut}(\mathbb{Q};<)$ such that $\alpha^{-1}(p_1,p_2) = (0,1)$, $\beta^{-1}(q) = 0$, and $\gamma(g(p_1,q), g(p_2,q)) = (1,0)$. Then $g'(0,0) = 1$ and $g'(1,0) = 0$. Defining $g''(x,y) \coloneqq g'(\mathrm{mi}(x,y),y)$ we get $g''(0,0) = g'(0,0) = 1$ and for all $c > 0$ we get $g''(c,0) = g'(1,0) = 0$ and $g''(0,c) = g'(2,c) \eqqcolon d > 1$. Defining $f(x,y) \coloneqq \mathrm{mi}(g''(y,x), g''(x,y))$ we get $f(0,0) = \mathrm{mi}(1,1) = 3$, and for all $c > 0$ we get $f(c,0) = \mathrm{mi}(d,0) = 1$, and $f(0,c) = \mathrm{mi}(0,d) = 2$. As $x \mapsto x-1$ is in $\mathrm{Aut}(\mathfrak{A})$, the function $(x,y) \mapsto f(x,y) - 1$ satisfies (10). □

The following proposition is similar to Proposition 10.5.13 in [2].

Proposition 33. *Let $\mathfrak{A}$ be a temporal structure preserved by* pp *such that $\leq$ does not have a pp-definition in $\mathfrak{A}$. Then $\mathfrak{A}$ is preserved by* mix.

Proof. Let R be a k-ary relation of $\mathfrak{A}$ and $r,s \in R$. We have to show that $t \coloneqq \mathrm{mix}(r,s)$ is in R. Let $\alpha,\beta,\gamma \in \mathrm{End}(\mathbb{Q};<)$ be from the definition of mix. Let $v_1 < \cdots < v_l$ be the shortest sequence of rational numbers such that $t_i \in \bigcup_{j \in [l]} \{\alpha(v_j), \beta(v_j), \gamma(v_j)\}$ for every $i \in [k]$. For every $j \in [l]$ we define

$$M_j \coloneqq \big\{ i \in [k] \mid t_i \in \{\alpha(v_j), \beta(v_j), \gamma(v_j)\} \big\}.$$

Observe that $M_1, \ldots, M_l$ is a partition of $[k]$ and therefore defines a partition on $\{t_1, \ldots, t_k\}$. Furthermore, for each $i \in M_j$ either $v_j = r_i \leq s_i$ or $v_j = s_i \leq r_i$ holds. This defines a partition of M_j into three parts:

$$\begin{aligned} M_j^{\alpha} &\coloneqq \{ i \in M_j \mid v_j = r_j < s_j \}, \\ M_j^{\beta} &\coloneqq \{ i \in M_j \mid v_j = r_j = s_j \}, \\ \text{and} \quad M_j^{\gamma} &\coloneqq \{ i \in M_j \mid v_j = s_j < r_j \}. \end{aligned}$$

Let $\alpha_1, \ldots, \alpha_l \in \mathrm{Aut}(\mathbb{Q}; <)$ be such that $\alpha_j(v_j) = 0$ for all $j \in [l]$. By Lemma 32 there is a binary $f \in \mathrm{Pol}(\mathfrak{A})$ satisfying (10). For each $j \in [l]$ we define

$$u^j := \mathrm{pp}\left(f(\alpha_j r, \alpha_j s), \mathrm{pp}(\alpha_j s, \alpha_j r)\right)$$

It is easy to verify that for all $i \in M_j$ and $w, w' > 0$

$$\begin{aligned}
\text{if } i \in M_j^\alpha \text{ then } u_i^j &= \mathrm{pp}(f(0,w), \mathrm{pp}(w',0)) = \mathrm{pp}(1, \mathrm{pp}(1,0)), \\
\text{if } i \in M_j^\beta \text{ then } u_i^j &= \mathrm{pp}(f(0,0), \mathrm{pp}(0,0)) = \mathrm{pp}(2, \mathrm{pp}(0,0)), \\
\text{and if } i \in M_j^\gamma \text{ then } u_i^j &= \mathrm{pp}(f(w,0), \mathrm{pp}(0,w')) = \mathrm{pp}(0,0).
\end{aligned}$$

In particular, u^j is constant on each of M_j^α, M_j^β, M_j^γ and $u_i^j > u_{i'}^j$ for $i \in M_j^\alpha$ and $i' \in M_j^\beta$. We apply f again to obtain $z^j := f(\alpha_j r, \beta_j u^j)$ where $\beta_j \in \mathrm{Aut}(\mathbb{Q}; <)$ is such that $\beta_j(\mathrm{pp}(2, \mathrm{pp}(0,0))) = 0$. Then we get for all $i \in M_j$ and $w > 0$ that

$$\begin{aligned}
\text{if } i \in M_j^\alpha \text{ then } z_i^j &= f(0,w) = 1, \\
\text{if } i \in M_j^\beta \text{ then } z_i^j &= f(0,0) = 2, \\
\text{and if } i \in M_j^\gamma \text{ then } z_i^j &= f(w,e) < f(0,e') = 0 \text{ for some } e' < e < 0.
\end{aligned}$$

Thus, we found $z^1, \ldots, z^l \in R$ such that for all $i \in M_j^\beta$, $i' \in M_j^\alpha$, and $i'' \in M_j^\gamma$ we have $z_i^j > z_{i'}^j > z_{i''}^j$. Take any $j, j' \in [l]$ such that $j < j'$ and choose $i \in M_j^\beta$ and $i' \in M_{j'}$. Then $v_j = r_i = s_i < v_{j'} = \min(s_{i'}, r_{i'})$ and therefore $z_i^j < z_{i'}^{j'}$ because f, pp, and all automorphisms preserve $<$. Therefore, we can apply Lemma 10.5.3 in [2] to $z^1, \ldots, z^l$ which yields the existence of a tuple $t^* \in R$ with satisfies $t_i^* < t_{i'}^*$ if and only if there exists $j < j'$ such that $i \in M_j, i' \in M_{j'}$, and $z_i^j < z_{i'}^{j'}$. However, this is the same ordering that t satisfies and hence, $t \in R$. ☐

Proposition 34. *Let $\mathfrak{A}$ be a first-order expansion of $(\mathbb{Q}; <)$ preserved by* mix *but not by* ll. *Then R^{mix} has a pp-definition in $\mathfrak{A}$.*

Proof. Let R be a relation in $\mathfrak{A}$ that is not preserved by ll. Lemma 29 implies that R can be defined by conjunctions of clauses the form (3) and (4). As R is not preserved by ll, any such definition must include at least on clause of the form (4). Consider a clause of the form (4) $\phi_n^{\mathrm{mix}}(x_1, x_2, x_3, \ldots, x_n) = C_x \wedge C_y$ with

$$C_x := \left(x \geq y \vee \bigvee_{i=1}^{n} x > z_i \right) \qquad C_y := \left(y \geq x \vee \bigvee_{i=1}^{n} y > z_i \right).$$

Claim 1. Suppose that the literal $x \geq y$ can be replaced by $x > y$ in C_x without changing the relation defined by ϕ. Then we can also replace the literal $y \geq x$ by $y > x$ in C_y without changing the relation defined by ϕ.

The assumption implies that if $x \geq y$ is satisfied by a tuple $t \in R$ then either t satisfies $x > y$, or t satisfies $x = y$ and there exists i such that t satisfies $x > z_i$. In the first case t satisfies $y > z_j$ (in order to satisfy C_y) and hence t still satisfies ϕ after replacing $y \geq x$ by $y > x$ in C_y. In the second case, t satisfies $y = x > z_i$ and thus again satisfies C_y after the same replacement.

Claim 2. Suppose that for some $i \in [n]$, the literal $x > z_i$ can be removed from C_x without changing the relation defined by ϕ. Then $y > z_i$ can be removed from C_y without changing the relation defined by ϕ.

Case 1: All tuples $t \in R$ satisfy $x \leq z_i$, i.e., $x > z_i$ is never true. If there is $t \in R$ such that t satisfies $y > z_i$, then t also satisfies $y > x$. Hence, we can also remove $y > z_i$ from C_y without altering the relation defined by the formula.

