Complexity Theory
Complexity

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ORDER ANALYSIS

Notion of “Order”

Throughout the complexity portion of this course, we will be interested in how long an algorithm takes on the instances of some arbitrary "size" **n**. Recognizing that different times can be recorded for two instance of size **n**, we only ask about the worst case.

We also understand that different languages, computers, and even skill of the implementer can alter the "running time."

Notion of “Order”

As a result, we really can never know "exactly" how long anything takes.

So, we usually settle for a substitute function, and say the function we are trying to measure is "of the order of" this new substitute function.

Notion of “Order”

"Order" is something we use to describe an upper bound upon something else (in our case, time, but it can apply to almost anything).

For example, let **f(n)** and **g(n)** be two functions. We say "**f(n)** is order **g(n)**" when there exists constants **c** and **N** such that **f(n) ≤ cg(n)** for all **n ≥ N**.

What this is saying is that when **n** is 'large enough,' **f(n)** is bounded above by a constant multiple of **g(n)**.

Notion of “Order”

This is particularly useful when **f(n)** is not known precisely, is complicated to compute, and/or difficult to use. We can, by this, replace **f(n)** by **g(n)** and know we aren't "off too far."

We say **f(n)** is "in the order of **g(n)**" or, simply, **f(n)** $\in$ **O(g(n))**.

Usually, **g(n)** is a simple function, like **nlog(n)**, $\mathbf{n^3}$, $\mathbf{2^n}$, etc., that's easy to understand and use.

Notion of "Order"

Order of an Algorithm: The maximum number of steps required to find the answer to any instance of size **n**, for any arbitrary value of **n**.

For example, if an algorithm requires at most $\mathbf{6n^2+3n-6}$ steps on any instance of size n, we say it is "order $\mathbf{n^2}$" or, simply, $\mathbf{O(n^2)}$.

Order

Let the order of algorithm **X** be in $\mathbf{O(f_x(n))}$.

Then, for algorithms **A** and **B** and their respective order functions, $\mathbf{f_A(n)}$ and $\mathbf{f_B(n)}$, consider the limit of $\mathbf{f_A(n)/f_B(n)}$ as **n** goes to infinity.

If this value is

0	**A** is faster than **B**
constant	**A** and **B** are "equally slow/fast"
infinity	**A** is slower than **B**.

Order of a Problem

Order of a Problem

The order of the fastest algorithm that can ever solve this problem. (Also known as the "Complexity" of the problem.)

Often difficult to determine, since this allows for algorithms not yet discovered.

Decision vs Optimization

Two types of problems are of particular interest:

Decision Problems ("Yes/No" answers)

Optimization problems ("best" answers)

(there are other types)

Vertex Cover (VC)

- Suppose we are in charge of a large network (a graph where edges are links between pairs of cities (vertices). Periodically, a line fails. To mend the line, we must call in a repair crew that goes over the line to fix it. To minimize down time, we station a repair crew at one end of every line. How many crews must you have and where should they be stationed?

- This is called the Vertex Cover Problem. (Yes, it sounds like it should be called the Edge Cover problem – something else already had that name.)

- An interesting problem – it is among the hardest problems, yet is one of the easiest of the hard problems.

VC Decision vs Optimization

- As a Decision Problem:

- Instances: A graph **G** and an integer **k**.
- Question: Does **G** possess a vertex Cover with at most **k** vertices?

- As an Optimization Problem:

- Instances: A graph **G**.
- Question: What is the smallest **k** for which **G** possesses a vertex cover?

Relation of VC Problems

- If we can (easily) solve either one of these problems, we can (easily) solve the other. (To solve the optimization version, just solve the decision version with several different values of **k**. Use a binary search on **k** between **1** and **n**. That is **log(n)** solutions of the decision problem solves the optimization problem. It's simple to solve the decision version if we can solve the optimization version.

- We say their time complexity differs by no more than a multiple of **log(n)**.
- If one is polynomial then so is the other.
- If one is exponential, then so is the other.

- We say they are equally difficult (both poly. or both exponential).

Smallest VC

- A "stranger version"

- Instances: A graph **G** and an integer **k**.
- Question: Does the smallest vertex cover of **G** have exactly **k** vertices?
- This is a decision problem. But, notice that it does not seem to be easy to verify either Yes or No instances!! (We can easily verify No instances for which the VC number is less than **k**, but not when it is actually greater than **k**.)

- So, it would seem to be in a different category than either of the other two. Yet, it also has the property that if we can easily solve either of the first two versions, we can easily solve this one.

Natural Pairs of Problems

Interestingly, these usually come in pairs

a *decision* problem, and

an *optimization* problem.

Equally easy, or equally difficult, to solve.

Both can be solved in polynomial time, or both require exponential time.

A Word about Time

An algorithm for a problem is said to be polynomial if there exists integers **k** and **N** such that **t(n)**, the maximum number of steps required on any instance of size **n**, is at most $\mathbf{n^k}$, for all $\mathbf{n \geq N}$.

Otherwise, we say the algorithm is exponential. Usually, this is interpreted to mean $\mathbf{t(n) \geq c^n}$ for an infinite set of size **n** instances, and some constant $\mathbf{c > 1}$ (often, we simply use $\mathbf{c = 2}$).

A Word about “Words”

Normally, when we say a problem is "easy" we mean that it has a polynomial algorithm.

But, when we say a problem is "hard" or “apparently hard" we usually mean no polynomial algorithm is known, and none seems likely.

It is possible a polynomial algorithm exists for "hard" problems, but the evidence seems to indicate otherwise.

A Word about Abstractions

Problems we will discuss are usually "abstractions" of real problems. That is, to the extent possible, non-essential features have been removed, others have been simplified and given variable names, relationships have been replaced with mathematical equations and/or inequalities, etc.

If an abstraction is hard, then the real problem is probably even harder!!

A Word about Toy Problems

This process, Mathematical Modeling, is a field of study in itself, and not our interest here.

On the other hand, we sometimes conjure up artificial problems to put a little "reality" into our work. This results in what some call "toy problems."

Again, if a toy problem is hard, then the real problem is probably harder.

Very Hard Problems

Some problems have no algorithm (e. g., Halting Problem.)

No mechanical/logical procedure will ever solve all instances of any such problem!!

Some problems have only exponential algorithms (provably so – they must take at least order $\mathbf{2^n}$ steps) So far, only a few have been proven, but there may be many. We suspect so.

Easy Problems

Many problems have polynomial algorithms (Fortunately).

Why fortunately? Because, most exponential algorithms are essentially useless for problem instances with **n** much larger than 50 or 60. We have algorithms for them, but the best of these will take 100's of years to run, even on much faster computers than we now envision.

Three Classes of Problems

Problems proven to be in these three groups (classes) are, respectively,

Undecidable, Exponential, and Polynomial.

Theoretically, all problems belong to exactly one of these three classes.

Unknown Complexity

Practically, there are a lot of problems (maybe, most) that have not been proven to be in any of the classes (Yet, maybe never will be).

Most currently "lie between" polynomial and exponential – we know of exponential algorithms, but have been unable to prove that exponential algorithms are necessary.

Some may have polynomial algorithms, but we have not yet been clever enough to discover them.

Why do we Care?

If an algorithm is **$O(n^k)$**, increasing the size of an instance by one gives a running time that is **$O((n+1)^k)$**

That's really not much more.

With an increase of one in an exponential algorithm, **$O(2^n)$** changes to **$O(2^{n+1}) = O(2*2^n) = 2*O(2^n)$** – that is, it takes about twice as long.

A Word about “Size”

Technically, the size of an instance is the minimum number of bits (information) needed to represent the instance – its "length."

This comes from early Formal Language researchers who were analyzing the time needed to 'recognize' a string of characters as a function of its length (number of characters).

When dealing with more general problems there is usually a parameter (number of vertices, processors, variables, etc.) that is polynomially related to the length of the instance. Then, we are justified in using the parameter as a measure of the length (size), since anything polynomially related to one will be polynomially related to the other.

The Subtlety of “Size”

But, be careful.

For instance, if the "value" (magnitude) of n is both the input and the parameter, the 'length' of the input (number of bits) is $\log_2(n)$. So, an algorithm that takes n time is running in $n = 2^{\log2(n)}$ time, which is exponential in terms of the length, $\log_2(n)$, but linear (hence, polynomial) in terms of the "value," or magnitude, of n.

It's a subtle, and usually unimportant difference, but it can bite you.

Subset Sum

- Problem – Subset Sum

- Instances: A list **L** of **n** integer values and an integer **B**.
- Question: Does **L** have a subset which sums exactly to **B**?

- No one knows of a polynomial (deterministic) solution to this problem.

- On the other hand, there is a very simple (dynamic programming) algorithm that runs in **O(nB)** time.

- Why isn't this "polynomial"?
- Because, the "length" of an instance is **nlog(B)** and
- **nB > (nlog(B))^k** for any fixed **k**.

Why do we Care?

When given a new problem to solve (design an algorithm for), if it's undecidable, or even exponential, you will waste a lot of time trying to write a polynomial solution for it!!

If the problem really is polynomial, it will be worthwhile spending some time and effort to find a polynomial solution.

You should know something about how hard a problem is before you try to solve it.

Research Territory

Decidable – vs – Undecidable
(area of Computability Theory)

Exponential – vs – polynomial
(area of Computational Complexity)

Algorithms for any of these
(area of Algorithm Design/Analysis)

Complexity Theory

Second Part of Course

Models of Computation

NonDeterminism

Since we can't seem to find a model of computation that is more powerful than a TM, can we find one that is 'faster'?

In particular, we want one that takes us from exponential time to polynomial time.

Our candidate will be the NonDeterministic Turing Machine (NDTM).

NDTM's

In the basic Deterministic Turing Machine (DTM) we make one major alteration (and take care of a few repercussions):

- **The 'transition functon' in DTM's is allowed to become a 'transition mapping' in NDTM's.**

This means that rather than the next action being totally specified (deterministic) by the current state and input character, we now can have many next actions - simultaneously. That is, a NDTM can be in many states at once. (That raises some interesting problems with writing on the tape, just where the tape head is, etc., but those little things can be explained away).

NDTM's

We also require that there be only one halt state - the 'accept' state. That also raises an interesting question - what if we give it an instance that is not 'acceptable'? The answer - it blows up (or goes into an infinite loop).

The solution is that we are only allowed to give it 'acceptable' input. That means

NDTM's are only defined for decision problems and, in particular, only for Yes instances.

NDTM's

We want to determine how long it takes to get to the accept state - that's our only motive!!

So, what is a NDTM doing?

In a normal (deterministic) algorithm, we often have a loop where each time through the loop we are testing a different option to see if that "choice" leads to a correct solution. If one does, fine, we go on to another part of the problem. If one doesn't, we return to the same place and make a different choice, and test it, etc.

NDTM's

If this is a Yes instance, we are guaranteed that an acceptable choice will eventually be found and we go on.

In a NDTM, what we are doing is making, and testing, all of those choices at once by 'spawning' a different NDTM for each of them. Those that don't work out, simply die (or something).

This is kind of like the ultimate in parallel programming.

NDTM's

To allay concerns about not being able to write on the tape, we can allow each spawned NDTM to have its own copy of the tape with a read/write head.

The restriction is that nothing can be reported back except that the accept state was reached.

NDTM's

Another interpretation of nondeterminism:

From the basic definition, we notice that out of every state having a nondeterministic choice, at least one choice is valid and all the rest sort of die off. That is they really have no reason for being spawned (for this instance - maybe for another). So, we station at each such state, an 'oracle' (an all knowing being) who only allows the correct NDTM to be spawned.

An 'Oracle Machine.'

NDTM's

This is not totally unreasonable. We can look at a non deterministic decision as a deterministic algorithm in which, when an "option" is to be tested, it is very lucky, or clever, to make the correct choice the first time.

In this sense, the two machines would work identically, and we are just asking "How long does a DTM take if it always makes the correct decisions?"

NDTM's

As long as we are talking magic, we might as well talk about a 'super' oracle stationed at the start state (and get rid of the rest of the oracles) whose task is to examine the given instance and simply tell you what sequence of transitions needs to be executed to reach the accept state.

He/she will write them to the left of cell 0 (the instance is to the right).

NDTM's

Now, you simply write a DTM to run back and forth between the left of the tape to get the 'next action' and then go back to the right half to examine the NDTM and instance to verify that the provided transition is a valid next action. As predicted by the oracle, the DTM will see that the NDTM would reach the accept state and can report the number of steps required.

NDTM's

All of this was originally designed with Language Recognition problems in mind. It is not a far stretch to realize the Yes instances of any of our more real word-like decision problems defines a language, and that the same approach can be used to "solve" them.

Rather than the oracle placing the sequence of transitions on the tape, we ask him/her to provide a 'witness' to (a 'proof' of) the correctness of the instance.

NDTM's

For example, in the SubsetSum problem, we ask the oracle to write down the subset of objects whose sum is B (the desired sum). Then we ask "Can we write a deterministic polynomial algorithm to test the given witness."

The answer for SubsetSum is Yes, we can, i.e., the witness is verifiable in deterministic polynomial time.

NDTM's - Witnesses

Just what can we ask and expect of a "witness"?

The witness must be something that

(1) we can verify to be accurate (for the given problem and instance) and

(2) we must be able to "finish off" the solution.

All in polynomial time.

NDTM's - Witnesses

The witness can be nothing!

Then, we are on our own. We have to "solve the instance in polynomial time."

The witness can be "Yes."

Duh. We already knew that. We have to now verify the yes instance is a yes instance (same as above).

The witness has to be something other than nothing and Yes.

NDTM's - Witnesses

The information provided must be something we could have come up with ourselves, but probably at an exponential cost. And, it has to be enough so that we can conclude the final answer Yes from it.

Consider a witness for the graph coloring problem:

Given: A graph G = (V, E) and an integer k.

Question: Can the vertices of G be assigned colors so that adjacent vertices have different colors and use at most k colors?

NDTM's - Witnesses

The witness could be nothing, or Yes.

But that's not good enough - we don't know of a polynomial algorithm for graph coloring.

It could be "vertex 10 is colored Red."

That's not good enough either. Any single vertex can be colored any color we want.

It could be a color assigned to each vertex.

That would work, because we can verify its validity in polynomial time, and we can conclude the correct answer of Yes.

NDTM's - Witnesses

What if it was a color for all vertices but one?

That also is enough. We can verify the correctness of the n-1 given to us, then we can verify that the one uncolored vertex can be colored with a color not on any neighbor, and that the total is not more than k.

What if all but 2, 3, or 20 vertices are colored

All are valid witnesses.

What if half the vertices are colored?

Usually, No. There's not enough information. Sure, we can check that what is given to us is properly colored, but we don't know how to "finish it off."

NDTM's - Witnesses

An interesting question: For a given problem, what are the limits to what can be provided that still allows a polynomial verification?

NDTM's

A major question remains: Do we have, in NDTMs, a model of computation that solves all deterministic exponential (DE) problems in polynomial time (nondeterministic polynomial time)??

It definitely solves some problems we *think* are DE in nondeterministic polynomial time.

NDTM's

But, so far, all problems that have been proven to require deterministic exponential time also require nondeterministic exponential time.

So, the jury is still out. In the meantime, NDTMs are still valuable, because they might identify a larger class of problems than does a deterministic TM - the set of decision problems for which Yes instances can be verified in polynomial time.

Problem Classes

We now begin to discuss several different classes of problems. The first two will be:

NP	**'Nondeterministic' Polynomial**
P	**'Deterministic' Polynomial,**
	The 'easiest' problems in NP

Their definitions are rooted in the depths of Computability Theory as just described, but it is worth repeating some of it in the next few slides.

Problem Classes

We assume knowledge of Deterministic and Nondeterministic Turing Machines. (DTM's and NDTM's)

The only use in life of a NDTM is to scan a string of characters X and proceed by state transitions until an 'accept' state is entered.

X must be in the language the NDTM is designed to recognize. Otherwise, it blows up!!

Problem Classes

So, what good is it?

We can count the number of transitions on the shortest path (elapsed time) to the accept state!!!

If there is a constant k for which the number of transitions is at most $|X|^k$, then the language is said to be 'nondeterministic polynomial.'

Problem Classes

The subset of YES instances of the set of instances of a decision problem, as we have described them above, is a language.

When given an instance, we want to know that it is in the subset of Yes instances. (All answers to Yes instances look alike - we don't care which one we get or how it was obtained).

This begs the question "What about the No instances?"

The answer is that we will get to them later. (They will actually form another class of problems.)

Problem Classes

This actually defines our first Class, NP, the set of decision problems whose Yes instances can be solved by a Nondeterministic Turing Machine in polynomial time.

That knowledge is not of much use!! We still don't know how to tell (easily) if a problem is in NP. And, that's our goal.

Fortunately, all we are doing with a NDTM is tracing the correct path to the accept state. Since all we are interested in doing is counting its length, if someone just gave us the correct path and we followed it, we could learn the same thing - how long it is.

Problem Classes

It is even simpler than that (all this has been proven mathematically). Consider the following problem:

> **You have a big van that can carry 10,000 lbs. You also have a batch of objects with weights w_1, w_2, …, w_n lbs. Their total sum is more than 10,000 lbs, so you can't haul all of them.**
>
> **Can you load the van with exactly 10,000 lbs?**
>
> **(WOW. That's the SubsetSum problem.)**

Problem Classes

Now, suppose it is possible (i.e., a Yes instance) and someone tells you exactly what objects to select.

We can add the weights of those selected objects and verify the correctness of the selection.

This is the same as following the correct path in a NDTM. (Well, not just the same, but it can be proven to be equivalent.)

Therefore, all we have to do is count how long it takes to verify that a "correct" answer" is in fact correct.

Class – NP

First Significant Class of Problems:

The Class NP

Class – NP

We have, already, an informal definition for the set NP. We will now try to get a better idea of what NP includes, what it does not include, and give a formal definition.

Class – NP

Consider two seemingly closely related statements (versions) of a single problem. We show they are actually very different. Let G = (V, E) be a graph.

Definition: $X \subseteq V(G)$ is a *vertex cover* if every edge in G has at least one endpoint in X.

Class – NP

Version 1. Given a graph G and an integer k.
Does G contain a vertex cover
with at most k vertices?

Version 2. Given a graph G and an integer k.
Does the smallest vertex cover of G
have exactly k vertices?

Class – NP

Suppose, for either version, we are given a graph G and an integer k for which the answer is "yes." Someone also gives us a set X of vertices and claims

"X satisfies the conditions."

Class – NP

In Version 1, we can fairly easily check that the claim is correct – in polynomial time.

That is, in polynomial time, we can check that X has k vertices, and that X is a vertex cover.

Class – NP

In Version 2, we can also easily check that X has exactly k vertices and that X is a vertex cover.

<u>But</u>, we don't know how to easily check that there is not a smaller vertex cover!!

That seems to require exponential time.

These are very similar *looking* "decision" problems (Yes/No answers), yet they are VERY different in this one important respect.

Class – NP

In the first: We can verify a correct answer in polynomial time.

In the second: We apparently can not verify a correct answer in polynomial time.

(At least, we don't know how to verify one in polynomial time.)

Class – NP

Could we have asked to be given something that would have allowed us to easily verify that X was the smallest such set?

No one knows what to ask for!!

To check all subsets of k or fewer vertices requires exponential time (there can be an exponential number of them).

Class – NP

Version 1 problems make up the class called NP

Definition: The *Class NP* is the set of all decision problems for which answers to Yes instances can be verified in polynomial time.

{Why not the NO instances? We'll answer that later.}

For historical reasons, NP means

"Nondeterministic Polynomial."

(Specifically, it does not mean "not polynomial").