Case 2: There exists $t \in R$ where $x > z_i$ holds. Suppose for contradiction that there exists a tuple $t_{y,i} \in R$ which does not satisfy C_y after deletion of $y > z_i$ in

	x	y	z_i	$z_{j\neq i}$
$t' := \alpha_1(t_{y,i})$	2	1	0	≥ 1
$t'_c := \alpha_2(t_c)$	1	1	≥ 1	≥ 1
$\mathrm{mix}(t', t'_c)$	3	5	1	≥ 3

TABLE 3. Calculation for Claim 2 (Case 2) in the proof of Proposition 34.

	x	y	z_j	z_i	z_k
$t'_c := \alpha_1(t_c)$	0	0	> 0	0	0
$t'_{x,i} := \alpha_2(t_{x,i})$	1	2	≥ 1	0	≥ 1
$t_c^{\{i,j\}} := \mathrm{mix}(t'_c, t'_{x,i})$	1	1	> 2	2	1

TABLE 4. Calculation of $t_c^{i,j}$ in the proof of Proposition 34.

C_y. Then

$$t_{y,i} \text{ satisfies } x > y \wedge y > z_i \wedge \bigwedge_{j\neq i} z_j \geq y.$$

As we already know that literal replacement can be applied to C (Claim 1), we can assume that no literal in ϕ can be replaced. Therefore, there exists $t_c \in R$ such that

$$t_c \text{ satisfies} \quad x = y \wedge \bigwedge_{i=1}^{n} x \leq z_i.$$

Then there exist $\alpha_1, \alpha_2 \in \mathrm{Aut}(\mathbb{Q}; <)$ such that $\mathrm{mix}(t_{y,i}, t_c)$ satisfies $y > x \wedge x > z_i \wedge \bigwedge_{j\neq i} z_j \geq x$ (see Table 3), contradicting the assumption that we can remove $x > z_i$.

Claims 1 and 2 imply that we may assume without loss of generality that the literal $x \geq y$ cannot be replaced by $x > y$, that the literal $x > z_i$ cannot be removed from C_x and, symmetrically, that $y > z_i$ cannot be removed from C_y without changing the relation defined by ϕ. Hence, there are $t_c, t_{x,i}, t_{y,i} \in R$ such that for all $1 \leq i \leq n$

$$t_c \text{ satisfies} \quad x = y \wedge \bigwedge_{i=1}^{n} x \leq z_i, \qquad t_{x,i} \text{ satisfies} \quad z_i < x < y \wedge \bigwedge_{j\neq i} x \leq z_j,$$

$$\text{and } t_{y,i} \text{ satisfies} \quad z_i < y < x \wedge \bigwedge_{j\neq i} y \leq z_j.$$

Now we apply mix to t_c, $t_{x,i}$, and $t_{y,i}$ to prove that tuples with more specific properties must be contained in R.

We first prove that R must contain a tuple t_c^* satisfying $x = y \wedge \bigwedge_{i\in[n]} x < z_i$. Suppose that $i, j, k \in [n]$ are such that t_c satisfies $z_j > z_i = z_k = x$. Then we get $t_c^{\{i,j\}} \in R$ satisfying $z_j > x \wedge z_i > x \wedge x = z_k$, i.e., we change $z_i = x$ to $z_i > x$ without changing $z_j > x$ or $z_k = x$ for any j, k. This can be done by appropriately choosing automorphisms $\alpha_1, \alpha_2 \in \mathrm{Aut}(\mathbb{Q}; <)$ as shown in Table 4. Using this argument repeatedly, we obtain a tuple t_c^* with the required property.

	x	y	z_i	$z_{j\neq i}$
$t'_{x,i} \coloneqq \alpha_1(t_{x,i})$	1	2	0	≥ 1
$t_c^{*\prime} \coloneqq \alpha_2(t_c^*)$	1	1	>1	>1
$h_{x,i} \coloneqq \mathrm{mix}(t'_{x,i}, t_c^{*\prime})$	5	3	1	≥ 4
$t'_{y,i} \coloneqq \alpha_3(t_{y,i})$	2	1	0	≥ 1
$t_c^{*\prime}$	1	1	>1	>1
$h_{y,i} \coloneqq \mathrm{mix}(t'_{y,i}, t_c^{*\prime})$	3	5	1	≥ 4
$h'_{x,i} \coloneqq \alpha_4(h_{x,i})$	3	1	0	≥ 2
$h'_{y,i} \coloneqq \alpha_5(h_{y,i})$	1	3	0	≥ 2
$t^*_{x,i} \coloneqq \mathrm{mix}(h'_{x,i}, h'_{y,i})$	3	4	2	≥ 6
$h'_{y,i}$	1	3	0	≥ 2
$h'_{x,i}$	3	1	0	≥ 2
$t^*_{y,i} \coloneqq \mathrm{mix}(h'_{y,i}, h'_{x,i})$	4	3	2	≥ 6

TABLE 5. Calculation of $t^*_{x,i}$ and $t^*_{y,i}$ in the proof of Proposition 34.

Our next goal is to prove the existence of $t^*_{x,i}, t^*_{y,i} \in R$ such that

$$\begin{aligned} & t^*_{x,i} \text{ satisfies} \quad z_i < x < y \wedge \bigwedge_{j\neq i} y < z_j \\ \text{and} \quad & t^*_{y,i} \text{ satisfies} \quad z_i < y < x \wedge \bigwedge_{j\neq i} x < z_j . \end{aligned}$$

Using t_c^* and appropriately chosen $\alpha_1, \ldots, \alpha_5 \in \mathrm{Aut}(\mathbb{Q}; <)$ we may first produce $h_{x,i}, h_{y,i} \in R$ and combine them to get $t^*_{x,i}, t^*_{y,i} \in R$ as shown in Table 5.

Without loss of generality we may assume that $x, y, z_1, \ldots, z_n$ correspond to the first $n+2$ coordinates in R. Let $u_1, \ldots, u_m$ be fresh variables such that the arity of R is $2+n+m$ and define

$$\begin{aligned} \psi(x, y, z_1, \ldots, z_n) &\coloneqq R(x, y, z_1, \ldots, z_n, u_1, \ldots, u_m) \wedge \bigwedge_{i=2}^{n} x < z_i \wedge y < z_i \\ \text{and} \quad \psi'(x, y, z) &\coloneqq \exists z_1, \ldots, z_k, u_1, \ldots, u_m \big(\psi(x, y, z_1, \ldots, z_n) \wedge z < z_1\big). \end{aligned}$$

To show that ψ' defines R^{mix}, first notice that $t^*_{x,1}$, $t^*_{y,1}$, and t^*_c satisfy ψ and that ψ implies $x \geq y \vee x > z_1$ and $y \geq x \vee y > z_1$ because all disjuncts of C_x and C_y involving $z_2, \ldots, z_n$ do not hold. This in turn implies that the set of orbits of (x, y, z_1) in tuples that satisfy ψ is contained in R^{mix}. It follows that if (a, b, c) satisfies ψ', then either $a = b$, or there exists z_1 such that $c < z_1 < \min(a, b)$, so $(a, b, c) \in R^{\mathrm{mix}}$.

Conversely, let (a, b, c) be in R^{mix}. If $a = b$ and we may choose $\alpha \in \mathrm{Aut}(\mathfrak{A})$ such that $\alpha(t^*_c[x]) = a$ and $\alpha(t^*_c[z_1]) > c$, in which case $\alpha(t^*_c)$ yields values for $z_1, \ldots, u_m$ which prove that (a, b, c) satisfies ψ'. If $c < a < b$ then there exists $\alpha \in \mathrm{Aut}(\mathfrak{A})$ such that $\alpha(t^*_{x,1})[x] = a$, $\alpha(t^*_{x,1})[y] = b$ and $\alpha(t^*_{x,1})[z_1] > c$. Hence, $\alpha(t^*_{x,1})$ shows that (a, b, c) satisfies ψ'. The argument for $c < b < a$ works with $t^*_{y,1}$ in an analogous way. $\square$

Now we are ready to prove the main result of this subsection.

Proposition 35. *Let $\mathfrak{A}$ be a first-order expansion of $(\mathbb{Q}; <)$ which is preserved by* mi*, but not by* ll*. Then R^{mix} has a pp-definition in $\mathfrak{A}$.*

Proof. If $\leq$ is pp-definable in $\mathfrak{A}$, then Proposition 31 yields that R^{mix} is pp-definable in $\mathfrak{A}$. If $\leq$ is not pp-definable in $\mathfrak{A}$ then Proposition 33 yields that $\mathfrak{A}$ is preserved by mix. In this case Proposition 34 implies that R^{mix} is pp-definable. □

5.2. **Temporal Structures Preserved by** mx**.** In this section we consider first-order expansions of $(\mathbb{Q}; <)$ that are preserved by mx. We distinguish the case whether X is pp-definable in $\mathfrak{A}$ or not. Theorem 13 implies that if X is not pp-definable in $\mathfrak{A}$, then $\mathfrak{A}$ is also preserved by min. So we first consider the situation that $\mathfrak{A}$ is preserved by both mx and min. For $R \subseteq \mathbb{Q}^n$, $t = (t_1, \ldots, t_n) \in R$ and $I = \{i_1, \ldots, i_l\} \subseteq [n]$ we write $\pi_I(t)$ for the tuple $(t_{i_1}, \ldots, t_{i_l})$ where $i_1 < i_2 < \cdots < i_l$ and $\pi_I(R)$ for the relation $\{\pi_I(t) \mid t \in R\}$.

Proposition 36. *Let $\mathfrak{A}$ be a first-order expansion of $(\mathbb{Q}; <)$ that is preserved by* mx *and* min*. Then $\mathfrak{A}$ is preserved by* mi*.*

Proof. Let R be a relation in $\mathfrak{A}$. The proof proceeds by induction on the arity n of R. For $n = 1$, or if R is empty, there is nothing to be shown. Suppose that the statement holds for all relations of arity less than n and that R is not empty. For every $I \subseteq [n]$ we fix a homogeneous system $A_I^R x = 0$ of Boolean linear equations with solution space $\chi_0(\pi_I(R))$, which exists due to case (4) in Theorem 16. As R is preserved by min, the Boolean maximum operation preserves $\chi_0(\pi_I(R))$. Furthermore, the solution space of a system of homogeneous linear equations over GF_2 is also preserved by the operation $(x, y, z) \mapsto x + y + z \mod 2$ (because it is a subspace of GF_2^3), we get that $\chi_0(\pi_I(R))$ is also preserved by min because $\min(x, y) = \max(x, y) + x + y \mod 2$. For every pair $t, t' \in R$ we want to show that $\mathrm{mi}(t, t') \in R$. If $\min(t) = \min(t')$, we consider the set $S := \{i \in [n] \mid \chi(t)[i] = \chi(t')[i] = 1\}$ and distinguish two cases:

(1) If $S \neq \emptyset$ then $\chi(\mathrm{mi}(t, t')) = \min(\chi(t), \chi(t')) \in \chi(R)$.
(2) If $S = \emptyset$, then $\chi(\mathrm{mi}(t, t')) = \chi(t') \in \chi(R)$.

If $\min(t) \neq \min(t')$, then $\chi(\mathrm{mi}(t, t')) \in \{\chi(t), \chi(t')\} \subseteq \chi(R)$.

Thus, there exists a tuple $c \in R$ with $\chi(c) = \chi(\mathrm{mi}(t, t'))$. Let $I := \{i \mid \chi(c)[i] = 1\}$ and observe that I is non-empty. By induction hypothesis, the statement holds for $\pi_{[n]\setminus I}(R)$ and we have $\pi_{[n]\setminus I}(\mathrm{mi}(t, t')) = \mathrm{mi}(\pi_{[n]\setminus I}(t), \pi_{[n]\setminus I}(t')) \in \pi_{[n]\setminus I}(R)$. Therefore, there exists $r \in R$ with $\pi_{[n]\setminus I}(\mathrm{mi}(t, t')) = \pi_{[n]\setminus I}(r)$. We can apply an automorphism of $(\mathbb{Q}; <)$ to r to obtain a tuple $r' \in R$ where all entries are positive. We can also apply an automorphism to c to obtain a tuple $c' \in R$ so that its minimal entries are 0 and for every other entry $i \in [n] \setminus I$ it holds that $c'[i] > r'[i]$. Then $\mathrm{mx}(c', r')$ yields a tuple in R which is minimal at the coordinates in I and all other coordinates are ordered like the coordinates in r, i.e., $\mathrm{mx}(c', r')$ is equal to $\mathrm{mi}(t, t')$ under an automorphism. Hence, $\mathrm{mi}(t, t') \in R$, i.e., R is preserved by mi. □

Proposition 37. *Let $\mathfrak{A}$ be a first-order expansion of $(\mathbb{Q}; <)$ that is is preserved by* mx *but not by* mi*. Then R^{mix} is pp-definable in in $\mathfrak{A}$.*

Proof. If X is pp-definable in $\mathfrak{A}$, then Lemma 26 implies that the relation R^{mix} has a pp-definition in $\mathfrak{A}$. Otherwise, Theorem 13 implies that $\mathfrak{A}$ is also preserved by min, and hence by mi by Proposition 36, which contradicts our assumptions. □

5.3. **Temporal Structures Preserved by** min**.** This section treats first-order expansions of $(\mathbb{Q}; <)$ that are preserved by min but not by mi and mx. We first show that we may assume that $\leq$ has a pp-definition in $\mathfrak{A}$.

Lemma 38. *Let $\mathfrak{A}$ be a first-order expansion of $(\mathbb{Q}; <)$ which is preserved by* pp *and does not admit a pp-definition of $\leq$. Then $\mathfrak{A}$ is preserved by* mi *or by* mx*.*

Proof. By Theorem 7 there exists an $f \in \mathrm{Pol}(\mathfrak{A})$ that does not preserve $\leq$. As $\leq$ is a union of two orbits of $\mathrm{Aut}(\mathbb{Q};<) = \mathrm{Aut}(\mathfrak{A})$, there is a binary polymorphism f' of $\mathfrak{A}$ that does not preserve $\leq$ by Lemma 10 in [11]. As $\mathfrak{A}$ is also preserved by pp, Lemma 35 in [11] implies that $\mathfrak{A}$ is preserved by an operation providing *min-intersection closure* or *min-xor closure*. Then $\mathfrak{A}$ is preserved by mi or by mx by Proposition 27 and Proposition 29 in [11], respectively. $\square$

Proposition 39. *Let $\mathfrak{A}$ be a first-order expansion of $(\mathbb{Q};<)$ preserved by* min *but not by* mi *and not by* mx. *Then $R^{\min}_{\leq}$, R^{mi}, and R^{mix} have a pp-definition in $\mathfrak{A}$.*

Proof. Let R be a relation of $\mathfrak{A}$ that is not preserved by mi and let n be the arity of R. As R is preserved by min, it is definable by a conjunction ϕ of formulas where each conjunct is of the form as described in Proposition 16 (2). Furthermore, there must be a clause C in ϕ that is not preserved by mi. By Proposition 16 (3) C is of the form

$$x > x_1 \vee \cdots \vee x > x_\ell \vee x \geq y_1 \vee \cdots \vee x \geq y_k$$

with $k > 1$. Furthermore, we can assume that ϕ is in reduced CNF. Hence, there exist tuples $t_1, t_2 \in R$ witnessing that the literals $x \geq y_1$ and $x \geq y_2$ cannot be replaced by $x > y_1$ and by $x > y_2$, respectively, i.e.,

$$t_1 \text{ satisfies} \quad x = y_1 \wedge x < y_2 \wedge \bigwedge_{i=1}^{\ell} x \leq x_i \wedge \bigwedge_{i=3}^{k} x < y_i,$$

$$t_2 \text{ satisfies} \quad x < y_1 \wedge x = y_2 \wedge \bigwedge_{i=1}^{\ell} x \leq x_i \wedge \bigwedge_{i=3}^{k} x < y_i.$$

Let $z_1, \dots, z_m$ be all the variables from ϕ that do not occur in C. Without loss of generality, we may assume that the coordinates of R are in the following order: $x, x_1, \dots, x_\ell, y_1, \dots, y_k, z_1, \dots, z_m$. As $\mathfrak{A}$ is not preserved by mx, Lemma 38 implies that $\leq$ has a pp-definition in $\mathfrak{A}$; so we may assume that $\leq$ is among the relations of $\mathfrak{A}$. We claim that $R^{\min}_{\leq}$ can be defined over $\mathfrak{A}$ by the primitive positive formula $\phi(x,u,v)$ given as follows.

$$\exists z_1, \dots, z_m, x_1, \dots, x_\ell, y_1, \dots, y_k \Big(R(x, x_1, \dots, x_\ell, y_1, \dots, y_k, z_1, \dots, z_m) \\ \wedge y_1 \geq u \wedge y_2 \geq v \wedge \bigwedge_{i=1}^{\ell} x \leq x_i \wedge \bigwedge_{i=3}^{k} x < y_i \Big)$$

To prove the claim, let $(a,b,c) \in R^{\min}_{\leq}$. Assume that $a \geq b$. There exists $\alpha \in \mathrm{Aut}(\mathfrak{A})$ such that $t_1' \coloneqq \alpha(t_1)$ satisfies $t_1'[x] = a$ and $t_1'[y_2] > \max(a,c)$. Now we extend t_1' by two coordinates, named u and v such that $t_1'[u] = b$ and $t_1'[v] = c$. Then $\pi_{\{x,u,v\}}(t_1') = (a,b,c)$ and t_1' satisfies the quantifier-free part of ϕ. Therefore, $\phi(a,b,c)$ holds. The case where $a \geq c$ holds is handled analogously using t_2 instead of t_1.