Class – NP

Version 2 of the Vertex Cover problem is not unique. There are other versions that exhibit this same property. For example,

Version 3: Given: A graph G = (V, E) and an integer k.

Question: Do all vertex covers of G have more than k vertices?

What would/could a 'witness' for a Yes instance be?

Class – NP

Again, no one knows except to list all subsets of at most k vertices. Then we would have to check each of the possible exponential number of sets.

Further, this is not isolated to the Vertex Cover problem. Every decision problem has a 'Version 3,' also known as the 'complement' problem (we will discuss these further at a later point).

Class – NP

All problems in NP are *decidable*.

That means there is an algorithm.

And, the algorithm is no worse than $O(2^n)$.

Class – NP

Version 2 and 3 problems are apparently not in NP.

So, where are they??

We need more structure! {Again, later.}

First we look inward, within NP.

Class – P

Second Significant Class of Problems: The Class P

Class – P

Some decision problems in NP can be solved (without knowing the answer in advance) - in polynomial time. That is, not only can we verify a correct answer in polynomial time, but we can actually compute the correct answer in polynomial time - from "scratch."

These are the problems that make up the class P.

P is a subset of NP.

Class – P

Problems in P can also have a witness – we just don't need one. But, this line of thought leads to an interesting observation. Consider the problem of searching a list L for a key X.

Given: A list L of n values and a key X.

Question: Is X in L?

Class – P

We know this problem is in P. But, we can also envision a nondeterministic solution. An oracle can, in fact, provide a "witness" for a Yes instance by simply writing down the index of where X is located.

We can verify the correctness with one simple comparison and reporting, Yes the witness is correct.

Class – P

Now, consider the complement (Version 3) of this problem:

Given: **A list L of n values and a key X.**

Question: Is X not in L?

Here, for any Yes instance, no 'witness' seems to exist, but if the oracle simply writes down "Yes" we can verify the correctness in polynomial time by comparing X with each of the n values and report "Yes, X is not in the list."

Class – P

Therefore, both problems can be verified in polynomial time and, hence, both are in NP.

This is a characteristic of any problem in P - both it and its complement can be verified in polynomial time (of course, they can both be 'solved' in polynomial time, too.)

Therefore, we can again conclude P $\subseteq$ NP.

Class – P

There is a popular conjecture that if any problem and its complement are both in NP, then both are also in P.

This has been the case for several problems that for many years were not known to be in P, but both the problem and its complement were known to be in NP.

For example, Linear Programming (proven to be in P in the 1980's), and Prime Number (proven in 2006 to be in P).

A notable 'holdout' to date is Graph Isomorphism.

Class – P

There are a lot of problems in NP that we do not know how to solve in polynomial time. Why?

Because they really don't have polynomial algorithms?

Or, because we are not yet clever enough to have found a polynomial algorithm for them?

Class – P

At the moment, no one knows.

Some believe all problems in NP have polynomial algorithms. Many do not (believe that).

The fundamental question in theoretical computer science is:

Does P = NP?

There is an award of one million dollars for a proof.

– Either way, True or False.

Other Classes

We now look at other classes of problems.

Hard appearing problems can turn out to be easy to solve. And, easy looking problems can actually be very hard (Graph Theory is rich with such examples).

We must deal with the concept of "as hard as," "no harder than," etc. in a more rigorous way.

"No harder than"

Problem A is said to be 'no harder than' problem B when the smallest class containing A is a subset of the smallest class containing B.

Recall that $f_X(n)$ is the order of the smallest complexity class containing problem X.

If, for some constant α,

$$f_A(n) \leq n^{\alpha} \; f_B(n),$$

the time to solve A is no more than some polynomial multiple of the time required to solve B, i.e., A is 'no harder than' B.

"No harder than"

The requirement for determining the relative difficulty of two problems A and B requires that we know, at least, the order of the fastest algorithm for problem B and the order of some algorithm for Problem A.

We may not know either!!

In the following we exhibit a technique that can allow us to determine this relationship without knowing anything about an algorithm for either problem.

The "Key" to Complexity Theory

'Reductions,'
'Reductions,'
'Reductions.'

Reductions

For any problem X, let X(I_X, $Answer_X$) represents an algorithm for problem X – even if none is known to exist.

I_X is an arbitrary instance given to the algorithm and $Answer_X$ is the returned answer determined by the algorithm.

Reductions

Definition: For problems A and B, a (*Polynomial) Turing Reduction* is an algorithm A(I_A, $Answer_A$) for solving all instances of problem A and satisfies the following:

(1) Constructs zero or more instances of problem B and invokes algorithm B(I_B, $Answer_B$), on each.

(2) Computes the result, $Answer_A$, for I_A.

(3) Except for the time required to execute algorithm B, the execution time of algorithm A must be polynomial with respect to the size of I_A.

Reductions

Proc A(I_A, $Answer_A$)

For i = 1 to alpha

- **Compute I_B**
-

B(I_B, $Answer_B$)

-

End For

Compute $Answer_A$

End proc

Reductions

We may <u>assume</u> a 'best' algorithm for problem B without actually knowing it.

If $A(I_A, Answer_A)$ can be written without algorithm B, then problem A is simply a polynomial problem.

Poly Turing Reductions

The existence of a Turing reduction is often stated as:

"Problem A *reduces* to problem B" or, simply,

"$A \leq_P B$"

PolyTime Reductions

Theorem. If $A \leq_P B$ and problem B is polynomial, then problem A is polynomial.

Corollary. If $A \leq_P B$ and problem A is exponential, then problem B is exponential.

PT Reductions

The previous theorem and its corollary do not capture the full implication of Turing reductions.

Regardless of the complexity class problem B is in, a Turing reduction implies problem A is in a subclass.

Regardless of the class problem A might be in, problem B is in a super class.

PT Reductions

Theorem. If $A \leq_P B$, then problem A is "no harder than" problem B.

Proof: Let $t_A(n)$ and $t_B(n)$ be the maximum times for algorithms A and B per the definition. Thus, $f_A(n) \leq t_A(n)$. Further, since we assume the best algorithm for B, $t_B(n) = f_B(n)$. Since $A \leq_P B$, there is a constant k such that $t_A(n) \leq n^k t_B(n)$. Therefore, $f_A(n) \leq t_A(n) \leq n^k t_B(n) = n^k f_B(n)$. That is, A is no harder than B.

PT Reductions

Theorem.

If $A \leq_P B$ and $B \leq_P C$ then $A \leq_P C$.

Definition.

If $A \leq_P B$ and $B \leq_P A$, then A and B are *polynomially equivalent*.

Polynomial Reductions

$A \leq_P B$ means:

'Problem A is *no harder within a polynomial factor than* problem B,' and

'Problem B is *as hard within a polynomial factor as* is problem A.'

An Aside

Without condition (3) of the definition, a simple Reduction results.

If problem B is decidable,

then so is problem A.

Equivalently,

If problem A is undecidable,

then problem B is undecidable.

NP–Complete

Third Significant Class of Problems:
The Class NP–Complete

NP–Complete

- **Polynomial Transformations enforce an equivalence relationship on all decision problems, particularly, those in the Class NP. Class P is one of those classes and is the "easiest" class of problems in NP.**
- **Is there a class in NP that is the hardest class in NP?**
- **A problem B in NP such that $A \leq_P B$ for every A in NP.**

NP–Complete

In 1971, Stephen Cook proved there was. Specifically, a problem called

Satisfiability **(or, SAT).**

Satisfiability

$U = \{u_1, u_2, \ldots, u_n\}$, Boolean variables.

$C = \{c_1, c_2, \ldots, c_m\}$, "OR clauses"

For example:

$$c_i = (u_4 \vee u_{35} \vee \sim u_{18} \vee u_3 \ldots \vee \sim u_6)$$

Satisfiability

Can we assign Boolean values to the variables in U so that every clause is TRUE?

There is no known polynomial time algorithm!!

NP–Complete

Cooks Theorem:

1) SAT is in NP

2) For every problem A in NP,

$A \leq_P SAT$

Thus, SAT is as hard as every problem in NP.

NP–Complete

Since SAT is itself in NP, that means SAT is a hardest problem in NP (there can be more than one.).

A hardest problem in a class is called the "completion" of that class.

Therefore, SAT is NP–Complete.

NP–Complete

Today, there are 1,000's of problems that have been proven to be NP–Complete. (See Garey and Johnson, *Computers and Intractability: A Guide to the Theory of NP–Completeness*, for a list of over 300 as of the early 1980's).

P = NP?

If P = NP then all problems in NP are polynomial problems.

If P ≠ NP then all NP–C problems are at least super-polynomial and perhaps exponential. That is, NP-C problems could require sub-exponential super-polynomial time. (Example of super-polynomial, sub-exponential is $\mathbf{o(2^{o(n)})}$, e.g., $\mathbf{2^{\sqrt[3]{n}}}$

P = NP?

Why should P equal NP?

- There seems to be a huge "gap" between the known problems in P and Exponential. That is, almost all known polynomial problems are no worse than n^3 or n^4.
- Where are the $O(n^{50})$ problems?? $O(n^{100})$? Maybe they are the ones in NP–Complete?
- It's awfully hard to envision a problem that would require n^{100}, but surely they exist?
- Some of the problems in NP–C just look like we should be able to find a polynomial solution (looks can be deceiving, though).

P ≠ NP?

Why should P not equal NP?

- P = NP would mean, for any problem in NP, that it is just as easy to solve an instance form "scratch," as it is to verify the answer if someone gives it to you. That seems a bit hard to believe.
- There simply are a lot of awfully hard looking problems in NP–Complete (and Co–NP-Complete) and some just don't seem to be solvable in polynomial time.
- Many smart people have tried for a long time to find polynomial algorithms for some of the problems in NP-Complete - with no luck.

NP-Complete; NP-Hard

A decision problem, *C*, is NP-complete if:

C is in NP and

C is NP-hard. That is, every problem in NP is polynomially reducible to C.

D polynomially reduces to *C* means that there is a deterministic polynomial-time many-one algorithm, *f*, that transforms each instance *x* of *D* into an instance f(x) of *C*, such that the answer to *f(x)* is YES if and only if the answer to *x* is YES.

To prove that an NP problem *A* is NP-complete, it is sufficient to show that an already known NP-complete problem polynomially reduces to *A*. By transitivity, this shows that *A* is NP-hard.

A consequence of this definition is that if we had a polynomial time algorithm for any NP-complete problem *C*, we could solve all problems in NP in polynomial time. That is, P = NP.

Note that NP-hard does not necessarily mean NP-complete, as a given NP-hard problem could be outside NP.

Returning to SAT

- SAT is the problem to decide of an arbitrary Boolean formula (wff in the propositional calculus) whether or not this formula is satisfiable (has a set of variable assignments that evaluate the expression to true).
- SAT clearly can be solved in time $k2^n$, where k is the length of the formula and n is the number of variables in the formula.
- What we now show is that SAT is NP-complete, providing us our first concrete example of an NP-complete decision problem.

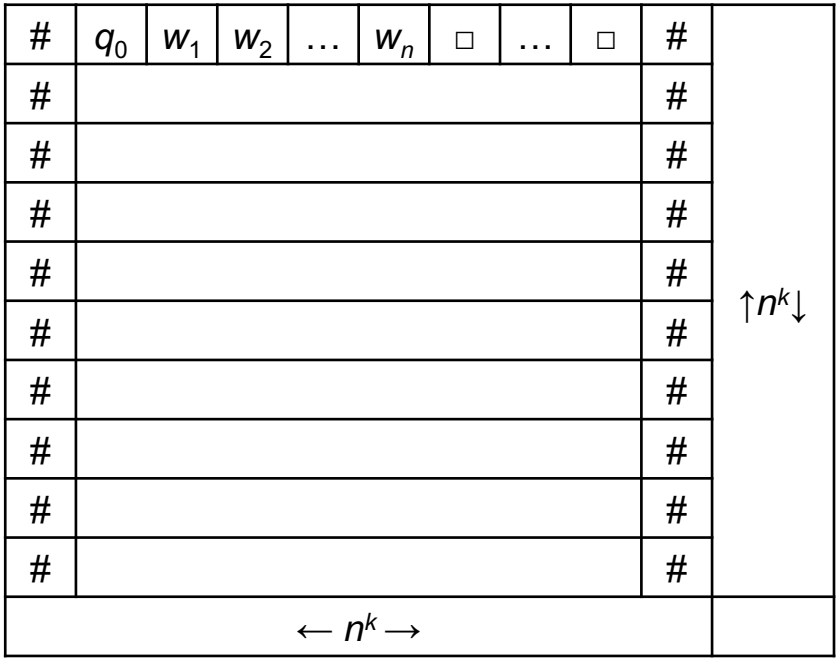
Simulating NDTM

- Given a NDTM, **M**, and an input **w**, we need to create a formula, $\varphi_{\mathbf{M},\mathbf{w}}$, containing a polynomial number of terms that is satisfiable just in case **M** accepts **w** in polynomial time.
- The formula must encode within its terms a trace of configurations that includes
  - A term for the starting configuration of the TM
  - Terms for all accepting configurations of the TM
  - Terms that ensure the consistency of each configuration
  - Terms that ensure that each configuration after the first follows from the prior configuration by a single move

Tableaus

A **tableau** is an array of tape alphabet symbols.

- It represents a configuration history of **one branch** of our NDTM's nondeterminism.
- If the NDTM runs in n^k time, the tableau is an ($n^k \times n^k$) tableau.
  - It's big enough downward because, well, the TM runs in n^k.
  - …and rightward because the TM can only *count* to n^k.
- We assume that every configuration starts and ends with a # symbol.
- We think of our tableau as looking like this in the "beginning": the starting configuration across the top, and the other configurations blank.
  - (We quote "beginning" because SAT isn't really a stateful algorithm, but just go with it for now.)
- But we've assumed that we can "represent" alphabet symbols. How do we do that, in *SAT*?



Encoding the Tableau: Basics

Consider a set comprised of:

- The tape alphabet
- The state set
- The separator character

$$C = \Gamma \cup Q \cup \{ \# \}$$

Consider a cell variable:

$$x_{i,j,c}$$

Turning this variable on *corresponds to* ***setting cell (i, j) = c***, for some $c \in C$.

	1	2	3	4	5	6	7	8	9	10
1	#	q_0	w_1	w_2	…	w_n	□	…	□	#
2	#									#
3	#									#
4	#									#
5	#									#
6	#									#
7	#									#
8	#									#
9	#									#
10	#									#

Encoding the Tableau: Cells

Consider our tableau alphabet:

$$C = \Gamma \cup Q \cup \{ \# \}$$

Consider a cell and corresponding variable:

$$x_{i,j,c}$$

Now we need to make sure the tableau is consistently encoded.

Create a clause for **each cell (*i*, *j*)**.

$$\phi_{\text{encode}}(i,j) = \left[\left(\bigvee_{c \in C} x_{i,j,c} \right) \wedge \left(\bigwedge_{\substack{c,d \in C \\ c \neq d}} \left(\overline{x_{i,j,c}} \vee \overline{x_{i,j,d}} \right) \right) \right]$$

The left demands $x_{i,j,c}$ be true for **some *c*.**

The right demands $x_{i,j,c}$ be true for **only one *c*.**

	1	2	3	4	5	6	7	8	9	10
1	#	q_0	w_1	w_2	…	w_n	□	…	□	#
2	#									#
3	#									#
4	#									#
5	#									#
6	#									#
7	#									#
8	#									#
9	#									#
10	#									#

Encoding the Tableau: The Tableau

Tableau alphabet: $C = \Gamma \cup Q \cup \{ \# \}$

Cell variable: $x_{i,j,c}$

Create an encoding clause for each cell (i, j).

$$\phi_{\text{encode}}(i,j) = \left[\left(\bigvee_{c \in C} x_{i,j,c} \right) \wedge \left(\bigwedge_{\substack{c,d \in C \\ c \neq d}} \left(\overline{x_{i,j,c}} \vee \overline{x_{i,j,d}} \right) \right) \right]$$

Now repeat the clause across the tableau.

$$\phi_{\text{cells}} = \bigwedge_{1 \le i,j \le n^k} \phi_{\text{encode}}(i,j)$$

This is our *cell formula*. It ensures that each cell in the tableau is assigned a single symbol.

	1	2	3	4	5	6	7	8	9	10
1	#	q_0	w_1	w_2	…	w_n	□	…	□	#
2	#									#
3	#									#
4	#									#
5	#									#
6	#									#
7	#									#
8	#									#
9	#									#
10	#									#

Encoding the Tableau: Complexity

$$\phi_{\text{encode}}(i,j) = \left[\left(\bigvee_{c \in C} x_{i,j,c}\right) \wedge \left(\bigwedge_{\substack{c,d \in C \\ c \neq d}} (\overline{x_{i,j,c}} \vee \overline{x_{i,j,d}})\right)\right]$$

We can create the single-cell encoding formula in polynomial time with a $|C|^2$ iteration.

$$\phi_{\text{cells}} = \bigwedge_{1 \leq i,j \leq n^k} \phi_{\text{encode}}(i,j)$$

We can create the *entire* cell formula in polynomial time with an n^{2k} iteration around that.

So we can say that **ϕ_{cells} is satisfied by, and only by, a properly encoded tableau, and is created in polynomial time.**

	1	2	3	4	5	6	7	8	9	10
1	#	q_0	w_1	w_2	…	w_n	□	…	□	#
2	#									#
3	#									#
4	#									#
5	#									#
6	#									#
7	#									#
8	#									#
9	#									#
10	#									#

Starting and Accepting

Starting and accepting are (comparatively) easy.

To start, take the start configuration padded to n^k length with blanks…

$S = \#q_0 w_1 w_2 \ldots w_n \square \ldots \square \#$ so that $|S| = n^k$

…and **require the first row be equal to the start configuration**:

$$\phi_{\text{start}} = \bigwedge_{1 \le j \le n^k} [x_{1,j,s_j}]$$

Then to accept, just **require an accept state somewhere in the tableau.**

$$\phi_{\text{accept}} = \bigvee_{1 \le i,j \le n^k} [x_{i,j,q_A}]$$

	1	2	3	4	5	6	7	8	9	10
1	#	q_0	w_1	w_2	…	w_n	□	…	□	#
2	#									#
3	#									#
4	#									#
5	#	w_1	w_2	…	q_A	…	□	…	□	#
6	#									#
7	#									#
8	#									#
9	#									#
10	#									#

Starting and Accepting

$$\phi_{\text{start}} = \bigwedge_{1 \le j \le n^k} [x_{1,j,s_j}]$$

$$\phi_{\text{accept}} = \bigvee_{1 \le i,j \le n^k} [x_{i,j,q_A}]$$

We can generate the start and accept formulas in n^k and $(n^k)^2$ time, both polynomial.

So now we can say that:

ϕ_{start} is satisfied by, and only by, a tableau with the starting configuration of *M* on *w* encoded as its first row, and is created in polynomial time.

…and…

ϕ_{accept} is satisfied by, and only by, a tableau encoding an accepting configuration as one of its rows, and is created in polynomial time.

	1	2	3	4	5	6	7	8	9	10
1	#	q_0	w_1	w_2	…	w_n	□	…	□	#
2	#									#
3	#									#
4	#									#
5	#	z_1	z_2	…	q_A	…	□	…	□	#
6	#									#
7	#									#
8	#									#
9	#									#
10	#									#

Transitions

Now, for transitions. Recall the discussions we had about ID changes being limited to three characters or six, when looking at transitions..

- A given 2x3 **window** is **legal** if it does not violate our machine's transition function.
- Given the linear sets of states and tape symbols, and the finite size of 2x3 windows, we can make a **polynomial-sized set of all legal windows**.