Now suppose that (a,b,c) satisfies $\phi(x,u,v)$ and let t^* be any tuple which satisfies the quantifier-free part of ϕ such that $\pi_{\{x,u,v\}}(t^*) = (a,b,c)$. Then t^* satisfies C, and hence t^* satisfies $x \geq y_1 \vee x \geq y_2$. Therefore, t^* satisfies $x \geq u \vee x \geq v$, i.e., $t \in R^{\min}_{\leq}$. It is easy to check that the formula $\exists h\, (\phi(x,h,y) \wedge h > z)$ is a pp-definition of R^{mi} in $\mathfrak{A}$. Therefore, R^{mix} is pp-definable in $\mathfrak{A}$ as well (see note below Proposition 16). $\square$

5.4. **Definability Dichotomy.** In this section we prove Theorem 30, following the strategy outlined earlier, and subsequently we prove Theorem 4

Proposition 40. *A temporal relation has a primitive positive definition in* $(\mathbb{Q}; T_3)$ *if and only if it is preserved by* pp.

Proof. By Proposition 17, it suffices to prove that the relations $\neq$, $R^{\text{min}}_{\leq}$, and S^{mi} are pp-definable in $(\mathbb{Q}; T_3)$. Clearly, $x \leq y$ is equivalent to $\exists z. T_3(x, y, z)$ and $x \neq y$ is equivalent to $\exists z. T_3(z, x, y)$. We claim that the following pp-formula is a pp-definition of $R^{\text{min}}_{\leq}$ in $(\mathbb{Q}; T_3, \leq)$.

$$\phi(x, y, z) := \exists x', y', z' \big(T_3(x', y', z') \wedge x \geq x' \wedge y \leq y' \wedge z \leq z'\big)$$

Suppose that $(a, b, c) \in R^{\text{min}}_{\leq}$ holds. By the symmetry of the second and third argument in $R^{\text{min}}_{\leq}$ we may assume that $a \geq b$ holds. Choose $a' = b'$ such that $b \leq a' = b' \leq a$ holds and $c' > \max(a', b', c)$. Then $T_3(a', b', c') \wedge a \geq a' \wedge b \leq b' \wedge c < c'$ holds and therefore (a, b, c) satisfies ϕ. For the converse direction, suppose for contradiction that (a, b, c) is not in $R^{\text{min}}_{\leq}$ but $\phi(a, b, c)$ holds. Then we have $a < b \wedge a < c$. The quantifier-free part of ϕ implies $x' \leq a < b \leq y'$ and therefore $x' = z' < y'$. However, $c \leq z' = x' \leq a$ follows, contradicting $a < c$.

Finally, we claim that the formula

$$\psi(x, y, z) := \exists u, v \big(T_3(x, u, v) \wedge (u \neq y) \wedge (v \geq z)\big)$$

defines S^{mi}. If (a, b, c) satisfies ψ we either have $a = u \neq b$ or $a = v \geq c$. Therefore (a, b, c) satisfies S^{mi}. If (a, b, c) satisfies S^{mi} we have two cases. If $a \neq b$, we choose $u = a$ and $v > \max(c, a)$. Then $b \neq a = u < v$ and $v > c$ holds and therefore $\psi(a, b, c)$ holds. If $c \leq a$ holds, then we choose $v = a$ and $u > \max(a, b)$. Then $c \leq a = v < u \neq b$ holds, i.e., $\psi(a, b, c)$ holds. $\square$

Proof of Theorem 30. $\Longrightarrow$: Suppose that R^{mix} has a pp-definition in $\mathfrak{A}$. Then $\mathfrak{A}$ is not preserved by ll because R^{mix} is not preserved by lex: consider for instance $\text{lex}((0, 0, 1), (2, 3, 0))$, which is in the same orbit as $(0, 1, 2)$ and therefore not in R^{mix}.

$\Longleftarrow$: Suppose that $\mathfrak{A}$ is not preserved by ll. If the relation T_3 is pp-definable in $\mathfrak{A}$, then so is R^{mix} by Proposition 40 because R^{mix} is preserved by pp and we are done. Otherwise, Lemma 15 implies that $\mathfrak{A}$ is preserved by mi, mx, or min. If $\mathfrak{A}$ is preserved by mi, then R^{mix} is pp-definable in $\mathfrak{A}$ by Proposition 35. If $\mathfrak{A}$ is preserved by mx but not by mi, then R^{mix} is pp-definable in $\mathfrak{A}$ by Proposition 37. If $\mathfrak{A}$ is preserved by min but neither by mi nor by mx, then $\mathfrak{A}$ pp-defines R^{mix} by Proposition 39. $\square$

Proof of Theorem 4. Suppose that $\mathfrak{A}$ does not have a binary injective polymorphism. Then $\mathfrak{A}$ is preserved by min, mi or mx or their duals. Therefore, $\mathfrak{A}$ is preserved by pp or dual-pp by the inclusions presented in Section 2.5.1. If $\mathfrak{A}$ is preserved by pp, then Theorem 30 implies that R^{mix} is pp-definable in $\mathfrak{A}$. If $\mathfrak{A}$ is preserved by dual-pp, the dual of $\mathfrak{A}$, i.e., the structure obained from $\mathfrak{A}$ by substituting all relations by their duals, has pp as a polymorphism. Hence, R^{mix} has a pp-definition in the dual of $\mathfrak{A}$ and therefore $-R^{\text{mix}}$ has a pp-definition in $\mathfrak{A}$.

It remains to show that the two cases of the theorem are mutually exclusive. Suppose that $\mathfrak{A}$ has a binary injective polymorphism f; we may also assume without loss of generality that $f(0, 1) > f(1, 0)$. Then $f(1, 0) < f(0, 1)$ and $f(0, 1) \neq f(0, 2)$, so $(f(0, 1), f(0, 2), f(1, 0)) \notin R^{\text{mix}}$. As $(0, 0, 1), (1, 2, 0) \in R^{\text{mix}}$ we have that J is not preserved by f. The dual case works analogously. Therefore pp-definability of R^{mix} in $\mathfrak{A}$ and binary injective polymorphisms in $\text{Pol}(\mathfrak{A})$ are mutually exclusive by Theorem 7. $\square$

6. Combinations of Temporal CSPs

In this section we prove that the every generic combination of the structure $(\mathbb{Q}; <, R^{\text{mix}})$ with another structure that can prevent crosses has an NP-hard CSP (Theorem 3). We then derive our complexity classification for the CSP of combinations of temporal structures (Theorem 1). In our NP-hardness proof we use the following.

Proposition 41 (Corollary 6.1.23 in [3])**.** *Let $\mathfrak{A}$ be a countably infinite ω-categorical structure with finite relational signature and without constant polymorphisms. If all polymorphisms of $\mathfrak{A}$ are essentially unary then* $\mathrm{CSP}(\mathfrak{A})$ *is NP-hard.*

The next definition introduces the key property of the polymorphisms of $(\mathbb{Q}; R^{\text{mix}})$.

Definition 42. For all $n, i \in \mathbb{N}$, $1 \le i \le n$, $a \in \mathbb{Q}^n$, and operations $f\colon \mathbb{Q}^n \to \mathbb{Q}$ we define

$$\begin{aligned} H(a,i) &\coloneqq \{b \in \mathbb{Q}^n \mid \text{for all } j \in [n] \setminus \{i\} \text{ we have } b_j > a_j \text{ and } b_i = a_i\} \\ \text{and} \quad I_f(a) &\coloneqq \{i \in \mathbb{N} \mid f \text{ is constant on } H(a,i)\}\,. \end{aligned}$$

Let $\mathscr{K}$ be the set of all operations $f\colon \mathbb{Q}^n \to \mathbb{Q}$ with $n \ge 1$ where $I_f(a) \neq \emptyset$ for all $a \in \mathbb{Q}^n$.

Examples of operations in $\mathscr{K}$ are min, mi, mix, mx, pp, and all unary operations. Non-examples are max and ll.

Lemma 43. *All polymorphisms of $(\mathbb{Q}; R^{\text{mix}})$ are in $\mathscr{K}$.*

Proof. Let $f\colon \mathbb{Q}^n \to \mathbb{Q}$ be a polymorphism of $(\mathbb{Q}; R^{\text{mix}})$. We proceed by induction on $n \in \mathbb{N}$. If $n = 1$, the statement is trivial. For $n \ge 2$, assume towards a contradiction that $f \notin \mathscr{K}$. Then there exists $c \in \mathbb{Q}^n$ such that for every $k \in [n]$ there exists $a^k, b^k \in H(c,k)$ such that $f(a^k) < f(b^k)$. Without loss of generality we may assume that $\max(f(b^1), \ldots, f(b^n)) = f(b^1)$. If there exists $k \neq 1$ and $e > b^1_1$ such that $f(a^k) \neq f(e, a^k_2, \ldots, a^k_n)$, then $(a^k_1, e, b^1_1) \in R^{\text{mix}}$ and $(a^k_l, a^k_l, b^1_l) \in R^{\text{mix}}$ for all $l \in \{2, \ldots, n\}$, but $(f(a^k), f(e, a^k_2, \ldots, a^k_n), f(b^1)) \notin R^{\text{mix}}$ because $f(b^1) \ge f(a^k) \neq f(e, a^k_2, \ldots, a^k_n)$, contradicting the assumption that f preserves R^{mix}. Similarly, if there exists $k \neq 1$ and $e > b^1_1$ such that $f(b^k) \neq f(e, b^k_2, \ldots, b^k_n)$, then $(b^k_1, e, b^1_1) \in R^{\text{mix}}$ and $(b^k_l, b^k_l, b^1_l) \in R^{\text{mix}}$ for all $l \in \{2, \ldots, n\}$, but $(f(b^k), f(e, b^k_2, \ldots, b^k_n), f(b^1)) \notin R^{\text{mix}}$. Hence, for every $k \neq 1$ and every $e > b^1_1$ we have

$$f(e, a^k_2, \ldots, a^k_n) = f(a^k) < f(b^k) = f(e, b^k_2, \ldots, b^k_n).$$

Choose $e > b^1_1$ and define $f'\colon \mathbb{Q}^{n-1} \to \mathbb{Q}$ as $(x_2, \ldots, x_n) \mapsto f(e, x_2, \ldots, x_n)$; as R^{mix} is preserved by all constant polymorphisms, f' is a composition of polymorphisms of $(\mathbb{Q}; R^{\text{mix}})$ and hence a polymorphism of $(\mathbb{Q}; R^{\text{mix}})$. Then for all $k \in \{2, \ldots, n\}$ we have

$$\begin{gathered} (b^k_2, \ldots, b^k_n), (a^k_2, \ldots, a^k_n) \in H((c_2, \ldots, c_n), k) \quad \text{and} \\ f'(a^k_2, \ldots, a^k_n) = f(e, a^k_2, \ldots, a^k_n) < f(e, b^k_2, \ldots, b^k_n) = f'(b^k_2, \ldots, b^k_n). \end{gathered}$$

Therefore, f' is an $(n-1)$-ary polymorphism of $\mathfrak{A}$ which is not in $\mathscr{K}$, a contradiction to the induction hypothesis. ☐

Lemma 44. *Let $f \in \mathscr{K} \cap \mathrm{Pol}(\mathbb{Q}; <)$ be of arity $n \ge 2$. Let $a, b \in \mathbb{Q}^n$ and $i \in I_f(a)$.*

(1) If $b_i < a_i \wedge \bigwedge_{k \neq i} b_k > a_k$, then $I_f(b) = \{i\}$.
(2) If $b_i \le a_i \wedge \bigwedge_{k \neq i} b_k \ge a_k$, then $i \in I_f(b)$.

Proof. To prove (1), suppose to the contrary that there exists $b \in \mathbb{Q}^n$ satisfying $b_i < a_i \wedge \bigwedge_{k \neq i} b_k > a_k$ and $j \in I_f(b)$ with $j \neq i$. Then there exists $c \in H(b,j)$ such that $b_i < c_i < a_i$. Now consider $d, e \in \mathbb{Q}^n$ such that

$$d_j = c_j \wedge d_i = a_i \wedge \bigwedge_{k \notin i,j} d_k > c_k \quad \text{and} \quad e_i = a_i \wedge \bigwedge_{k \neq i} e_k > d_k.$$

Then $e_j > d_j = c_j = b_j > a_j$, $d_i = a_i = e_i > c_i > b_i$, and $e_k > d_k > c_k > b_k > a_k$ for $k \in [n] \setminus \{i,j\}$. Hence, $d \in H(b,j) \cap H(a,i)$ and $e \in H(a,i)$. This implies that $f(c) = f(d) = f(e)$, which contradicts the assumption that f preserves $<$, because $c_k < e_k$ for every $k \in [n]$.

To prove (2), first consider the case that $b_i = a_i$. Then $H(b,i) \subseteq H(a,i)$ and therefore $i \in I_f(b)$. If $b_i < a_i$, choose $u, v \in H(b,i)$. Then there exists $b' \in H(b,i)$ such that for each $k \neq i$ we have $b'_k < \min(u_k, v_k)$. Then $u, v \in H(b',i)$ and b' satisfies $b'_i < a_i \wedge \bigwedge_{k \neq i} b'_k > a_j$. Hence, we have $I_f(b') = \{i\}$ by the first claim of the statement and therefore $f(u) = f(v)$. As $H(b',i) \subseteq H(b,i)$ we conclude that $i \in I_f(b)$. $\square$

Proof of Theorem 3. Let $\mathfrak{B}$ be the generic combination of $(\mathbb{Q}; <, R^{\text{mix}})$ and $\mathfrak{A}$, which exists by Proposition 10. Without loss of generality we may assume that the domain of $\mathfrak{B}$ is $\mathbb{Q}$ and that $\mathfrak{A}$ and $(\mathbb{Q}; <, R^{\text{mix}})$ are reducts of $\mathfrak{B}$. Let $f \in \text{Pol}(\mathfrak{B})$. Our goal is to show that f is essentially unary; the NP-hardness of $\text{CSP}(\mathfrak{B})$ then follows from Proposition 41.

By Lemma 43 we have $\text{Pol}(\mathfrak{B}) \subseteq \text{Pol}(\mathbb{Q}; <, R^{\text{mix}}) \subseteq \mathscr{K}$ and therefore $f \in \mathscr{K}$. Suppose for contradiction that there are $a, b \in \mathbb{Q}^n$ such that $i \in I_f(a)$, $j \in I_f(b)$, and $i \neq j$. We will treat the case that $i = 1$ and $j = 2$; all other cases can be treated analogously. Let ϕ be a cross prevention formula of $\mathfrak{A}$. Now consider the following first-order formulas $\psi_1(\bar{x}, \bar{y}, \bar{u}, \bar{v})$ with parameters $a_1, \dots, a_n, b_1, \dots, b_n$ and $\psi_2(\bar{x}, \bar{y}, \bar{u}, \bar{v})$:

$$\begin{aligned}
\psi_1 \coloneqq \quad & x_1 < a_1 \wedge x_1 = y_1 \wedge \bigwedge_{k \in [n] \setminus \{1\}} (x_k > a_k \wedge y_k > a_k) \\
& \wedge u_2 < b_2 \wedge u_2 = v_2 \wedge \bigwedge_{k \in [n] \setminus \{2\}} (u_k > b_k \wedge v_k > b_k) \\
\psi_2 \coloneqq \quad & \bigwedge_{k=1}^{n} \phi(x_k, y_k, u_k, v_k)
\end{aligned}$$

For $k \in [n]$ let $\psi_k(x_k, y_k, u_k, v_k)$ be the conjunction of all atomic formulas in ψ_1 that contain x_k, y_k, u_k, or v_k. Notice that every atomic formula in ψ_1 only contains variables from $\{x_k, y_k, u_k, v_k\}$ for a fixed k. Hence, $\psi_1(\bar{x}, \bar{y}, \bar{u}, \bar{v}) \equiv \bigwedge_{k=1}^{n} \psi_k(x_k, y_k, u_k, v_k)$. Let $\delta(z_1, z_2, z_3, z_4)$ be the first-order formula $z_1 = z_2 \wedge \bigwedge_{1 < i < j} z_i \neq z_j$. For each k there exist assignments s_1 and s_2 of variables to values in $\mathbb{Q}$ such that either

$$\begin{aligned}
\psi_k(s_1(x_k), s_1(y_k), s_1(u_k), s_1(v_k)) \wedge \delta(s_1(x_k), s_1(y_k), s_1(u_k), s_1(v_k)) \quad & \text{and} \\
\phi(s_2(x_k), s_2(y_k), s_2(u_k), s_2(v_k)) \wedge \delta(s_2(x_k), s_2(y_k), s_2(u_k), s_2(v_k)), \quad & \text{or} \\
\psi_k(s_1(x_k), s_1(y_k), s_1(u_k), s_1(v_k)) \wedge \delta(s_1(u_k), s_1(v_k), s_1(x_k), s_1(y_k)) \quad & \text{and} \\
\phi(s_2(x_k), s_2(y_k), s_2(u_k), s_2(v_k)) \wedge \delta(s_2(u_k), s_2(v_k), s_2(x_k), s_2(y_k)) \quad &
\end{aligned}$$