Let a sequence $A = (a_1, \ldots, a_6)$ be a 2x3 window, with a_1 the top left cell, a_2 the top middle, etc.

- We say that ***A* is *legal*** if it **represents a legal window**. Here we have $\mathbf{q_0}$ **a R** $\mathbf{q_1}$

	1	2	3	4	5	6	7	8	9	10
1	#	q_0	*a*	*b*	c	*a*	□	□	□	#
2	#	*a*	q_1	*b*	c	*a*	□	□	□	#
3	#									#
4	#									#
5	#									#
6	#									#
7	#									#
8	#									#
9	#									#
10	#									#

Transitions

A given 2x3 **window** is **legal** if it does not violate our machine's transition function. We have a **polynomial-sized set of all legal windows**.

Let a sequence $A = (a_1, \ldots, a_6)$ be a 2x3 window. ***A* is *legal*** if it **represents a legal window**.

Now we can come up with a formula to say that the window top-centered at cell (i, j) is legal.

$$\phi_{\text{legal}}(i,j) = \bigvee_{\substack{A=(a_1,\ldots,a_6) \\ \text{is legal}}} \left[\begin{matrix} x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j+1,a_3} \wedge \\ x_{i+1,j-1,a_4} \wedge x_{i+1,j,a_5} \wedge x_{i+1,j+1,a_6} \end{matrix} \right]$$

	1	2	3	4	5	6	7	8	9	10
1	#	q_0	*a*	*b*	c	*a*	□	□	□	#
2	#	*a*	q_1	*b*	c	*a*	□	□	□	#
3	#									#
4	#									#
5	#									#
6	#									#
7	#									#
8	#									#
9	#									#
10	#									#

Don't be intimidated by this formula!

It's just **counting off the six cells of the window** and demanding that each be **equal to the corresponding cell** in **some legal window**.

Transitions

A given 2x3 **window** is **legal** if it does not violate our machine's transition function.

We have a **polynomial-sized set of all legal windows**.

Let a sequence $A = (a_1, \ldots, a_6)$ be a 2x3 window. ***A* is *legal*** if it **represents a legal window**.

$$\phi_{\text{legal}}(i,j) = \bigvee_{\substack{A=(a_1,\ldots,a_6) \\ \text{is legal}}} \left[\begin{matrix} x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j+1,a_3} \wedge \\ x_{i+1,j-1,a_4} \wedge x_{i+1,j,a_5} \wedge x_{i+1,j+1,a_6} \end{matrix} \right]$$

	1	2	3	4	5	6	7	8	9	10
1	#	q_0	*a*	*b*	c	*a*	□	□	□	#
2	#	*a*	q_1	*b*	c	*a*	□	□	□	#
3	#									#
4	#									#
5	#									#
6	#									#
7	#									#
8	#									#
9	#									#
10	#									#

Since we have a polynomial number of legal windows, this formula is also polynomial. So we can say:

ϕ_{legal} (*i*, *j*) is satisfied by, and only by, a tableau whose window top-centered at (*i*, *j*) is legal; and is created in polynomial time.

Windows and Configurations

Consider any **upper** and **lower** configuration in the tableau, so that the lower configuration is the one immediately below – that is, following – the upper.

If all the windows top-centered on cells in the upper configuration are legal, then:

- The legality of the windows that don't involve the state symbol easily ensures the legality of the configuration below them.
- The window top-centered on the state symbol in the upper configuration is sufficient to ensure that the state symbol in the lower configuration makes a legal move.

The upper configuration yields the lower one if and only if all the windows top-centered on cells in the upper configuration are legal – and that holds all the way down the tableau.

	1	2	3	4	5	6	7	8	9	10
1	#	q_0	a	b	c	a	□	□	□	#
2	#	a	q_1	b	c	a	□	□	□	#
3	#									#
4	#									#
5	#									#
6	#									#
7	#									#
8	#									#
9	#									#
10	#									#

Windows and Configurations

$$\phi_{\text{legal}}(i,j) = \bigvee_{\substack{A=(a_1,\ldots,a_6)\\ \text{is legal}}} \begin{bmatrix} x_{i,j-1,a_1} \wedge x_{i,j,a_2} \wedge x_{i,j+1,a_3} \wedge \\ x_{i+1,j-1,a_4} \wedge x_{i+1,j,a_5} \wedge x_{i+1,j+1,a_6} \end{bmatrix}$$

ϕ_{legal} (i, j) is satisfied by, and only by, a tableau whose window top-centered at (i, j) is legal; and is created in polynomial time.

An upper configuration yields a lower one iff all the windows top-centered within the upper are legal.

This holds all the way down the tableau.

Then we have:

$$\phi_{\text{move}} = \bigwedge_{\substack{1\le i<n^k,\\ 1<j<n^k}} \phi_{\text{legal}}(i,j)$$

And can say ϕ_{move} **is satisfied by, and only by, a tableau that does not violate the machine's transition function; and is created in polynomial time.**

	1	2	3	4	5	6	7	8	9	10
1	#	q_0	*a*	*b*	c	*a*	□	□	□	#
2	#	*a*	q_1	*b*	c	*a*	□	□	□	#
3	#									#
4	#									#
5	#									#
6	#									#
7	#									#
8	#									#
9	#									#
10	#									#

Pulling It Together

$$\phi_{\text{cells}} = \bigwedge_{1 \le i,j \le n^k} \phi_{\text{encode}}(i,j)$$

$$\phi_{\text{start}} = \bigwedge_{1 \le j \le n^k} \left[x_{1,j,s_j} \right]$$

$$\phi_{\text{accept}} = \bigvee_{1 \le i,j \le n^k} \left[x_{i,j,q_A} \right]$$

$$\phi_{\text{move}} = \bigwedge_{\substack{1 \le i < n^k, \\ 1 < j < n^k}} \phi_{\text{legal}}(i,j)$$

$$\phi_{\text{NDTM}} = \left(\phi_{\text{cells}} \wedge \phi_{\text{start}} \wedge \phi_{\text{accept}} \wedge \phi_{\text{move}} \right)$$

We have:

- ϕ_{cells} is satisfied by, and only by, a properly encoded tableau.
- ϕ_{start} is satisfied by, and only by, a tableau with the starting configuration of *M* on *w* encoded as its first row.
- ϕ_{accept} is satisfied by, and only by, a tableau encoding an accepting configuration as one of its rows.
- ϕ_{move} is satisfied by, and only by, a tableau that does not violate the machine's transition function.
- All are created in polynomial time.

Then **ϕ_{NDTM} is satisfied by, and only by, a tableau encoding an accepting computation history of *M* on *w*, and is created in polynomial time.**

SAT is NP-Complete

$$\phi_{\text{NDTM}} = (\phi_{\text{cells}} \wedge \phi_{\text{start}} \wedge \phi_{\text{accept}} \wedge \phi_{\text{move}})$$

ϕ_{NDTM} created from NDTM *M* and input *w* is satisfied by, and only by, a tableau encoding an accepting computation history of *M* on *w*, and is created in polynomial time.

This means that:

> *SAT* accepts ϕ_{NDTM} if and only if such a tableau exists…
>
> …if and only if the NDTM we are encoding into ϕ_{NDTM} accepts *w*.

We've just polynomially reduced every possible NP language to *SAT*.

Let's convince ourselves of that a bit more.

> By definition, any NP language has an NDTM *M* that decides it in polynomial time.

We can decide any NP language with a result from *SAT* using the following algorithm:

On input <*M*, *w*>:

> Create ϕ_{NDTM} from *M* and *w*.
>
> Run the decider for *SAT* on ϕ_{NDTM}.
>
> Accept if *SAT* accepts, reject if it rejects.

***SAT* is NP-complete.**

Cook's Theorem

- $\varphi_{M,w} = \phi_{cells} \wedge \phi_{start} \wedge \phi_{accept} \wedge \phi_{move}$
- **See the following for another detailed description and discussion of the four terms that make up this formula.**
- http://www.cs.tau.ac.il/~safra/Complexity/Cook.ppt

NP–Complete

Within a year, Richard Karp added 22 problems to this special class.

We will focus on:

- 3-SAT
- Integer Linear Programming
- SubsetSum
- Partition
- Vertex Cover
- Independent Set
- K-Color
- Multiprocessor Scheduling

Co-NP

- A problem is in co-NP if its complement is in NP – this is like co-RE, with respect to RE problems.
- An example is the problem to determine if a Boolean expression is a tautology.
  - If the answer to the problem "is B in TAUT ?" is NO, then ¬A is in SAT.
- A more direct example of a co-NP problem is to determine if a Boolean expression is self-contradictory.
  - This is the complement of satisfiability.
- Both of the above are co-NP Complete

SAT to 3SAT

- 3-SAT means that each clause has exactly three terms
- If one term, e.g., (p), expand to (p∨p∨p)
- If two terms, e.g., (p∨q), expand to (p∨q∨p)
- Any clause with three terms is fine
- If n > three terms, can reduce to two clauses, one with three terms and one with n-1 terms, e.g., (p1∨p2∨…∨pn) to (p1∨p2∨z) & (p3∨…∨pn∨~z), where z is a new variable. If n=4, we are done, else apply this approach again with the clause having n-1 terms

Linear Programming (LP)

- Linear Programming (LP) is like solving a set of linear equations but allows just just equality (=) but also inequality (>, <, ≥, ≤)
- In actual fact, LP usually also includes an optimization function, but we are limiting ourselves to decision problems
- Example:
  $x + y > 7$
  $x - y \geq 4$
  Has many solutions, some of which are integral, e.g., x=7, y=1

Integer LP (ILP)

- Integer Linear Programming (ILP) just constrains the solutions to an LP problem to be integral values
- This constraint, on the surface, may seem to make the problem easier but, in fact, makes it harder
- This is even true when we view this as a decision problem where we just ask
  "Is there a solution to this instance of ILP"
- We will see this complexity play out in the next few slides

0-1 ILP

- 0-1 ILP says constrains the solution space to variable values of 1 and 1
- Start with an instance of SAT (or 3SAT), assuming variables v1,…, vn and clauses c1,…, cm
- For each variable vi, have the constraint that $0 \leq vi \leq 1$
- For each clause we provide a constraint that it must be satisfied (evaluate to at least 1). For example, if clause cj is
  v2 ∨ ~v3 ∨ v5 ∨ v6 then add the constraint
  $v2 + (1-v3) + v5 + v6 \geq 1$
- A solution to this set of integer linear constraints implies a solution to the instance of SAT and vice versa
- Note this works for any SAT instance not just 3SAT

0-1 ILP is NP-Complete

- Previous page just show 0-1 ILP is NP-Hard
- Must show it is in NP
- Can do by trying all 2^k 0-1 assignments to k variables
- Or can show that verifying a solution is in P – it's really just linear

0-1 ILP Example

- **Original SAT: E = (a+b+~c+d+e)(~b)(~a+~d)(b+c+~e)**
- **$0 \le a \le 1$; $0 \le b \le 1$; $0 \le c \le 1$;**
  $0 \le d \le 1$; $0 \le e \le 1$
- **$a+b+(1-c)+d+e \ge 1$;**
  alternatively, $a+b-c+d+e \ge 0$
- **$1-b \ge 1$;**
  alternatively, $b = 0$
- **$(1-a)+(1-d) \ge 1$;**
  alternatively, $a+d \le 1$
- **$b+c+(1-e) \ge 1$;**
  alternatively, $b+c-e \ge 0$

What about ILP?

- As we said, ILP just constrains the solution to integers not to binary values
- Clearly ILP is NP-Hard as the constrained version of 0-1 ILP is NP-Hard
- Showing ILP is in NP is easy using a verifier; you give me a proposed solution and I can check it in linear time

What about Linear Programming (LP) #1?

- Linear programming just requires that solution be real number values
- The only constraints are in the simultaneous inequalities (and equalities)
- If you limit LP to equalities, then it has a well-known complexity of $O(N^3)$ using Gaussian Elimination or one of its variants

What about Linear Programming (LP) #2?

- The problem of solving LP appeared to be exponential for a long time and was, and still is, generally attacked using the Simplex Method which involves adding slack variables, e.g.,
  $x + y < 7$ iff $x + y + e = 7$ for some $e > 0$ and
  $x + y \leq 7$ iff $x + y + f = 7$ for some $f \geq 0$
- One can show cases where the Simplex Method takes exponential time, but its average case is $O(N^{\sqrt{d}})$ time where N is the number of variables and d is bounded above by the size of the input in bits

What about Linear Programming (LP) #3?

- In 1984, LP was shown to be of polynomial complexity
- Complexity is $O(N^{3.5} L \lg L \lg\lg L)$ where N is number of variables and L is the size of the input in bits
- Simplex is still used much as QuickSort is used for sorting even though its worst case is $O(N^2)$ as its expected performance is O(N lg N) and it often converges in O(N)

SubsetSum

S = {s_1, s_2, …, s_n}

set of positive integers

and an integer **G.**

Question: Does **S** have a subset whose values sum to the goal **G?**

Note: This is really a Bag (Multiset) not a Set

No one knows of a polynomial algorithm.

{No one has proven there isn't one, either!!}

SubsetSum is in NP

- You give me a “solution” to [**{s_1, s_2, …, s_n}, G**]
- Your solution is just a subset of the set of integers **{1 .. n}**
- I make sure that each number you give me is unique and in the correct range (that takes me n units of time). If not, I reject your “solution”
- I then add the selected numbers together. That takes the sum of the log based 2 of the numbers you selected. I then check that the sum equals **G**. If so, I verify; if not, I reject.
- Note that the original representation is of length the sum of the log based 2 of the $\mathbf{s_i}$’s and **G** so my growth of time is linear
- Thus, I can verify in polynomial time

Example SubsetSum

- Instance
  [(15, 17, 27, 11, 4, 12, 33, 5, 6, 21, 2), 57]
- A solution is 15, 17, 11, 12, 2 (or with indices just 1,2,4,6,11)
- Note that an item can only be chosen once
- Note that we can try a heuristic like sorting low to high
  [(2, 4, 5, 6, 11, 12, 15, 17, 21, 27, 33), 57]
- But, an attack with that might have us choose
  2, 4, 5, 6, 11, 12, 15 and we are stuck
- In above one can backtrack to remove 15 and replace by 17 works, but backtracking is in general exponential
- Try high to low [(33, 27, 21, 17, 15, 12, 11, 6, 5, 4, 2), 57]
  33, 27 (Fail), 33, 21, 17 (Fail), 33, 21, 15 (Fail), etc.
- Clearly all these are potentially exponential approaches

SAT $\leq_P$ 3SAT $\leq_P$ SubsetSum $\equiv_p$ Partition

Theorem. SAT $\leq_P$ ILP $\leq_P$ LP

Theorem. SAT $\leq_P$ 3SAT

Theorem. 3SAT $\leq_P$ SubsetSum

Theorem. SubsetSum $\leq_P$ Partition

Theorem. Partition $\leq_P$ SubsetSum

Therefore, not only is SAT in NP–Complete, but so are ILP, LP, 3SAT, Partition, and SubsetSum.

3SAT $\leq_p$ SubsetSum

Assuming a **3SAT expression (a + ~b + c) (~a + b + ~c)**

	a	b	c	a+~b+c	~a+b+~c
a	1	0	0	1	0
~a	1	0	0	0	1
b	0	1	0	0	1
~b	0	1	0	1	0
c	0	0	1	1	0
~c	0	0	1	0	1
C1	0	0	0	1	0
C1’	0	0	0	1	0
C2	0	0	0	0	1
C2’	0	0	0	0	1
	1	1	1	3	3

SubsetSum Matrix

- One column per variable and one column per clause
- Two rows per variable (true/false so only one can be chosen per variable) and two rows per clause (optional pads to get to 3's in clause columns, provided we are already at least a 1).
- Each row is a number and summing them never results in carry to next column, so each column is independent of other and only influenced by rows we select.
- Goal of 1 … 1 3… 3 forces one choice (true or false per variable) and satisfiability for every clause (must have a 1 in at least one variable row of each clause column)

How it works

- Satisfying (a + ~b + c) (~a + b + ~c)
- Make a, b and c true (satisfies)
  Rows a, b and c get us 11121
  Padding with C1, C2 and C2' gets 11133
- Make a, ~b and c true (does not satisfy)
  Rows a, ~b and c get us 11130
  No amount of padding can get the last column to be 3 (2 is max)

Partition

- Given a Multiset S = {s_1, s_2, …, s_n}, where each s_i is a positive integer, can we partition it into two sub-bags P1, P2 such that P1 ∪ P2 = S and P1 ∩ P2 = ∅ ?
- Note: If S contains multiple copies of some integer, each is considered distinct and thus does not unduly influence the intersection and union operators above

SubsetSum$\equiv_p$Partition Details

- Partition is polynomial equivalent to SubsetSum
  - Let $i_1, i_2, .., i_n$, G be an instance of SubsetSum. This instance has answer "yes" iff
    $i_1, i_2, .., i_n$, 2*Sum($i_1, i_2, .., i_n$) – G,Sum($i_1, i_2, .., i_n$) + G
    has answer "yes" in Partition. Here we assume that
    G ≤ Sum($i_1, i_2, .., i_n$), for, if not, the answer is "no."
  - Let $i_1, i_2, .., i_n$ be an instance of Partition. This instance has answer "yes" iff
    $i_1, i_2, .., i_n$, Sum($i_1, i_2, .., i_n$)/2
    has answer "yes" in SubsetSum

SubsetSum $\equiv_p$ Partition

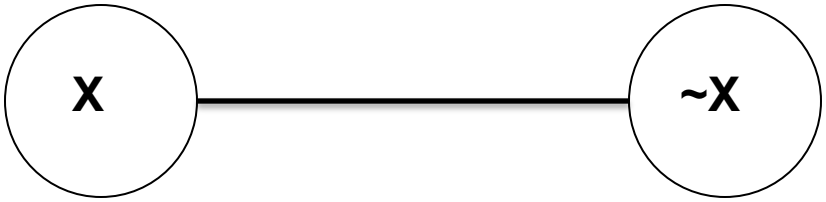
- [(15, 17, 27, 11, 4, 12, 33, 5, 6, 21, 2), 57]
- A solution is 15, 17, 11, 12, 2
- Sum of all is 153
- Mapping to Partition is
  - (15, 17, 27, 11, 4, 12, 33, 5, 6, 21, 2, 306-57, 153+57)
  - (15, 17, 27, 11, 4, 12, 33, 5, 6, 21, 2, 249, 210)
  - (15+17+11+12+2+249) = 306
  - (27+4+33+5+6+21+210) = 306
- Going other direction map above to
  - [(15, 17, 27, 11, 4, 12, 33, 5, 6, 21, 2, 249, 210), 306]

VERTEX COVERING (VC) DECISION PROBLEM IS NP-HARD

3SAT to Vertex Cover

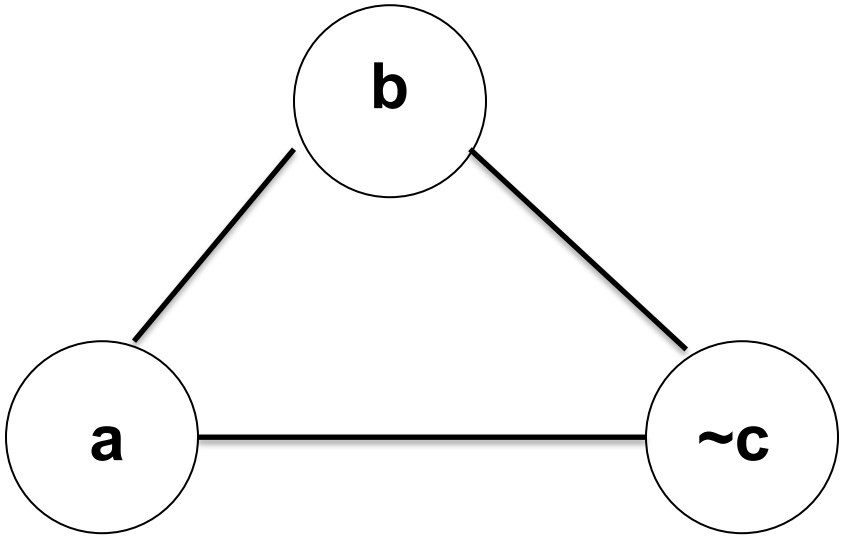
- Vertex cover seeks a set of vertices that cover every edge in some graph
- Let $I_{3\text{-SAT}}$ be an arbitrary instance of 3-SAT. For integers n and m, $U = \{u_1, u_2, \ldots, u_n\}$ and $C_i = \{z_{i1}, z_{i2}, z_{i3}\}$ for $1 \le i \le m$, where each z_{ij} is either a u_k or u_k' for some k.
- Construct an instance of VC as follows.
- For each i, $1 \le i \le n$, construct two vertices, u_i and u_i' with an edge between them.
- For each clause $C_i = \{z_{i1}, z_{i2}, z_{i3}\}$, $1 \le i \le m$, construct three vertices z_{i1}, z_{i2}, and z_{i3} and form a "triangle on them. Each z_{ij} is one of the Boolean variables u_k or its complement u_k'. Draw an edge between z_{ij} and the Boolean variable (whichever it is). Each z_{ij} has degree 3. Finally, set $k = n+2m$.
- Theorem. The given instance of 3-SAT is satisfiable if and only if the constructed instance of VC has a vertex cover with at most k vertices.

VC Variable Gadget



To cover the edge in between x and ~x, at least one of these must be chosen

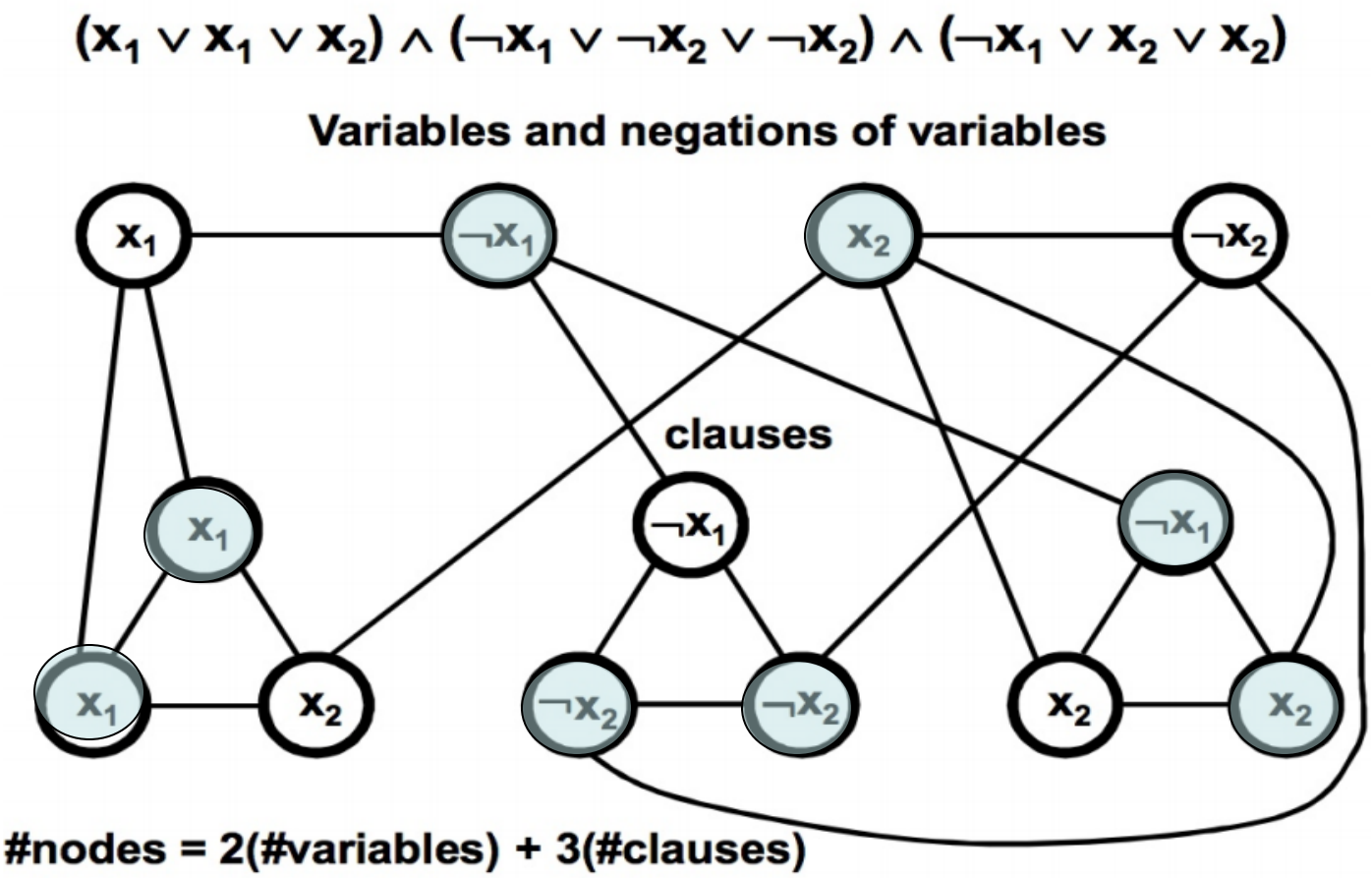
VC Clause Gadget



a + b + ~c

To cover the edges here, at least two of three vertices must be chosen

VC Gadgets Combined

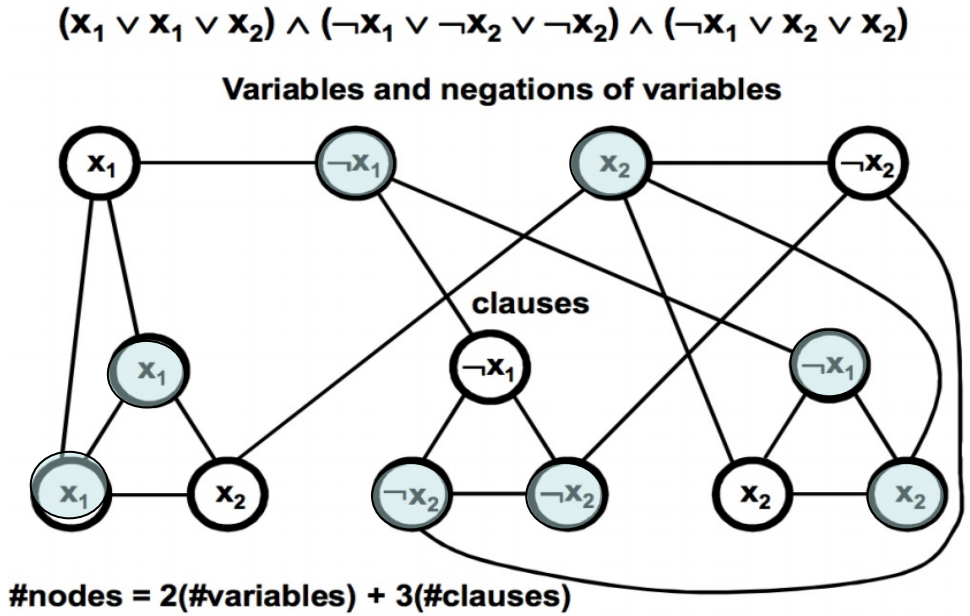


Goal is to cover all variables (really edges) with **v + 2c** nodes (minimum needed); **v**=#variables; **c** = #clauses; for above that is 2 + 2*3 = 8: Light blue are choices

Why VC Gadgets Work

- For each variable gadget, we must either choose the variable or its complement to cover the edge connecting them; Choosing both is wasteful
- For each clause gadget, we must cover its internal edges. This requires 2 per clause, but also need to cover all edges entering from variable gadgets, if not already covered by the selection of the corresponding variable gadget
- If we can cover all edges, with just v+2c nodes then we have attained the minimum possible and guaranteed that each clause has at least one of its literals true. This allows the corresponding variable selection (true/false) to cover the incoming edge allowing the three internal edges to be covered by the other two variable nodes in the clause. If more than one literal is covered, you have a choice of which covered internal node to not choose for the clause gadget

Explaining Example

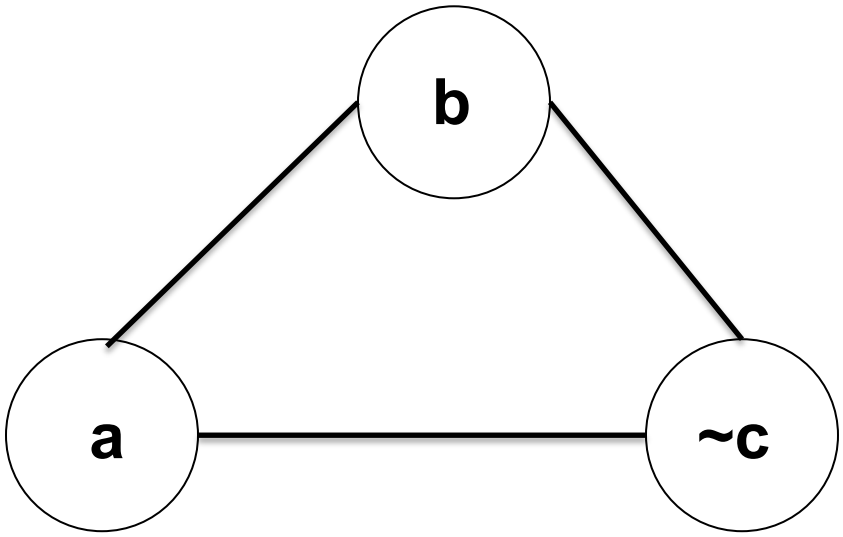


Solution choses ~x1, x2; By choosing those variable gadget nodes we cover both variable gadgets. We know we must cover all clause gadgets and the edges entering them from the variable gadgets. For the first clause, we do not need to cover the edge entering from variable gadget x2, but we must cover the other two external edges and the three internal edges. Choosing the two x1 nodes in clause 1 covers all external and internal edges. Had we chosen ~x1 in the x1 variable gadget and ~x2 in the x2 variable gadget we would have had to chose x2 in the first clause gadget thereby exceeding our quota of v + 2c. I leave it to you to see why the others work and to understand why we had no actual constraints in the third clause gadget (any two would have worked).

Independent Set

- Independent Set
  - Given Graph G = (V, E), a subset S of the vertices is independent if there are no edges between vertices in S
  - The k-IS problem is to determine for a k>0 and a graph G, whether or not G has an independent set of k nodes
- Note there is a related NP-Hard optimization problem to find a Maximum Independent Set. It is even hard to approximate a solution to the Maximum Independent Set Problem.

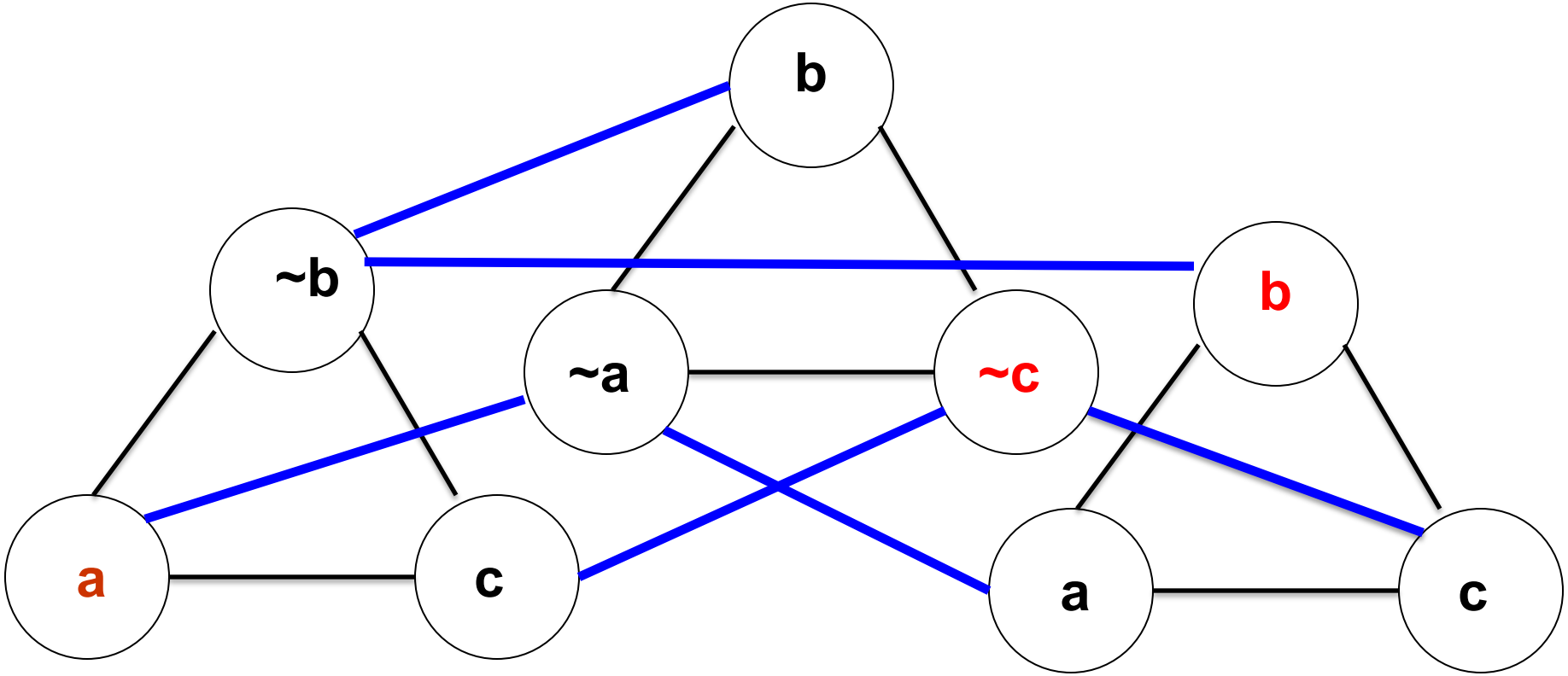
IS (VC) Clause Gadget



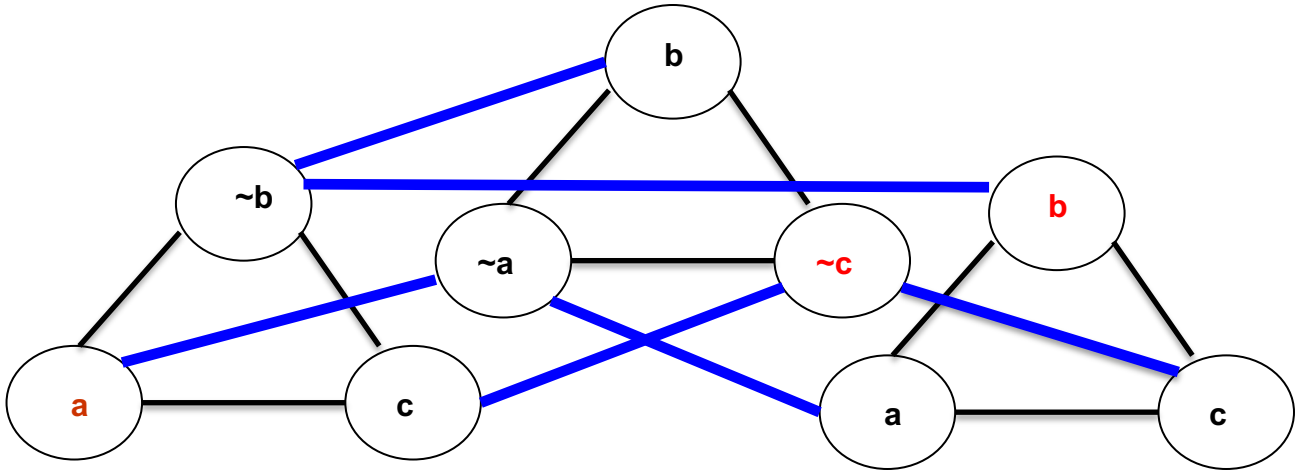
a + b + ~c

3SAT to IS

(a + ~b + c) (~a + b + ~c)(a + b + c), k=3
(k=number of clauses, not variables)



Explaining the example



In each clause gadget we can only chose one node, else we would be choosing two nodes with a shared edge. Assume we select a in clause 1, we cannot choose ~a in any other clause as they have a shared edge. After choosing a in clause 1, we could choose either b or ~c in clause 2. Assume we select ~c in clause 2, we cannot choose c in any other clause as they have a shared edge. To reach three (the number of clauses, we need a choice left for clause 3. Fortunately we have two choices, either a or c.

K-COLOR (KC) DECISION PROBLEM IS NP-HARD

K-Coloring

Given:

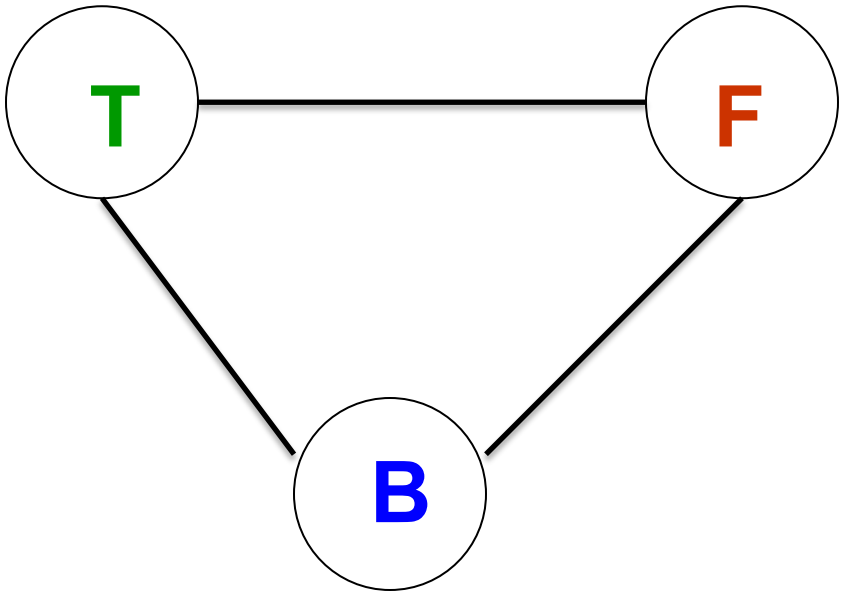
A graph G = (V, E) and an integer k.