holds in $(\mathbb{Q}; <)$ and $\mathfrak{A}$ respectively. Furthermore, s_1 and s_2 can be chosen such that their images are disjoint to the set of all entries of $c \coloneqq (a_1, \dots, a_n, b_1, \dots, b_n)$ because both $(\mathbb{Q}; <)$ and $\mathfrak{A}$ do not have algebraicity. Now we apply the first statement of Lemma 2.7 in [6] to c, $t_{k,1} \coloneqq (s_1(x_k), s_1(y_k), s_1(u_k), s_1(v_k))$ and $t_{k,2} \coloneqq (s_2(x_k), s_2(y_k), s_2(u_k), s_2(v_k))$ for each $k \in [n]$. This yields, for each $k \in [n]$, the existence of a tuple $t_k = ((s(x_k), s(y_k), s(u_k), s(v_k))$ in $\mathfrak{B}$ which is in the same $\mathrm{Aut}(\mathbb{Q}; <, c)$-orbit as $t_{k,1}$ and in the same $\mathrm{Aut}(\mathfrak{A}, c)$-orbit as $t_{k,2}$. Therefore, $\psi_k(t_k) \wedge \phi(t_k)$ holds in $\mathfrak{B}$. Hence, $\psi_1 \wedge \psi_2$ is satisfiable in $\mathfrak{B}$ and s is a satisfying assignment.

Let $s(x)$ denote $(s(x_1), \ldots, s(x_n))$ and likewise for $s(y), s(u), s(v)$. Let a', b' be the componentwise minimum of $a, s(x), s(y)$ and $b, s(u), s(v)$, respectively. Then $s(x), s(y) \in H(a', 1)$ and $s(u), s(v) \in H(b', 2)$. We apply Case (2) of Lemma 44 to a and a' (in the role of b) and $i = 1$ and get $1 \in I_f(a')$. Similarly, we apply Case (2) of Lemma 44 to b, b' and $i = 2$ and get $2 \in I_f(b')$. Therefore, $f(s(x)) = f(s(y))$ and $f(s(u)) = f(s(v))$ must hold. However, as f preserves ϕ we must also have $\phi(f(s(x)), f(s(y)), f(s(u)), f(s(v))$, contradicting the fact that $\phi(x, y, u, v) \wedge x = y \wedge u = v$ is not satisfiable in $\mathfrak{A}$.

We conclude that there exists an $i \in [n]$ such that $I_f(a) = \{i\}$ for all $a \in \mathbb{Q}^n$. This implies that f only depends on the i-th coordinate: to prove this, let $a, b \in \mathbb{Q}^n$ be such that $a_i = b_i$. We choose any $c \in \mathbb{Q}^n$ such that $c_i = a_i$ and $c_j < \min(a_j, b_j)$ for every $j \in [n] \setminus \{i\}$. As $a, b \in H(c, i)$ and $i \in I_f(c)$ we have $f(a) = f(b)$, i.e., f can only depend on the i-th coordinate. The case that f is constant cannot happen, because f preserves $<$. Thus, f is essentially unary. □

Theorem 3 is applicable to countably infinite ω-categorical structures with finite relational signature which can prevent crosses and do not have algebraicity. Besides $(\mathbb{Q}; <)$, the following structures satisfy all of these conditions:

- the random graph with edge and non-edge relation [16]
- the univeral homogeneous K_n-free graph, for $n \geq 3$, also called Henson graph [13] with edge relation
- first-order expansions of the binary branching C-relation in [9]
- the Fraïssé-limit of all finite 3-uniform hypergraphs which do not embed a tetrahedron (see Chapter 6 in [24] for the construction method)

Proof of Theorem 1. Let $\mathfrak{B}$ be the generic combination of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, which exists by Proposition 10. We may assume that $\mathfrak{B}$, $\mathfrak{A}_1$, and $\mathfrak{A}_2$ all have the domain $\mathbb{Q}$ and that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are reducts of $\mathfrak{B}$. For $i = 1$ and $i = 2$, let $<_i$ be a linear order on $\mathbb{Q}$ such that all relations of $\mathfrak{A}_i$ are first-order definable in $(\mathbb{Q}; <_i)$; correspondingly $\mathrm{Betw}_i, \mathrm{Cycl}_i, \mathrm{Sep}_i, R_i^{\mathrm{mix}}$ are defined as the relations $\mathrm{Betw}, \mathrm{Cycl}, \mathrm{Sep}, R^{\mathrm{mix}}$ but with respect to $<_i$ instead of $<$. The same holds for $\min_i, \mathrm{mi}_i, \mathrm{mx}_i, \mathrm{ll}_i$ and their duals.

If both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have a constant polymorphism, then $\mathfrak{B}$ has a constant polymorphism, too, and in this case it $\mathrm{CSP}(\mathfrak{B}) = \mathrm{CSP}(T_1 \cup T_2)$ can be solved in constant time because only instances with an empty relation or $\perp$ as conjunct are unsatisfiable (item (2) of the statement). Hence, we may suppose without loss of generality that $\mathfrak{A}_1$ does not have a constant polymorphism. Then by Theorem 11, one of the following cases applies.

- $\mathfrak{A}_1$ is preserved by all permutations;
- the relation Betw_1, Cycl_1, or Sep_1 is pp-definable in $\mathfrak{A}_1$;
- the relation $<_1$ is pp-definable in $\mathfrak{A}_1$.

In the first case, $\mathfrak{B}$ itself is a temporal structure and $\mathrm{CSP}(\mathfrak{B})$ is in P (item (3) of the statement) or NP-complete by Theorem 14. If one of the relations Betw_1, Cycl_1, or Sep_1 is pp-definable in $\mathfrak{A}_1$, then $\mathrm{CSP}(\mathfrak{A}_1)$ is NP-hard and hence $\mathrm{CSP}(\mathfrak{B})$ is NP-hard. So we may assume in the following that $<_1$ is pp-definable in $\mathfrak{A}_1$. Hence, we can assume without loss of generality that $<_1$ is in the signature of $\mathfrak{A}_1$.

We now consider the case that both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have a binary injective polymorphism. If for some $i \in \{1, 2\}$ the problem $\mathrm{CSP}(\mathfrak{A}_i)$ is NP-hard, then clearly $\mathrm{CSP}(\mathfrak{B}) = \mathrm{CSP}(T_1 \cup T_2)$ is NP-hard as well. Otherwise, Theorem 14 implies that for $i = 1$ and $i = 2$, the structure $\mathfrak{A}_i$ is preserved by ll or by dual-ll (or P=NP, in which case Theorem 1 is trivial). Note that ll and dual-ll also preserve $\neq$, so we may add $\neq$ to the signature of $\mathfrak{A}_1$ and $\mathfrak{A}_2$. As $\neq$ is independent from T_1 and from T_2 by Proposition 19, the polynomial-time tractability of $\mathrm{CSP}(T_1 \cup T_2)$ then follows from Theorem 22.

If $\mathfrak{A}_1$ does not have a binary injective polymorphism, then $\mathrm{CSP}(\mathfrak{A}_1)$ and $\mathrm{CSP}(\mathfrak{B})$ are NP-hard unless mx_1, min_1, mi_1, or one of their duals is a polymorphism of $\mathfrak{A}_1$, by Theorem 14. We assume in the following that $\mathfrak{A}_1$ is preserved by mx_1, min_1, or mi_1; if $\mathfrak{A}_1$ is preserved by one of their duals, then the NP-hardness of $\mathrm{CSP}(T_1 \cup T_2)$ can be shown analogously. By Theorem 4, the relation R_1^{mix} has a pp-definition in $\mathfrak{A}_1$.

Now, we make a case distinction for $\mathfrak{A}_2$. If the structure $\mathfrak{A}_2$ is preserved by all permutations, we are again done (this is analogous to the situation that $\mathfrak{A}_1$ is preserved by all permutations, which was already treated above). Otherwise, we apply Theorem 11 to $(\mathfrak{A}_2; \neq)$ and obtain that a relation $R \in \{<_2, \mathrm{Betw}_2, \mathrm{Cycl}_2, \mathrm{Sep}_2\}$ has a pp-definition ϕ in $(\mathfrak{A}_2; \neq)$. Let E be the set of all sets $\{x_i, x_j\}$ such that $x_i \neq x_j$ appears in ϕ and n the arity of R. Then, for some $m \geq n$, the formula ϕ can be written in the following way: $\phi(x_1, \ldots, x_n) = \exists x_{n+1}, \ldots, x_m \big(\psi(x_1, \ldots, x_m) \wedge \bigwedge_{i,j \in E} x_i \neq x_j\big)$ where ψ is a primitive positive τ_2-formula. Notice that for all $i, j \in [n]$ with $i \neq j$ we may add $\{i, j\}$ to E because for any choice of R, all coordinates in tuples of R are pairwise distinct.

Consider the undirected graph $([m], E)$. We may choose any linear order E_d' on $[n]$ and extend E_d' to E_d on $[m]$ by choosing a direction for each edge in E such that $([m], E_d)$ is a cycle-free directed graph. Because $x < y \vee x > y$ defines $x \neq y$ we have

$$\phi(x_1, \ldots, x_n) \equiv \exists x_{n+1}, \ldots, x_m \Big((\psi(x_1, \ldots, x_m) \wedge \bigwedge_{(i,j) \in E_d} ((x_i <_1 x_j) \vee (x_j <_1 x_i)) \Big).$$

Now notice that $\exists x_{n+1}, \ldots, x_m \big(\psi(x_1, \ldots, x_m) \wedge \bigwedge_{(i,j) \in E_d} x_i <_1 x_j\big)$ is a pp-formula in $\mathfrak{B}$ which defines the same relation as the formula

$$R(x_1, \ldots, x_n) \wedge \bigwedge_{(i,j) \in E_d'} x_i <_1 x_j \tag{11}$$

in $\mathfrak{B}$. Now, we go through all possible choices for R and present pp-definitions for either $<_2$ or $\neq$ in $\mathfrak{B}$. In order to simplify the presentation, we will use conjuncts of the form (11) instead of their equivalent pp-definitions in $\mathfrak{B}$.

- If R equals $<_2$ then $\exists z \big((x <_2 z \wedge x <_1 z) \wedge (z <_2 y \wedge y <_1 z)\big)$ pp-defines $x <_2 y$. This is easy to see with the equivalent expression $\exists z \big((x <_2 z <_2 y) \wedge (x <_1 z) \wedge (y <_1 z)\big)$.
- If R equals Betw_2 we claim that

$$\begin{aligned}\exists u, v \big(&(\mathrm{Betw}_2(x, u, v) \wedge (x <_1 u) \wedge (u <_1 v)) \\ &\wedge (\mathrm{Betw}_2(u, v, y) \wedge (y <_1 u) \wedge (u <_1 v)) \big)\end{aligned}$$

  pp-defines $x \neq y$. Again we give an equivalent expression which helps to verify the claim: $\exists u, v \big(((x <_2 u <_2 v <_2 y) \vee (y <_2 v <_2 u <_2 x)) \wedge (x <_1 u <_1 v) \wedge (y <_1 u)\big)$. In the latter, it is clear that $x \neq y$ always holds and that all distinct x, y satisfy the formula.
- If R equals Cycl_2 we claim that

$$\begin{aligned}\exists u, v \big(&(\mathrm{Cycl}_2(x, u, v) \wedge (x <_1 u) \wedge (u <_1 v)) \\ &\wedge (\mathrm{Cycl}_2(u, y, v) \wedge (y <_1 u) \wedge (u <_1 v)) \big)\end{aligned}$$

pp-defines $x \neq y$. A case analysis of $\mathrm{Cycl}_2(x,u,v)$ yields that the given formula is equivalent to

$$\begin{aligned}\exists u,v\big(\big(\quad &(x <_2 u <_2 y <_2 v)\\ &\vee (u <_2 y <_2 v <_2 x)\\ &\vee (y <_2 v <_2 x <_2 u)\\ &\vee (v <_2 x <_2 u <_2 y)\big)\\ \wedge\ &(x <_1 u <_1 v) \wedge (y <_1 u)\big).\end{aligned}$$

In the latter formula, it is clear that $x \neq y$ must always hold and that for any distinct x, y the formula is satisfiable.

- If R equals Sep_2 we claim that

$$\begin{aligned}&\exists u,v,w\big((\mathrm{Sep}_2(x,u,v,w) \wedge (x <_1 u) \wedge (u <_1 v) \wedge (v <_1 w))\\ &\qquad \wedge (\mathrm{Sep}_2(u,v,w,y) \wedge (y <_1 u) \wedge (u <_1 v) \wedge (v <_1 w))\big)\end{aligned}$$

pp-defines $x \neq y$. Similarly to above, a case analysis of Sep_2 yields an equivalent expression

$$\begin{aligned}\exists u,v,w\big(\big(\quad &(x <_2 v <_2 y <_2 u <_2 w)\\ &\vee (x <_2 w <_2 u <_2 y <_2 v)\\ &\vee (u <_2 y <_2 v <_2 x <_2 w)\\ &\vee (u <_2 w <_2 x <_2 v <_2 y) \vee (y <_2 u <_2 w <_2 x <_2 v)\\ &\vee (v <_2 x <_2 w <_2 y <_2 u)\\ &\vee (v <_2 u <_2 y <_2 w <_2 x)\\ &\vee (w <_2 y <_2 u <_2 v <_2 x)\\ &\vee (w <_2 x <_2 v <_2 u <_2 y) \vee (y <_2 w <_2 x <_2 v <_2 u)\big)\\ \wedge\ &(x <_1 u <_1 v <_1 w) \wedge (y <_1 u)\big)\end{aligned}$$

for which the claim is easily verified because $x \neq y$ always holds and for any distinct x, y there exist u, v, w satisfying the formula.

Choose a relation S from $\{\neq, <_2\}$ which is pp-definable in $\mathfrak{B}$ and let $\mathfrak{A}_2'$ be the expansion of $\mathfrak{A}_2$ by S and $\mathfrak{B}' := \mathfrak{A}_1 * \mathfrak{A}_2'$. As S is pp-definable in $\mathfrak{B}$, it suffices to show NP-hardness of $\mathrm{CSP}(\mathfrak{B}')$ instead of $\mathrm{CSP}(\mathfrak{B}) = \mathrm{CSP}(T_1 \cup T_2)$.

If S is $<_2$, then $\mathfrak{A}_2'$ has cross prevention, so the NP-hardness of $\mathrm{CSP}(\mathfrak{B}')$ follows from Theorem 3. If S is $\neq$, then we may again apply Theorem 11 to $\mathfrak{A}_2'$ to conclude that the relation $<_2$, Betw_2, Cycl_2, or Sep_2 is pp-definable in $\mathfrak{A}_2'$. The first case has already been treated above. In the remaining cases we get NP-hardness of $\mathrm{CSP}(\mathfrak{A}_2')$ and hence of $\mathrm{CSP}(\mathfrak{B}')$. □

7. Conclusion and Outlook

Our results show that there are two temporal relations, namely R^{mix} and its dual, with the property that every first-order expansion of $(\mathbb{Q}; <)$ where the weakened Nelson-Oppen conditions do not apply, i.e., $\neq$ is not independent from their theory, can pp-define one of these relations. We also showed that $\mathrm{CSP}(\mathrm{Th}(\mathbb{Q}; R^{\mathrm{mix}}, <) \cup \mathrm{Th}(\mathfrak{A}))$ is NP-hard for structures $\mathfrak{A}$ that satisfy the fairly weak assumption of cross prevention and have a generic combination with $(\mathbb{Q}; <)$. These results can be used to prove a complexity dichotomy for combinations of temporal CSPs: they are either in P or NP-complete. Our results also motivate the following conjecture, which remains open in general.

Conjecture 45. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be countably infinite ω-categorical structures without algebraicity that are not preserved by all permutations and that have the cross prevention property. If*

- $\mathrm{CSP}(\mathfrak{A}_i)$ *is in P and $\mathfrak{A}_i$ has a binary injective polymorphism for both $i = 1$ and $i = 2$, or*
- *$\mathfrak{A}_i$ has a constant polymorphism for both $i = 1$ and $i = 2$,*

then $\mathrm{CSP}(\mathrm{Th}(\mathfrak{A}_1) \cup \mathrm{Th}(\mathfrak{A}_2))$ *is in P. Otherwise,* $\mathrm{CSP}(\mathrm{Th}(\mathfrak{A}_1) \cup \mathrm{Th}(\mathfrak{A}_2))$ *is NP-hard.*

References

[1] Franz Baader and Klaus U. Schulz. 'Combining Constraint Solving'. In: *H. Comon, C. March, and R. Treinen, editors, Constraints in Computational Logics*. LNCS 2002 (2001).

[2] Manuel Bodirsky. 'Complexity Classification in Infinite-Domain Constraint Satisfaction'. Mémoire d'habilitation à diriger des recherches, Université Diderot – Paris 7. Available at arXiv:1201.0856v8. 2012.

[3] Manuel Bodirsky. 'Complexity of Infinite-Domain Constraint Satisfaction'. Submitted for publication in the LNL Series, Cambridge University Press. 2020.