Question:

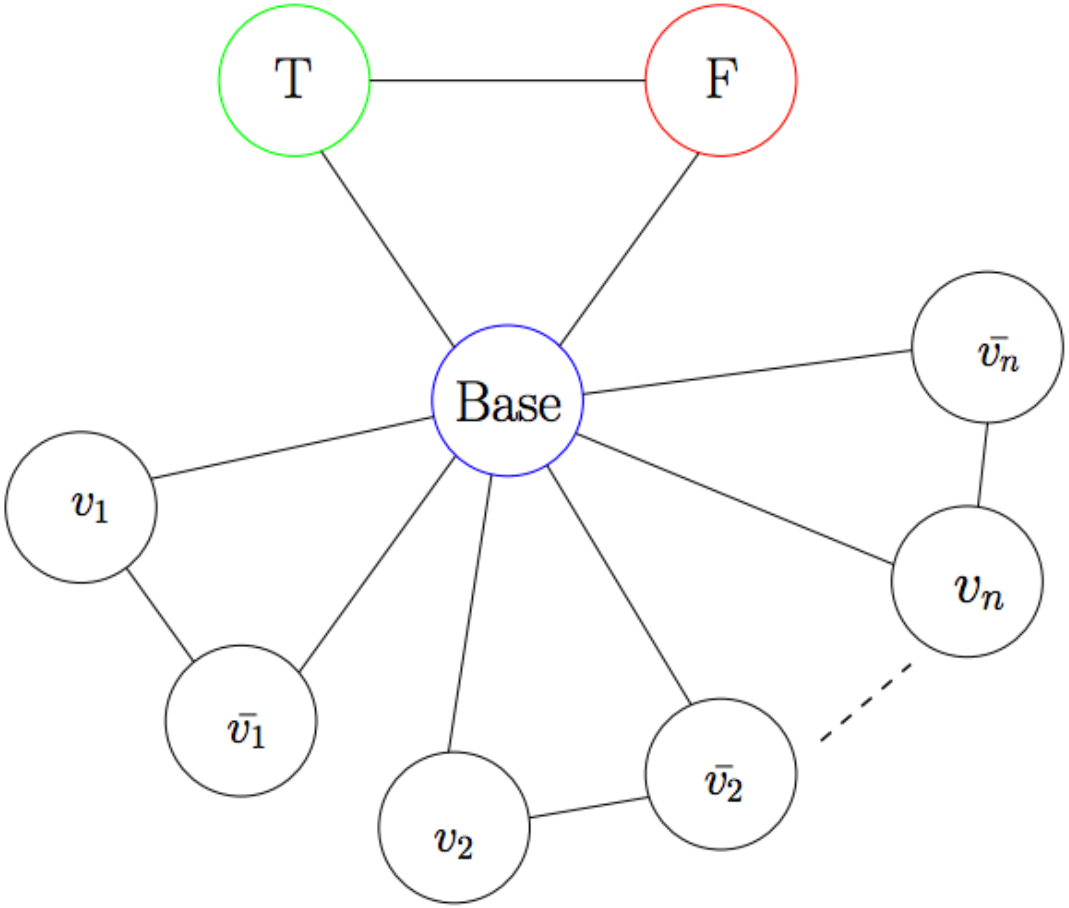
Can the vertices of G be assigned colors from a palette of size k, so that adjacent vertices have different colors and use at most k colors?

3Coloring (3C) uses k=3

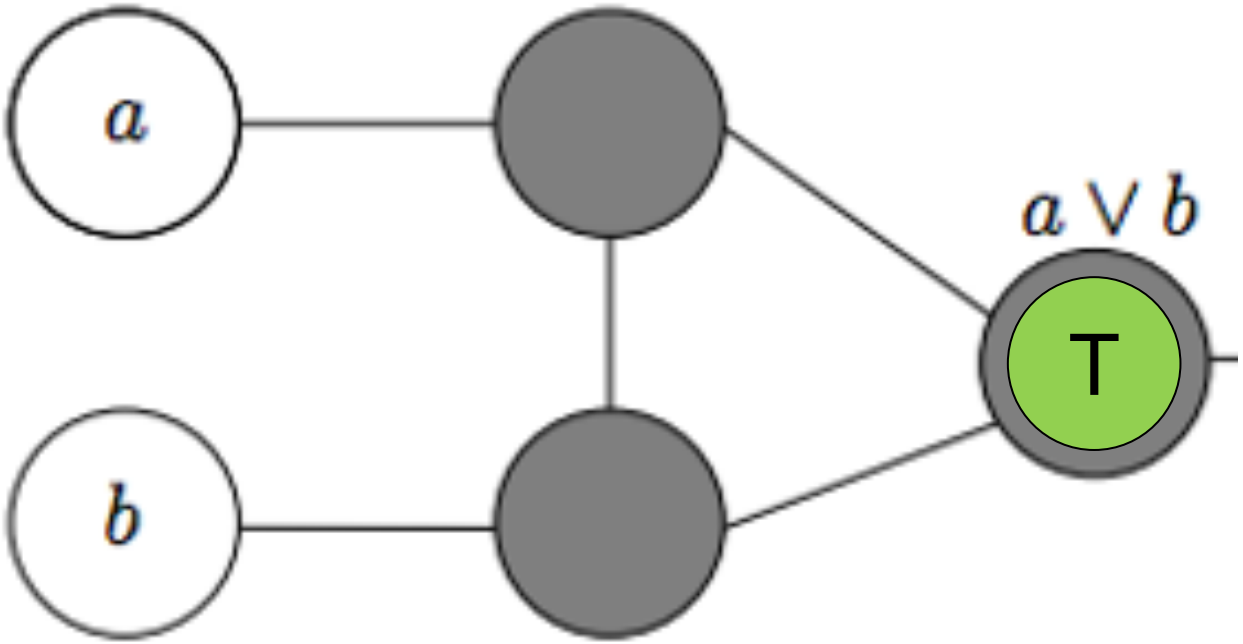
3C Super Gadget



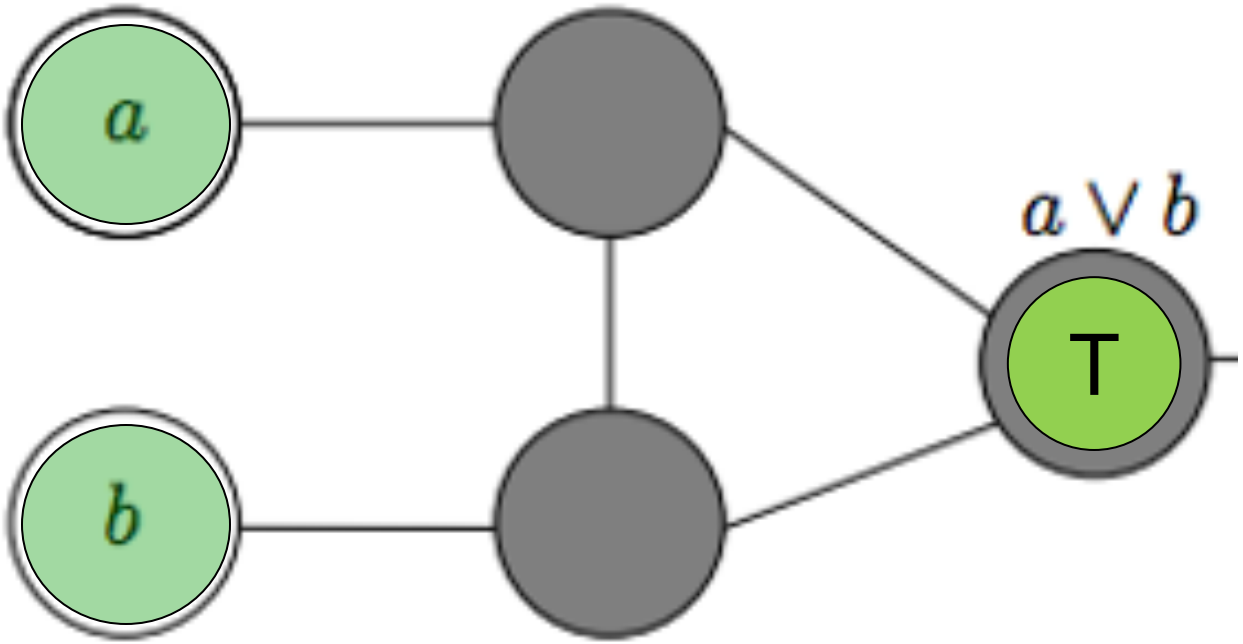
3C Super + Variables Gadget



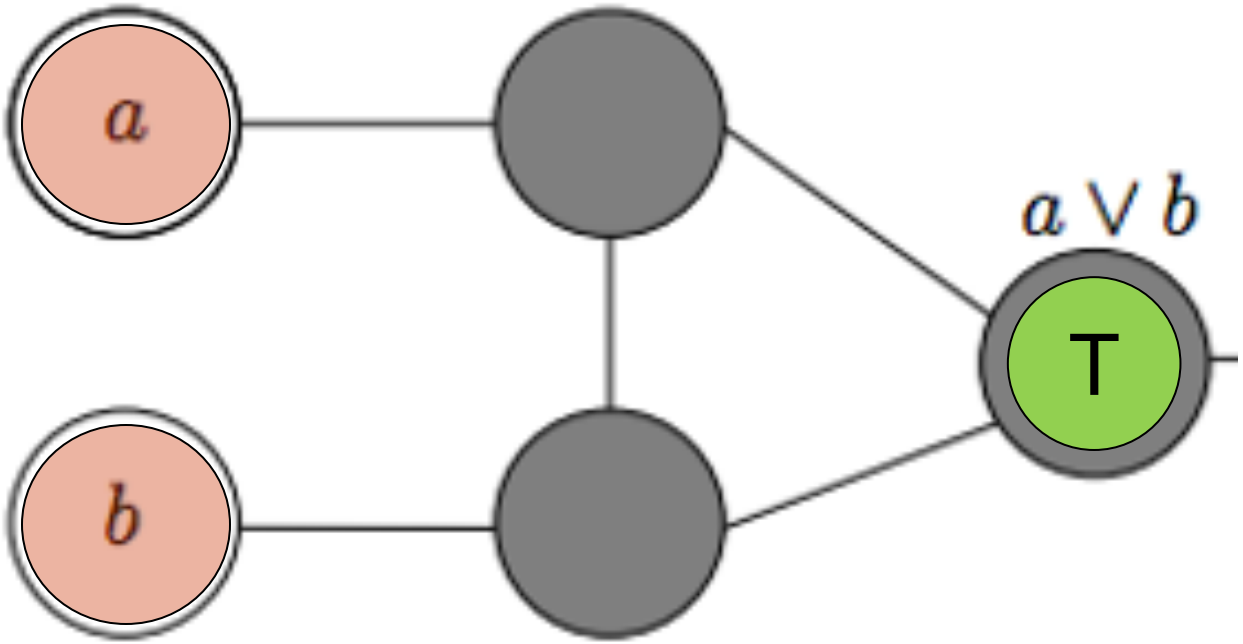
A Simple OR Gadget



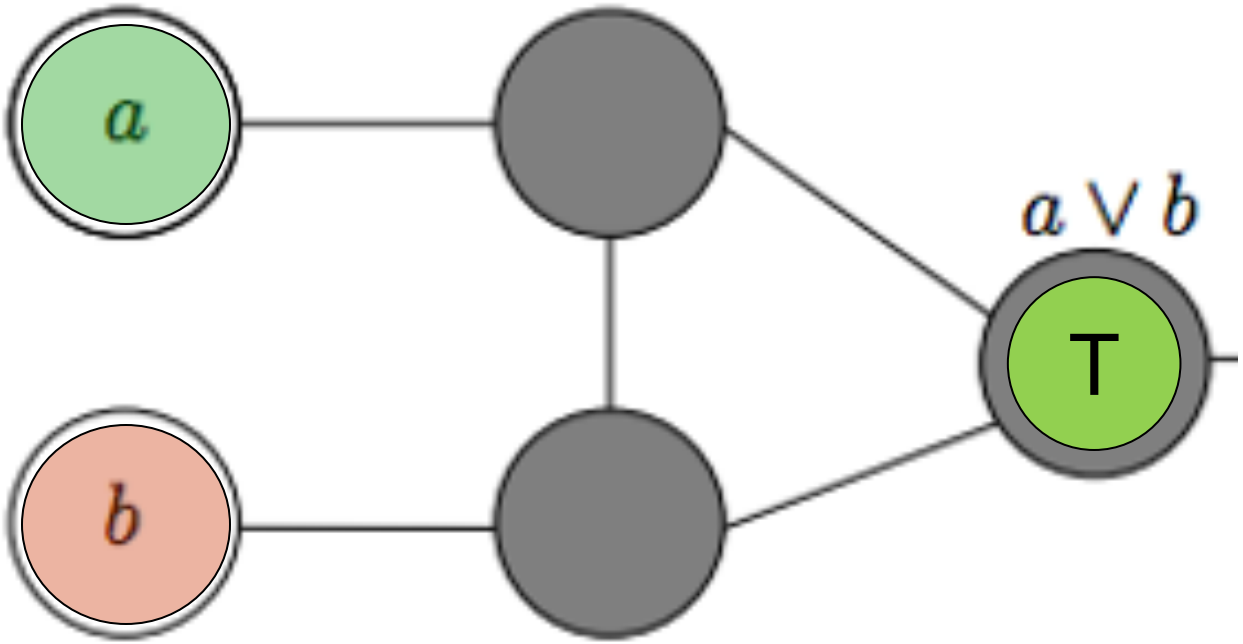
What if a, b?



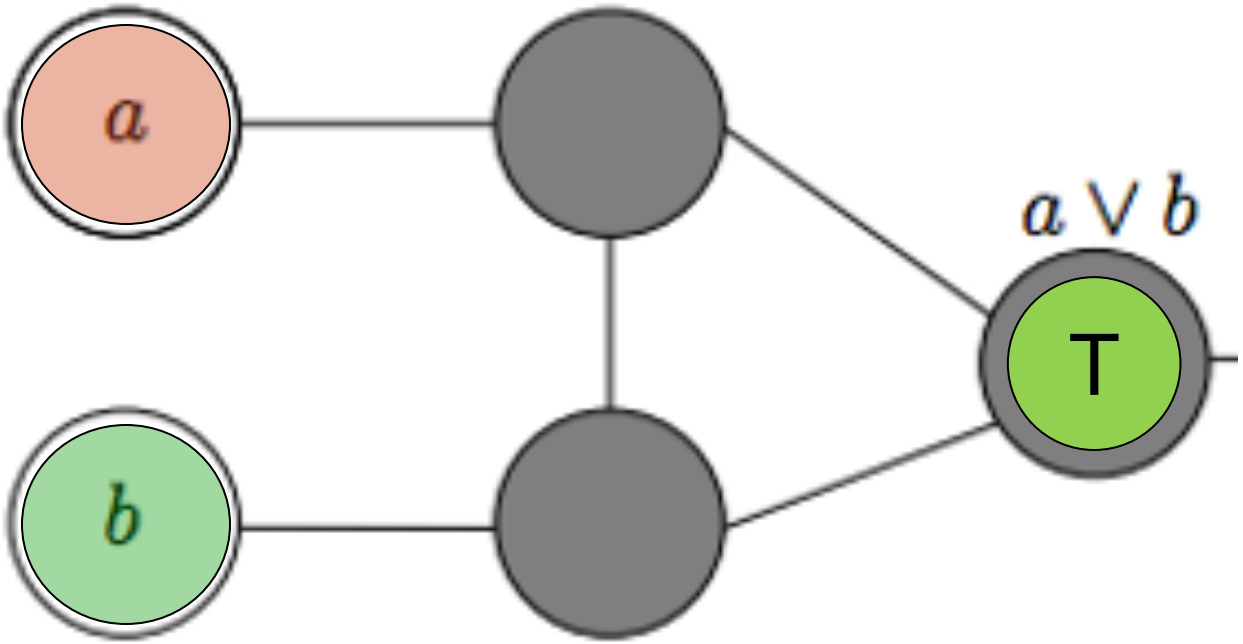
What if ~a, ~b?



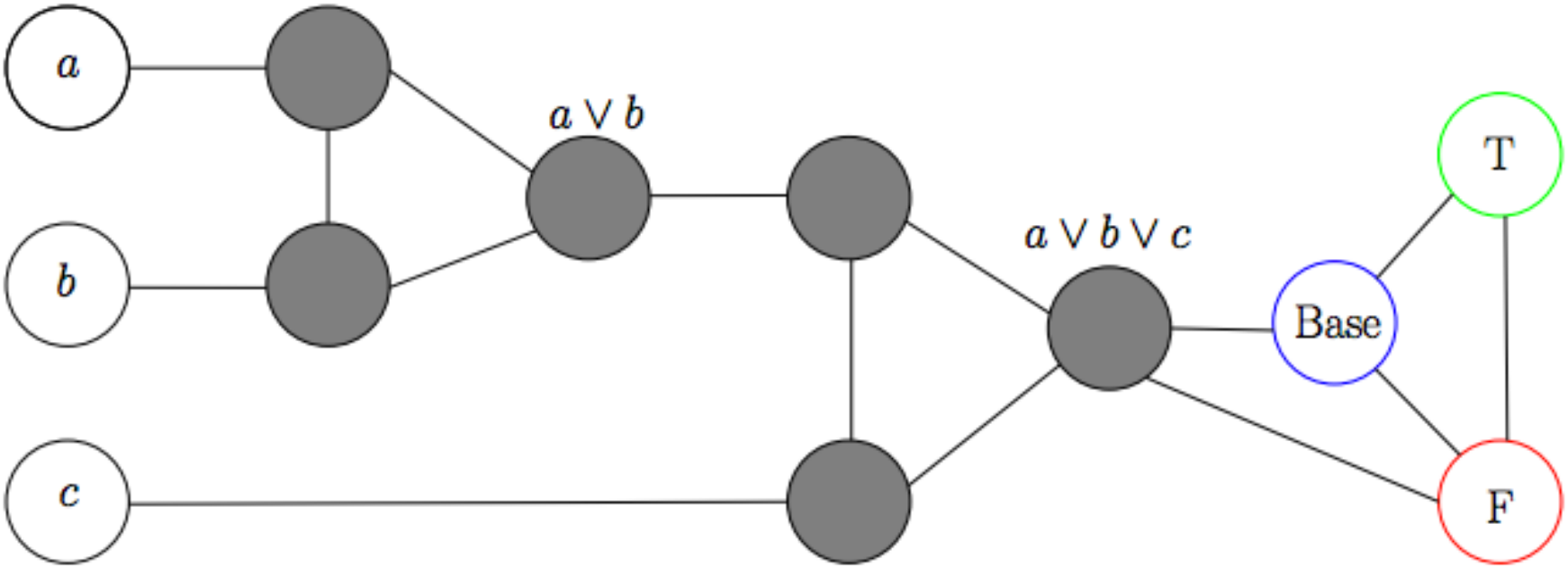
What if a, ~b?



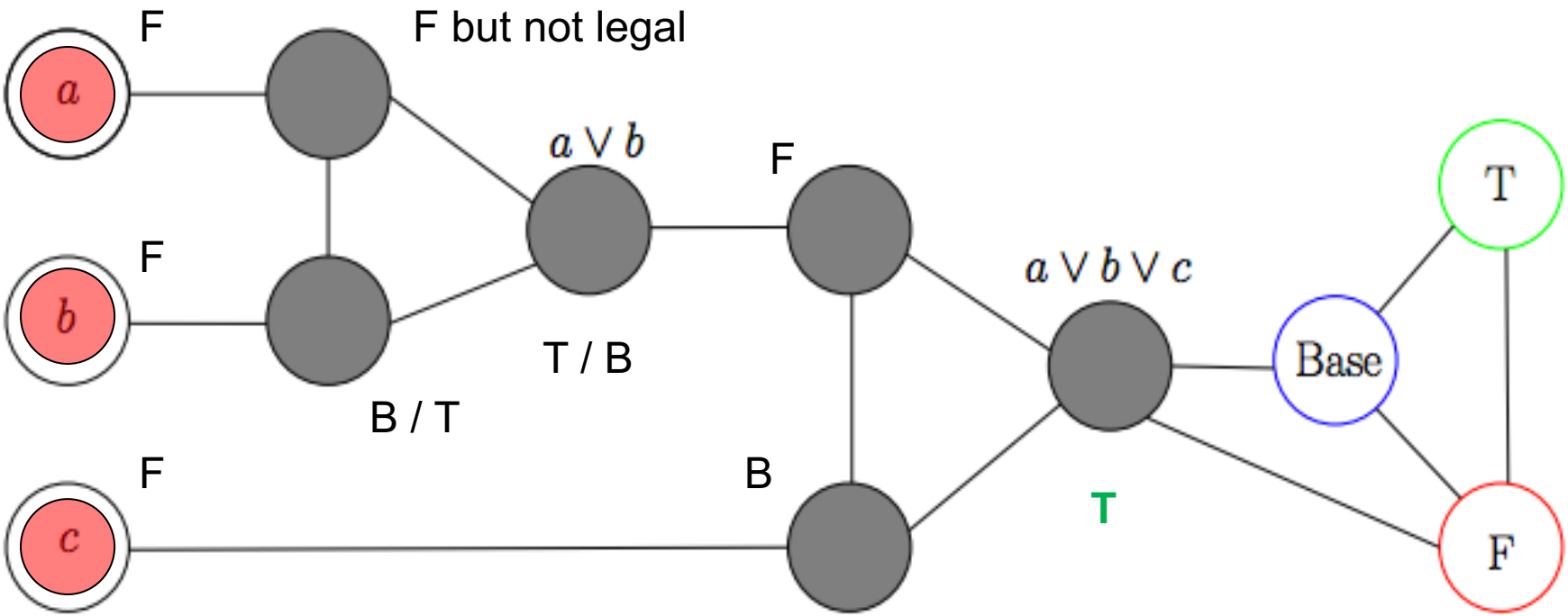
What if ~a, b?



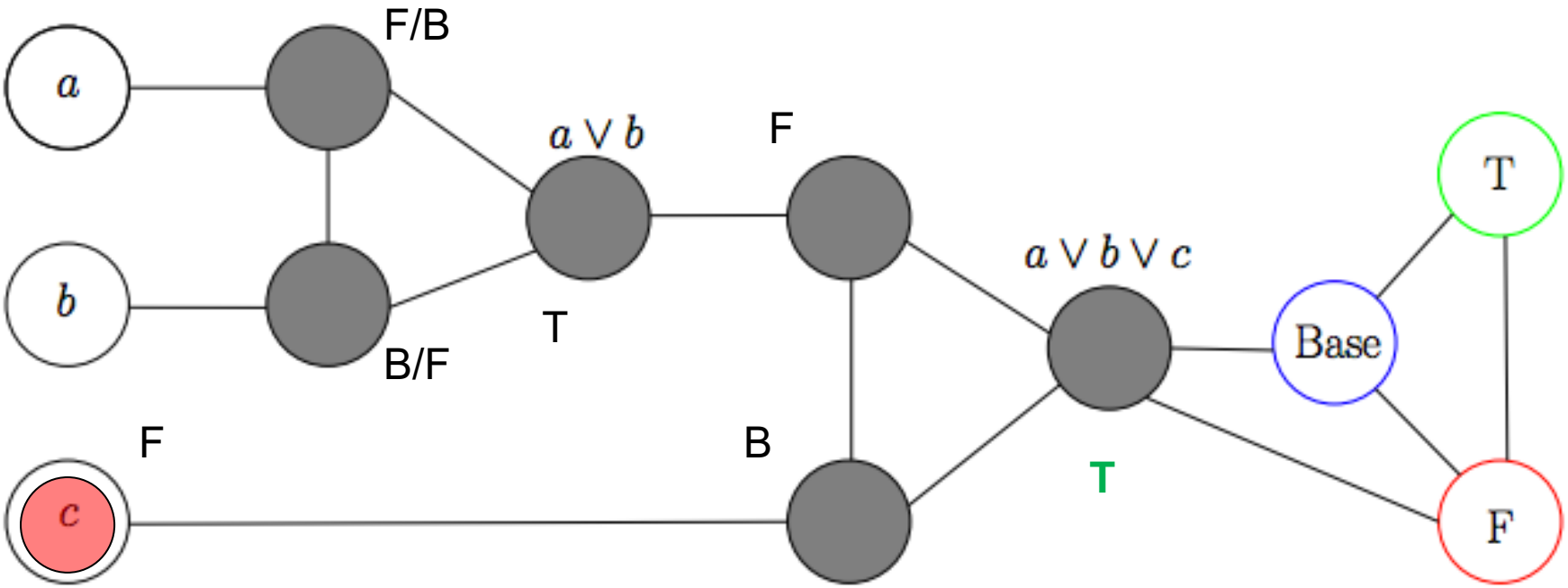
3C Clause Gadget



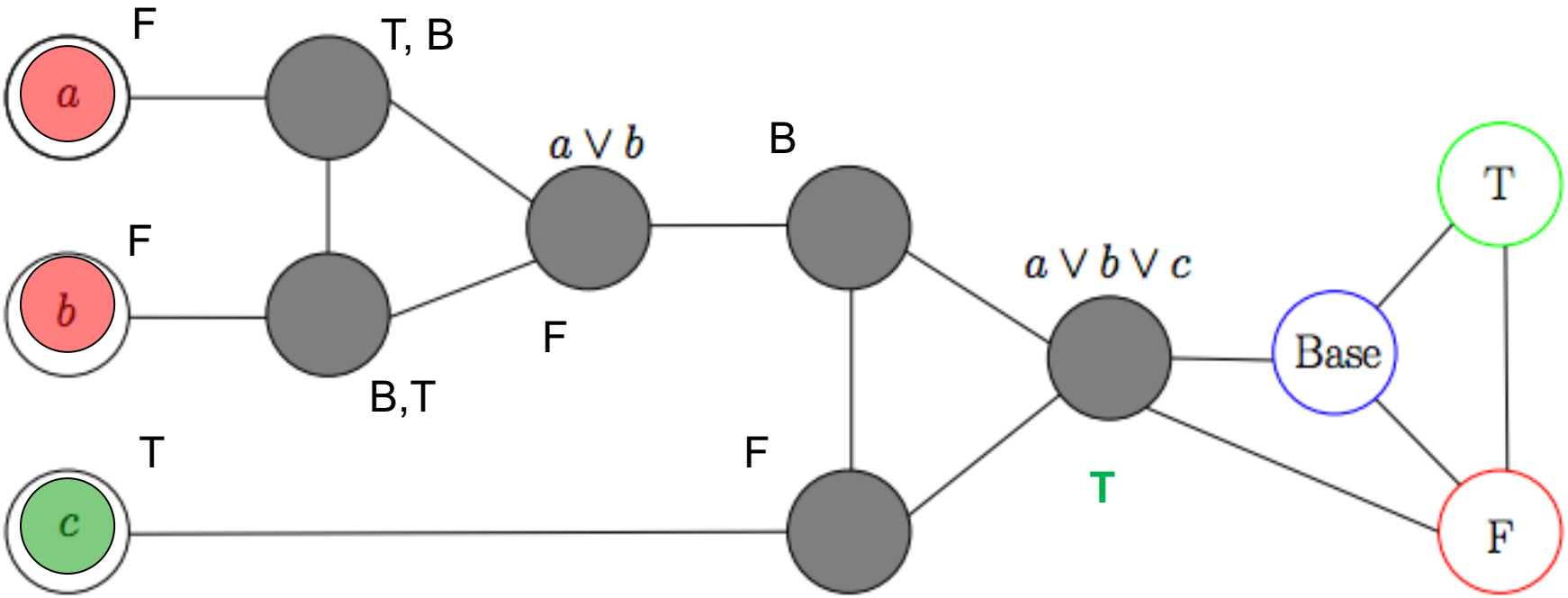
Consider ~a, ~b, ~c



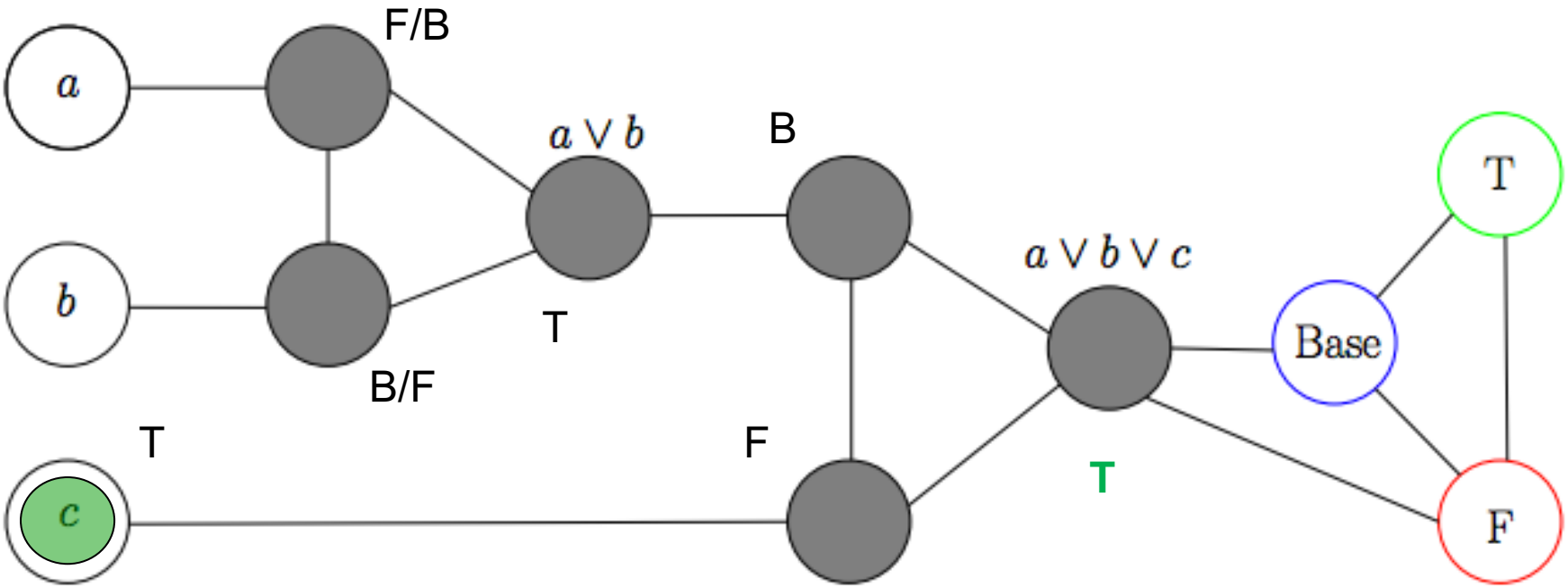
Consider a || b, ~c



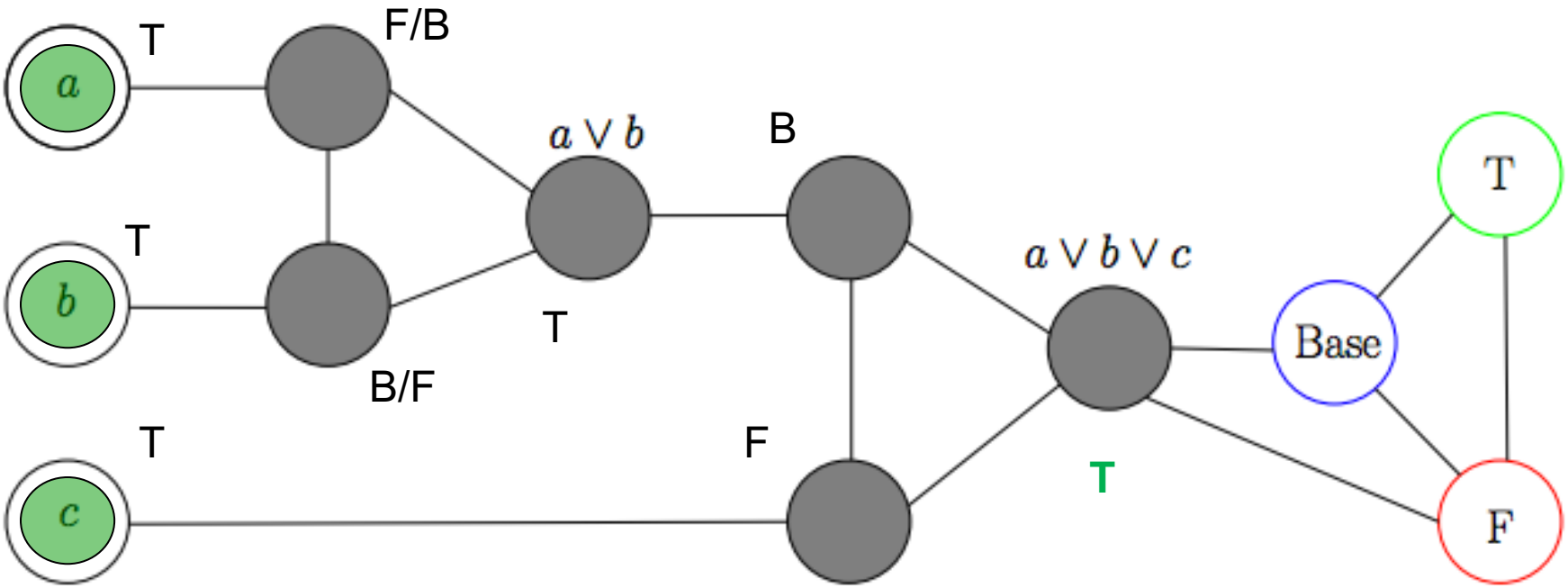
Consider ~a, ~b, c



Consider one of a || b, c

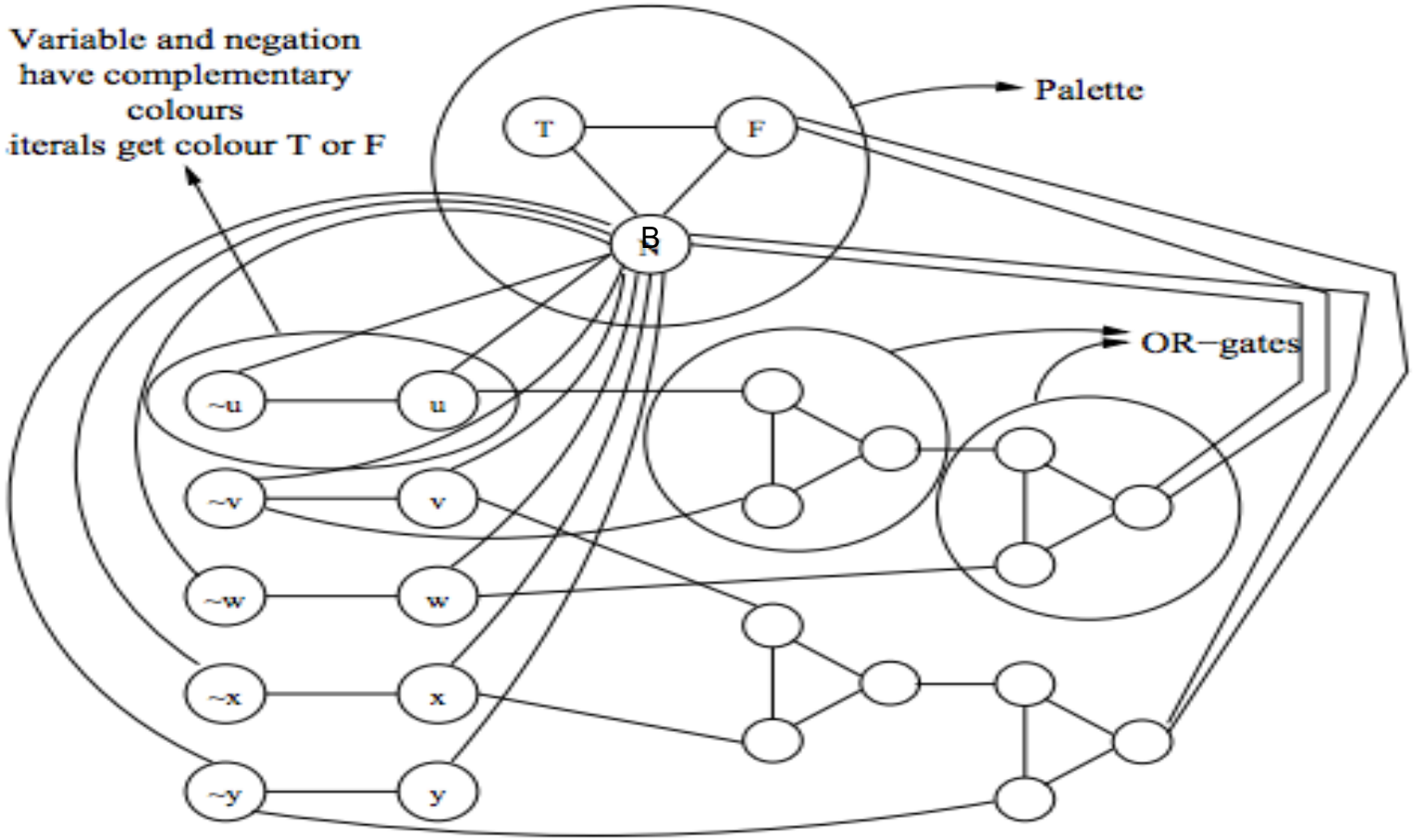


Consider a, b, c



KC Gadgets Combined

(u + ~v + w) (v + x + ~y)



K = 3

Register Allocation

- **Liveness: A variable is live if its current assignment may be used at some future point in a program's flow**
- **Optimizers often try to keep live variables in registers**
- **If two variables are simultaneously live, they need to be kept in separate registers**
- **Consider the K-coloring problem (can the nodes of a graph be colored with at most K colors under the constraint that adjacent nodes must have different colors?)**
- **Register Allocation reduces to K-coloring by mapping each variable to a node and inserting an edge between variables that are simultaneously live**
- **K-coloring reduces to Register Allocation by interpreting nodes as variables and edges as indicating concurrent liveness**
- **This is a simple mapping because it's an isomorphism**

Live Variable Analysis

Code

a = 1;

b = 2;

c = a * b;

d = a + 3;

e = a + b;

No Optimzation

T1 = 1

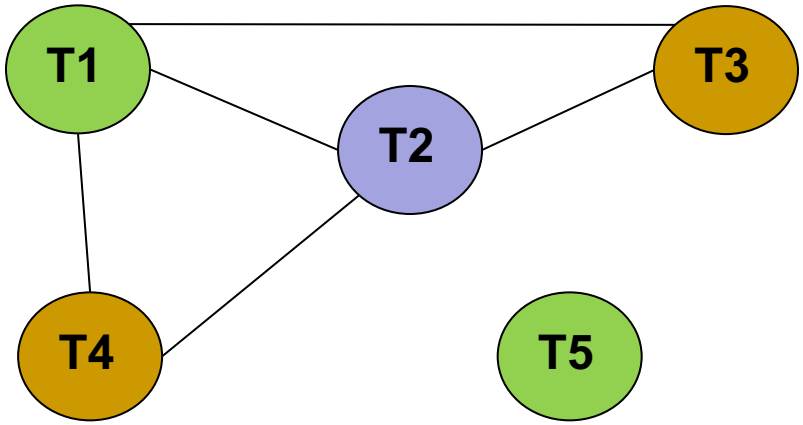
T2 = 2

T3 = T1 * T2

T4 = T1 + 3

T5 = T1 + T2

Minimum colors are 3 so need 3 registers to spilling to memory and reloading



PROCESSOR SCHEDULING IS NP-HARD

Processor Scheduling

- A Process Scheduling Problem can be described by
  - **m processors P_1, P_2, …, P_m,**
  - **processor timing functions S_1, S_2, …, S_m, each describing how the corresponding processor responds to an execution profile,**
  - **additional resources R_1, R_2, …, R_k, e.g., memory**
  - **transmission cost matrix C_{ij} ($1 \leq i, j \leq m$), based on proc. data sharing,**
  - **tasks to be executed T_1, T_2, …, T_n,**
  - **task execution profiles A_1, A_2, …, A_n,**
  - **a partial order defined on the tasks such that $T_i < T_j$ means that T_i must complete before T_j can start execution,**
  - **communication matrix D_{ij} ($1 \leq i, j \leq n$); D_{ij} can be non-zero only if $T_i < T_j$,**
  - **weights W_1, W_2, …, W_n -- cost of deferring execution of task.**

Complexity Overview

- **The intent of a scheduling algorithm is to minimize the sum of the weighted completion times of all tasks, while obeying the constraints of the task system. Weights can be made large to impose deadlines.**
- **The general scheduling problem is quite complex, but even simpler instances, where the processors are uniform, there are no additional resources, there is no data transmission, the execution profile is just processor time and the weights are uniform, are very hard.**
- **In fact, if we just specify the time to complete each task and we have no partial ordering, then finding an optimal schedule on two processors is an NP-complete problem. It is essentially the optimization version of the Partition or equally can be viewed as a SubsetSum problem.**

2 Processor Scheduling

The problem of optimally scheduling n tasks T_1, T_2, …, T_n onto 2 processors with an empty partial order < is the same as that of dividing a set of positive whole numbers into two subsets, such that the numbers are as close to evenly divided. So, for example, given the numbers

3, 2, 4, 1

we could try a "greedy" approach as follows:

put 3 in set 1

put 2 in set 2

put 4 in set 2 (total is now 6)

put 1 in set 1 (total is now 4)

This is not the best solution. A better option is to put 3 and 2 in one set and 4 and 1 in the other. Such a solution would have been attained if we did a greedy solution on a sorted version of the original numbers. In general, however, sorting doesn't work.