[4] Manuel Bodirsky, Hubie Chen and Michael Pinsker. 'The Reducts of Equality up to Primitive Positive Interdefinability'. In: *Journal of Symbolic Logic* 75.4 (2010), pp. 1249–1292.

[5] Manuel Bodirsky, Hubie Chen and Michał Wrona. 'Tractability of quantified temporal constraints to the max'. In: *International Journal of Algebra and Computation* 24.08 (2014), pp. 1141–1156.

[6] Manuel Bodirsky and Johannes Greiner. 'The Complexity of Combinations of Qualitative Constraint Satisfaction Problems'. In: *Logical Methods in Computer Science* Volume 16, Issue 1 (Feb. 2020). DOI: `10.23638/LMCS-16(1:21)2020`.

[7] Manuel Bodirsky and Johannes Greiner. 'Tractable Combinations of Theories via Sampling'. 2020. arXiv: `2012.01199`.

[8] Manuel Bodirsky and Martin Hils. 'Tractable Set Constraints'. In: *Journal of Artificial Intelligence Research* 45 (2012), pp. 731–759.

[9] Manuel Bodirsky, Peter Jonsson and Trung Van Pham. 'The Complexity of Phylogeny Constraint Satisfaction Problems'. In: *ACM Transactions on Computational Logic (TOCL)* 18.3 (2017). An extended abstract appeared in the conference STACS 2016.

[10] Manuel Bodirsky and Jan Kára. 'The Complexity of Equality Constraint Languages'. In: *Theory of Computing Systems* 3.2 (2008). A conference version appeared in the proceedings of Computer Science Russia (CSR'06), pp. 136–158.

[11] Manuel Bodirsky and Jan Kára. 'The Complexity of Temporal Constraint Satisfaction Problems'. In: *Journal of the ACM* 57.2 (2009). An extended abstract appeared in the Proceedings of the Symposium on Theory of Computing (STOC), pp. 1–41.

[12] Manuel Bodirsky, Florent Madelaine and Antoine Mottet. 'A universal-algebraic proof of the complexity dichotomy for Monotone Monadic SNP'. In: *Proceedings of the Symposium on Logic in Computer Science (LICS)*. Preprint arXiv:1802.03255. 2018, pp. 105–114.

[13] Manuel Bodirsky, Barnaby Martin, Michael Pinsker and András Pongrácz. 'Constraint Satisfaction Problems for Reducts of Homogeneous Graphs'. In: *SIAM Journal on Computing* 48.4 (2019). A conference version appeared in

the Proceedings of the 43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, pages 119:1–119:14, pp. 1224–1264.
[14] Manuel Bodirsky and Jaroslav Nešetřil. 'Constraint Satisfaction with Countable Homogeneous Templates'. In: *Journal of Logic and Computation* 16.3 (2006), pp. 359–373.
[15] Manuel Bodirsky, Wied Pakusa and Jakub Rydval. 'Temporal Constraint Satisfaction Problems in Fixed-Point Logic'. In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '20. Saarbrücken, Germany: Association for Computing Machinery, 2020, pp. 237–251. DOI: 10.1145/3373718.3394750.
[16] Manuel Bodirsky and Michael Pinsker. 'Schaefer's theorem for graphs'. In: *Journal of the ACM* 62.3 (2015). A conference version appeared in the Proceedings of STOC 2011, pages 655–664, 52 pages (article number 19).
[17] Manuel Bodirsky, Michael Pinsker and András Pongrácz. 'The 42 reducts of the random ordered graph'. In: *Proceedings of the LMS* 111.3 (2015). Preprint available from arXiv:1309.2165, pp. 591–632.
[18] Manuel Bodirsky and Michał Wrona. 'Equivalence Constraint Satisfaction Problems'. In: *Proceedings of Computer Science Logic*. Vol. 16. LIPICS. Dagstuhl Publishing, Sept. 2012, pp. 122–136.
[19] Maria Paola Bonacina, Silvio Ghilardi, Enrica Nicolini, Silvio Ranise and Daniele Zucchelli. 'Decidability and Undecidability Results for Nelson-Oppen and Rewrite-Based Decision Procedures'. In: *Automated Reasoning*. Ed. by Ulrich Furbach and Natarajan Shankar. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006, pp. 513–527.
[20] Mathias Broxvall, Peter Jonsson and Jochen Renz. 'Disjunctions, independence, refinements'. In: *Artificial Intelligence* 140.1–2 (2002), pp. 153–173. DOI: 10.1016/S0004-3702(02)00224-2.
[21] Peter J. Cameron. *Oligomorphic permutation groups*. Cambridge: Cambridge University Press, 1990.
[22] Zvi Galil and Nimrod Megiddo. 'Cyclic Ordering is NP-Complete'. In: *Theoretical Computer Science* 5.2 (1977), pp. 179–182.
[23] Silvio Ghilardi. 'Model-Theoretic Methods in Combined Constraint Satisfiability'. In: *Journal of Automated Reasoning* 33.3 (2004), pp. 221–249. DOI: 10.1007/s10817-004-6241-5.
[24] Wilfrid Hodges. *A shorter model theory*. Cambridge: Cambridge University Press, 1997.
[25] Alexander Kechris, Vladimir Pestov and Stevo Todorčević. 'Fraïssé Limits, Ramsey Theory, and topological dynamics of automorphism groups'. In: *Geometric and Functional Analysis* 15.1 (2005), pp. 106–189.
[26] Michael Kompatscher and Trung Van Pham. 'A Complexity Dichotomy for Poset Constraint Satisfaction'. In: *IfCoLog Journal of Logics and their Applications (FLAP)* 5.8 (2018), pp. 1663–1696.
[27] Sava Krstić and Amit Goel. 'Architecting Solvers for SAT Modulo Theories: Nelson-Oppen with DPLL'. In: *Proceedings of the 6th International Symposium on Frontiers of Combining Systems*. FroCoS '07. Berlin, Heidelberg: Springer-Verlag, 2007, pp. 1–27. DOI: 10.1007/978-3-540-74621-8_1.
[28] Julie Linman and Michael Pinsker. 'Permutations on the Random Permutation'. In: *Electronic Journal of Combinatorics* 22 (May 2014). DOI: 10.37236/4910.
[29] Rolf H. Möhring, Martin Skutella and Frederik Stork. 'Scheduling with AND/OR Precedence Constraints'. In: *SIAM Journal on Computing* 33.2 (2004), pp. 393–415.

[30] Leonardo de Moura. ‘Developing Efficient SMT Solvers’. 2007. URL: https://github.com/Z3Prover/z3/wiki/Slides.

[31] Bernhard Nebel and Hans-Jürgen Bürckert. ‘Reasoning about Temporal Relations: A Maximal Tractable Subclass of Allen’s Interval Algebra’. In: *Journal of the ACM* 42.1 (1995), pp. 43–66.

[32] Greg Nelson and Derek C. Oppen. ‘Simplification by Cooperating Decision Procedures’. In: *ACM Transactions on Programming Languages and Systems* 1.2 (Oct. 1979), pp. 245–257. DOI: 10.1145/357073.357079.

[33] Jaroslav Opatrny. ‘Total Ordering Problem’. In: *SIAM Journal on Computing* 8.1 (1979), pp. 111–114.

[34] Derek C. Oppen. ‘Complexity, Convexity and Combinations of Theories’. In: *Theoretical Computer Science* 12.3 (1980), pp. 291–302. DOI: 10.1016/0304-3975(80)90059-6.

[35] Emil L. Post. ‘The two-valued iterative systems of mathematical logic’. In: *Annals of Mathematics Studies* 5 (1941).

[36] Christophe Ringeissen. ‘Cooperation of Decision Procedures for the Satisfiability Problem’. In: *Frontiers of Combining Systems: First International Workshop, Munich, March 1996*. Ed. by Frans Baader and Klaus U. Schulz. Dordrecht: Springer Netherlands, 1996, pp. 121–139. DOI: 10.1007/978-94-009-0349-4_6.

[37] Klaus U. Schulz. ‘Why Combined Decision Problems Are Often Intractable’. In: *Frontiers of Combining Systems*. Ed. by Hélène Kirchner and Christophe Ringeissen. Berlin, Heidelberg: Springer Berlin Heidelberg, 2000, pp. 217–244.

[38] Cesare Tinelli and Christophe Ringeissen. ‘Unions of non-disjoint theories and combinations of satisfiability procedures’. In: *Theoretical Computer Science* 290.1 (2003), pp. 291–353. DOI: 10.1016/S0304-3975(01)00332-2.

[39] Marc Vilain, Henry Kautz and Peter van Beek. ‘Constraint propagation algorithms for temporal reasoning: A revised report’. In: *Reading in Qualitative Reasoning about Physical Systems* (1989), pp. 373–381.

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