2 Processor Nastiness

Try the unsorted list (2-1/m)

7, 7, 6, 6, 5, 4, 4, 5, 4

Greedy (Always in one that is least used)

7, 6, 5, 5 = 23

7, 6, 4, 4, 4 = 25

Optimal

7, 6, 6, 5 = 24

7, 4, 4, 4, 5 = 24

Sort it (non-increasing) (4/3-1/3m)

7, 7, 6, 6, 5, 5, 4, 4, 4

7, 6, 5, 4, 4 = 26

7, 6, 5, 4 = 22

Sort it (non-decreasing) (2-1/m)

4, 4, 4, 5, 5, 6, 6, 7, 7

4, 4, 5, 6, 7 = 26

4, 5, 6, 7 = 22

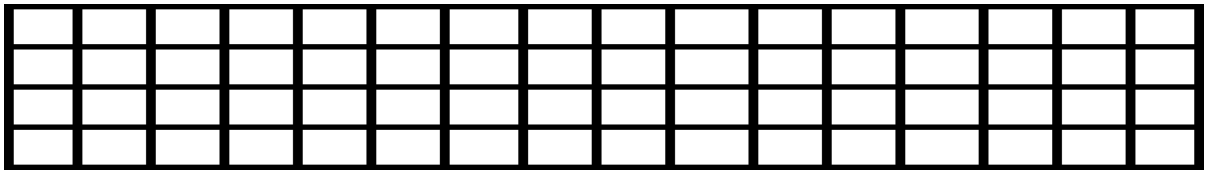
Both sorts are even worse than greedy unsorted !! (not a general result)

Challenge Problem

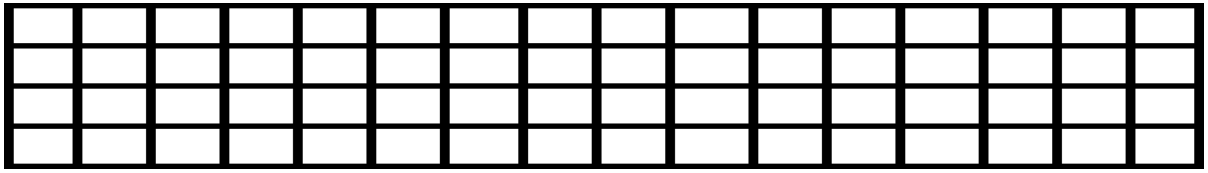
Consider the simple scheduling problem where we have a set of independent tasks running on a fixed number of processors, and we wish to minimize finishing time.

How would a list (first fit, no preemption) strategy schedule tasks with the following IDs and execution times onto four processors? Answer using Gantt chart.

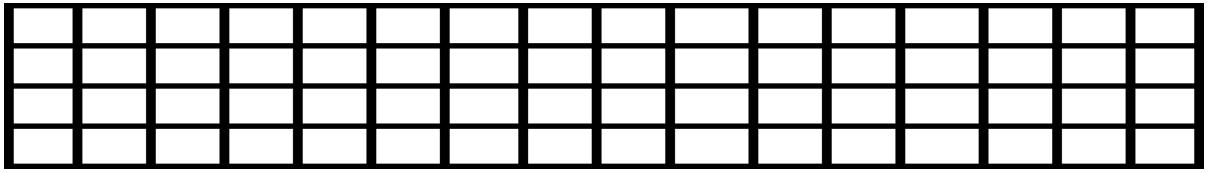
(T1,4) (T2,1) (T3,3) (T4,6) (T5,2) (T6,1) (T7,4) (T8,5) (T9,7) (T10,3) (T11,4) **(2-1/m)**



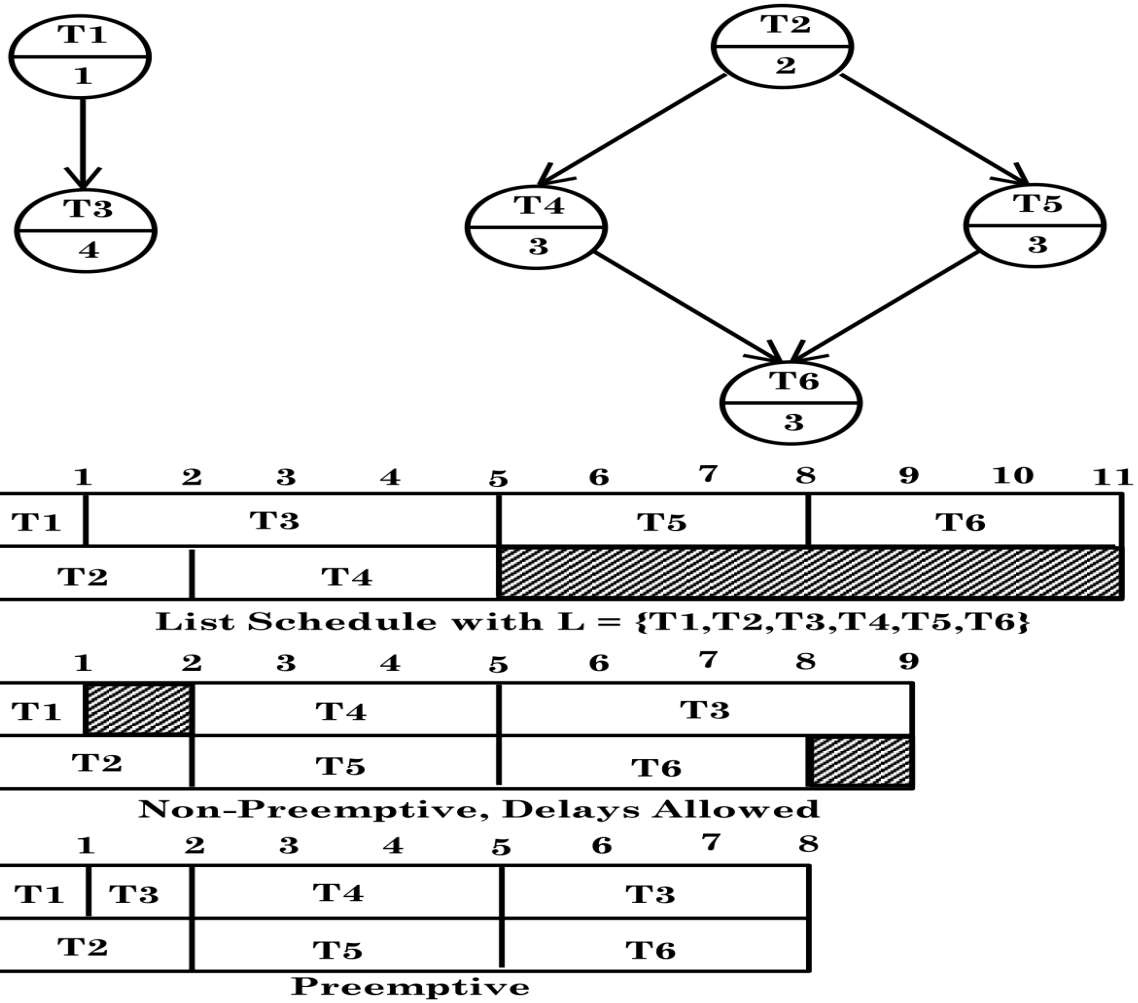
Now show what would happen if the times were sorted non-decreasing. **(2-1/m)**



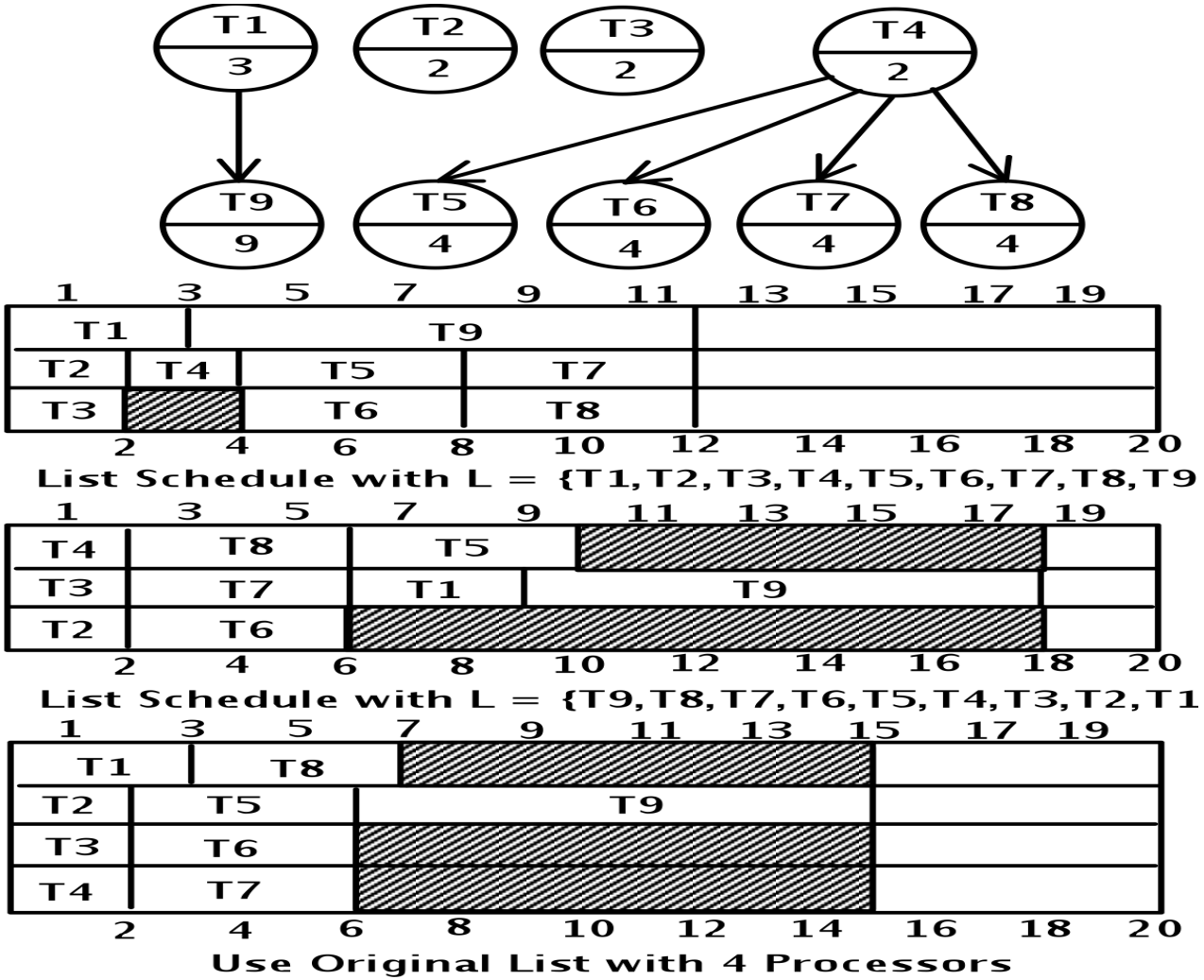
Now show what would happen if the times were sorted non-increasing. **(4/3-1/3m)**



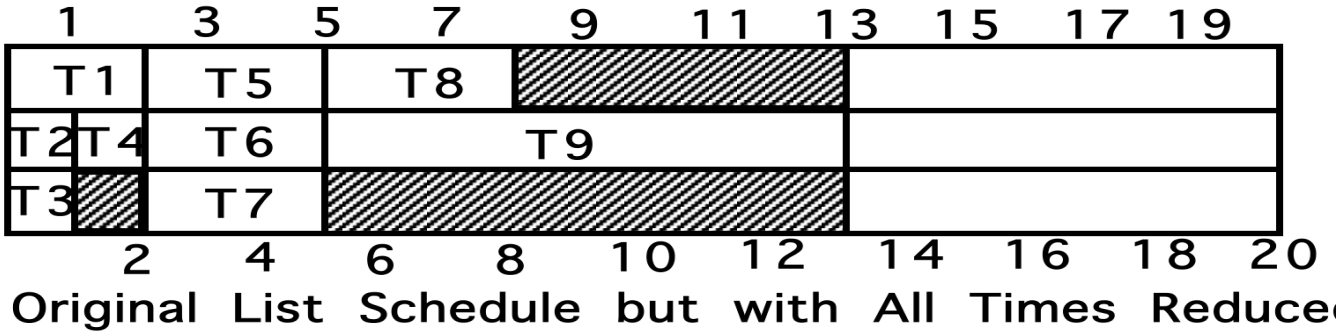
2 Processor with partial order

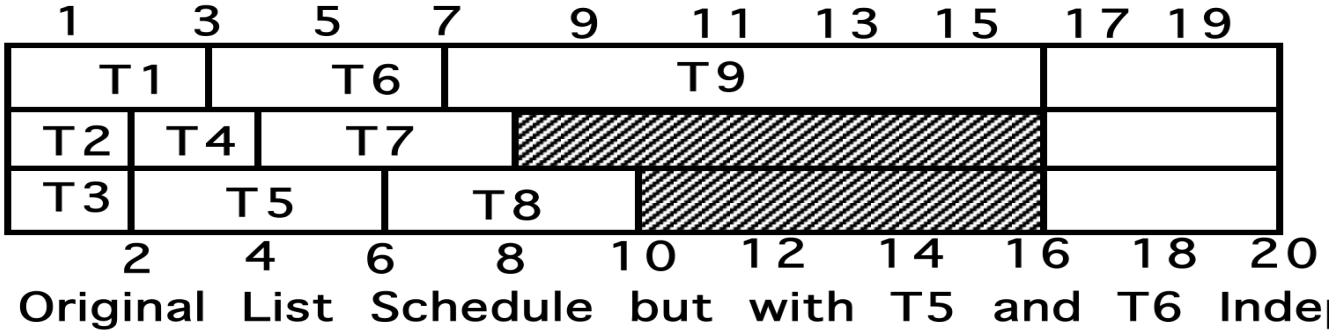


Anomalies everywhere



More anomalies





Critical path or level strategy

A UET is a Unit Execution Tree. Our Tree is funny. It has a single leaf by standard graph definitions.

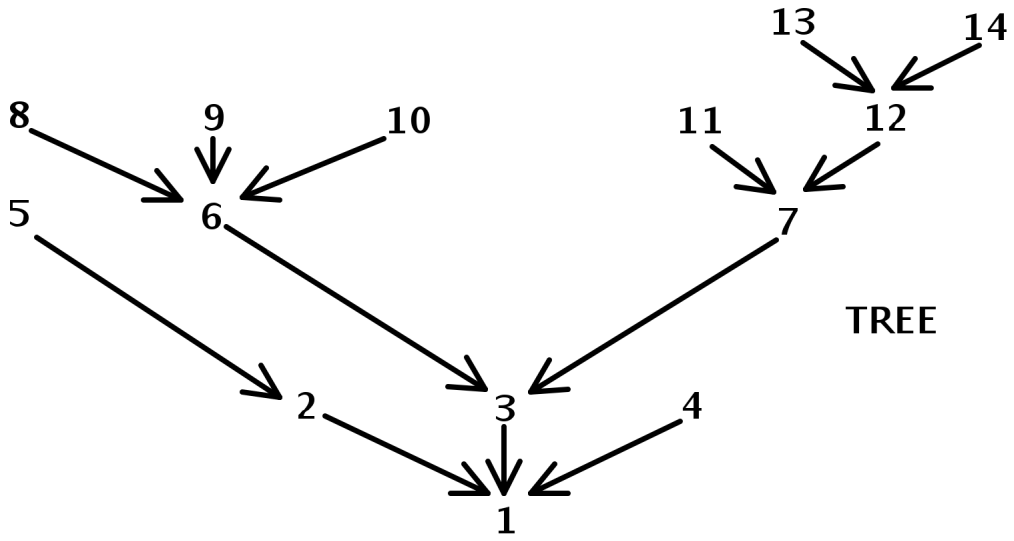
1. Assign L(T) = 1, for the leaf task T
2. Let labels 1, …, k-1 be assigned. If T is a task with lowest numbered immediate successor then define L(T) = k (non-deterministic)

   This is an order n labeling algorithm that can easily be implemented using a breadth first search.

Note: This can be used for a forest as well as a tree. Just add a new leaf. Connect all the old leafs to be immediate parents of the new one. Use the above to get priorities, starting at 0, rather than 1. Then delete the new node completely.

Note: This whole thing can also be used for anti-trees. Make a schedule, read it backwards. You cannot just reverse priorities.

Level strategy and UET



14	12	8	6	3	1
13	10	7	4		
11	9	5	2		

M=3

Theorem: Level Strategy is optimal for unit execution, m arbitrary, forest precedence

Level – DAG with unit time

1. Assign L(T) = 1, for an arbitrary leaf task T
2. Let labels 1, …, k-1 be assigned. For each task T such that

   { L(T') is defined for all T' in Successor(T) }

   Let N(T) be decreasing sequence of set members in
   {S(T') | T' is in S(T)}

   Choose T* with least N(T*).
   Define L(T*) = K.

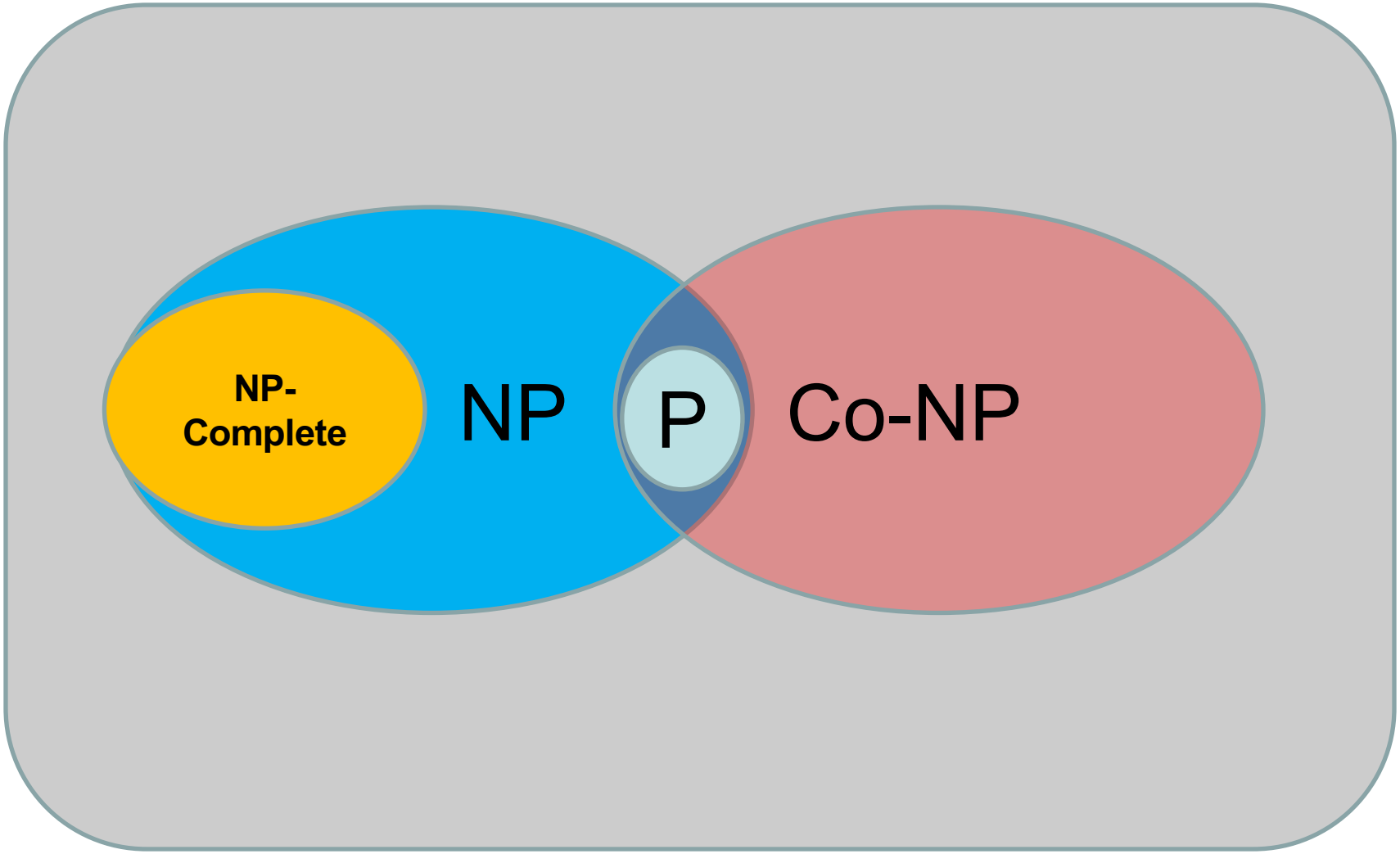
   This is an order n^2 labeling algorithm. Scheduling with it involves n union / find style operations. Such operations have been shown to be implementable in nearly constant time using an "amortization" algorithm.

Theorem: Level Strategy is optimal for unit execution, m=2, dag precedence.

Thought Experiment

Looking back at the UET example, consider adding two additional tasks numbered 15 and 16 that are siblings of 13 and 14. These four tasks must be completed before 12 is started.
a) Show the Gantt chart that reflects the new schedule associated with this enhanced tree
b) Show the Gantt chart that is associated with the corresponding anti-tree, in which all arcs are turned in the opposite direction. Use the technique of reversing the schedule from (a)
c) Show the Gantt chart associated with the anti-tree of b), where we now use the priorities obtained by treating lower numbered tasks as higher priority ones
d) Comment on the results seen in (b) versus (c), providing insight as to why they are different and why one is better than the other.

UNIVERSE OF SETS

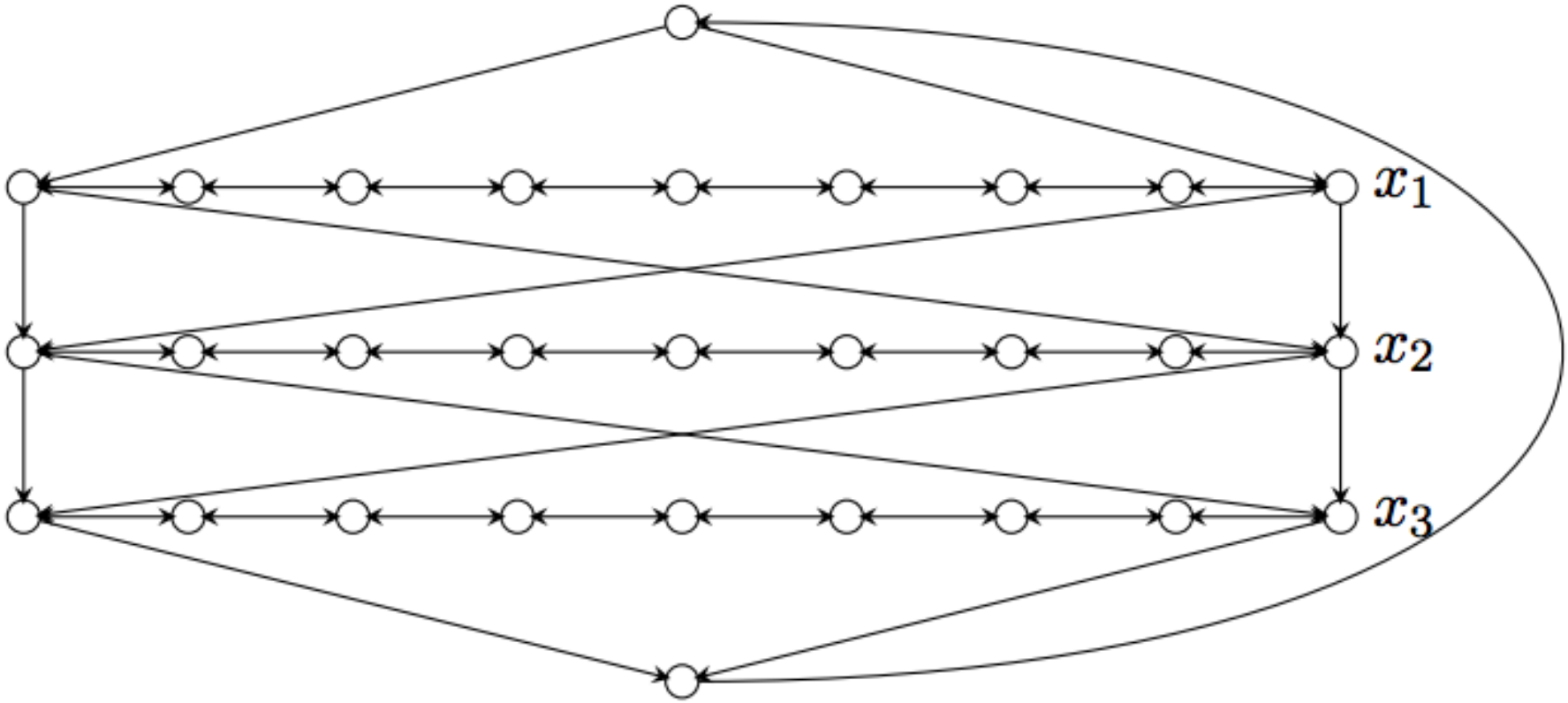


HAMILTONIAN CIRCUIT (HC) DECISION PROBLEM IS NP-HARD

Hamiltonian Path/Circuit

- A ***Hamiltonian Path*** is a path through a graph from one node ‘start’ to another node ‘end’ that visits every node in the graph just once.
- A ***Hamiltonian Circuit*** is a Hamiltonian Path whose end node is adjacent to its start node. It can also be viewed that the start and end nodes are the same and that the only repeated node is the start node with its only repetition being at the end of the path.

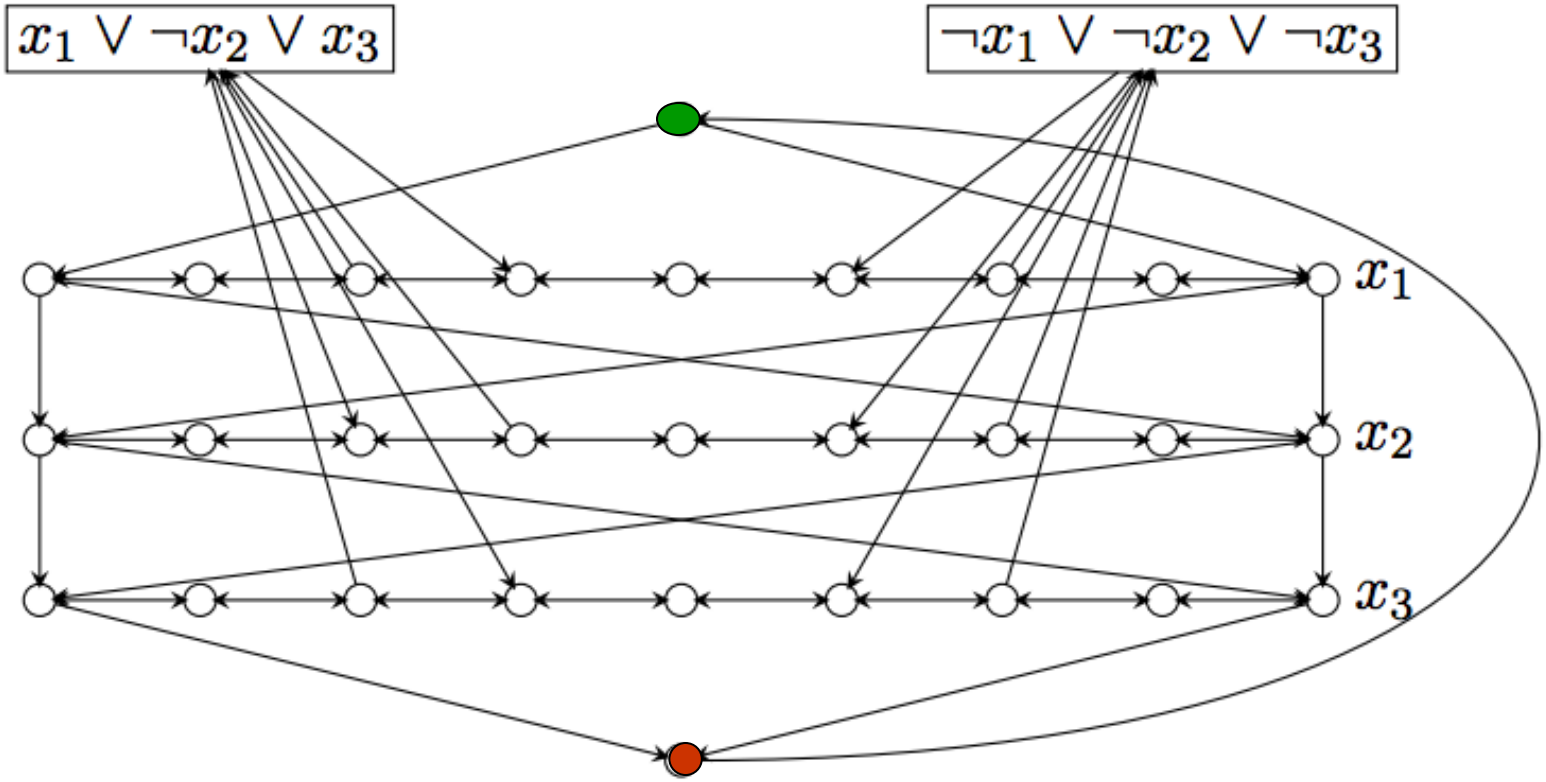
HC Variable Gadget



This has many Hamiltonian Circuits

HC Gadgets Combined

We will set convention on x_i true to be left to right and x_i false to be right to left (can fix for opposite)



This has a Hamiltonian Circuit iff all clauses are satisfied with consistent assignments to each variable. Note left to right assigns X_i as true; right to left assigns $\neg X_i$ as true. There are filler nodes on left and right and between clauses.

Hamiltonian Path

- Note we can split an arbitrary node, v, into two (v',v'') – one, v', has in-edges of v, other, v'', has out-edges. Path (not cycle) must start at v'' and end at v' and goal is still K (the number of vertices).

Travelling Salesman

- ***Travelling Salesman Problem:***
  Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city and returns to the origin city?
- This is a Hamiltonian Cycle with weights on edges and we seek ***minimum*** weight for cycle.
- The decision problem version involves setting a goal weight, ***L***, and asking is we can achieve it.

Travelling Salesman and HC

- Start with HC = (V,E), K=|V|
- Set edges from HC instance to 1
- Add edges between pairs that lack such edges and make those weights 2 (often people make these K+1); this means that the reverse of unidirectional links also get weight 2
- Goal weight is K for cycle

Knapsack 0-1 Problem

- The goal is to **maximize the value of a knapsack** that can hold at most W units (i.e. lbs or kg) worth of goods from a list of items $I_0, I_1, \ldots I_{n-1}$.
  - Each item has 2 attributes:
    1) Value – let this be v_i for item I_i
    2) Weight – let this be w_i for item I_i

Thanks to Arup Guha

Knapsack 0-1 Problem

- The difference between this problem and the fractional knapsack one is that you CANNOT take a fraction of an item.
  - You can either take it or leave it.
  - Hence the name Knapsack 0-1 problem.

Knapsack Optimize vs Decide

- The optimization problem is to have the sum of the chosen values, v_i, to be as large as possible with the constraint that the sum of the corresponding weights, w_i, cannot exceed W.
- We can restate as decision problem to determine if there exists a set of items, each with equal weights and values < W, that reaches some fixed goal value, W.

Knapsack and SubsetSum

- Let $v_i = w_i$ for each item I_i.
- By doing so, the value is maximized when the Knapsack is filled as close to capacity.
- The related decision problem is to determine if we can attain capacity (W).
- Clearly then, given an instance of the SubsetSum problem, we can create an instance of the Knapsack decision problem problem, such that we reach the goal sum, G, iff we can attain a Knapsack value of G.

Knapsack Decision Problem

- The reduction from SubsetSum shows that the Knapsack decision problem is at least as hard as SubsetSum, so it is NP-Complete if it is in NP.
- Think about whether or not it is in NP.
- Now, think about the optimization problem.

Related Bin Packing

- Have a bin capacity of B.
- Have item set **S = {s_1, s_2, …, s_n}**
- Use all items in **S**, minimizing the number of bins, while adhering to the constraint that any such subset must sum to **B** or less.
- This is similar to the processor scheduling problem without constraints, except we optimize on number of processors, not finishing time for all tasks. It is NP-Hard (WHY?)

Knapsack 0-1 Problem

- Brute Force
  - The naïve way to solve the 0-1 Knapsack problem is to cycle through all 2^n subsets of the n items and pick the subset with a legal weight that maximizes the value of the knapsack.
  - We can come up with a dynamic programming algorithm that is USUALLY faster than this brute force technique.

Knapsack 0-1 Problem

- We are going to solve the problem in terms of sub-problems and memoization (dynamic programming).
- Our first attempt might be to characterize a sub-problem as follows:
  - Let S_k be the optimal subset of elements from $\{I_0, I_1, \ldots, I_k\}$.
    - What we find is that the optimal subset from the elements $\{I_0, I_1, \ldots, I_{k+1}\}$ may not correspond to the optimal subset of elements from $\{I_0, I_1, \ldots, I_k\}$ in any regular pattern.
  - Basically, the solution to the optimization problem for S_{k+1} might NOT contain the optimal solution from problem S_k.

Knapsack 0-1 Problem

- Let's illustrate that point with an example:

Item	Weight	Value
I_0	3	10
I_1	8	4
I_2	9	9
I_3	8	11

- **The maximum weight the knapsack can hold is 20.**

- The best set of items from $\{I_0, I_1, I_2\}$ is $\{I_0, I_1, I_2\}$
- BUT the best set of items from $\{I_0, I_1, I_2, I_3\}$ is $\{I_0, I_2, I_3\}$.
  - In this example, note that this optimal solution, $\{I_0, I_2, I_3\}$, does NOT build upon the previous optimal solution, $\{I_0, I_1, I_2\}$.
    - (Instead it builds upon the solution, $\{I_0, I_2\}$, which is really the optimal subset of $\{I_0, I_1, I_2\}$ with weight 12 or less.)

Knapsack 0-1 problem

- So now we must re-work the way we build upon previous sub-problems…
  - Let **B[k, w]** represent the maximum total value of a subset S_k with weight w.
  - Our goal is to find **B[n, W],** where n is the total number of items and W is the maximal weight the knapsack can carry.

- So our recursive formula for subproblems:

$$B[k, w] = B[k-1, w], \text{ if } w_k > w$$
$$= \max\{ B[k-1, w], B[k-1, w-w_k] + v_k\}, \text{ otherwise}$$

- In English, this means that the best subset of S_k that has total weight w is:
  1) The best subset of S_{k-1} that has total weight w, or
  2) The best subset of S_{k-1} that has total weight $w-w_k$ plus the item k

Knapsack 0-1 Problem – Recursive Formula

$$B[k, w] = B[k-1, w], \quad \text{if } w_k > w$$
$$= \max\{B[k-1, w], B[k-1, w-w_k] + v_k\}, \quad \text{otherwise}$$

- The best subset of S_k that has the total weight w, either contains item k or not.

- **First case:** $w_k > w$
  - Item *k* can't be part of the solution! If it was the total weight would be > w, which is unacceptable.

- **Second case:** $w_k \leq w$
  - Then the item *k* can be in the solution, and we choose the case with greater value.

Knapsack 0-1 Algorithm

```
for w = 0 to W          // Initialize 1st row to 0's
   B[0,w] = 0
for i = 1 to n          // Initialize 1st column to 0's
   B[i,0] = 0
for i = 1 to n
   for w = 1 to W
        if w_i <= w     //item i can be in the solution
              if v_i + B[i-1,w-w_i] > B[i-1,w]
                    B[i,w] = v_i + B[i-1,w- w_i]
              else
                    B[i,w] = B[i-1,w]
        else B[i,w] = B[i-1,w] // w_i > w
}
```

Knapsack 0-1 Problem

- Let's run our algorithm on the following data:
  - n = 4 (# of elements)
  - W = 5 (max weight)
  - Elements (weight, value):
    (2,3), (3,4), (4,5), (5,6)

Knapsack 0-1 Example

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0					
2	0					
3	0					
4	0					

// Initialize the base cases

for w = 0 to W

B[0,w] = 0

for i = 1 to n

B[i,0] = 0

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0				
2	0					
3	0					
4	0					

$i = 1$
$v_i = 3$
$w_i = 2$
$w = 1$
$w\text{-}w_i = -1$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else **B[i,w] = B[i-1,w]** // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3			
2	0					
3	0					
4	0					

$i = 1$
$v_i = 3$
$w_i = 2$
w = 2
$w\text{-}w_i = 0$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else $B[i,w] = B[i\text{-}1,w]$ // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3		
2	0					
3	0					
4	0					

$i = 1$
$v_i = 3$
$w_i = 2$
w = 3
$w\text{-}w_i = 1$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else $B[i,w] = B[i\text{-}1,w]$ // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	
2	0					
3	0					
4	0					

$i = 1$
$v_i = 3$
$w_i = 2$
w = 4
$w\text{-}w_i = 2$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else $B[i,w] = B[i\text{-}1,w]$ // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0					
3	0					
4	0					

$i = 1$
$v_i = 3$
$w_i = 2$
w = 5
$w\text{-}w_i = 3$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else $B[i,w] = B[i\text{-}1,w]$ // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0				
3	0					
4	0					

$i = 2$
$v_i = 4$
$w_i = 3$
w = 1
$w\text{-}w_i = -2$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else $B[i,w] = B[i\text{-}1,w]$ // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3			
3	0					
4	0					

$i = 2$
$v_i = 4$
$w_i = 3$
w = 2
$w\text{-}w_i = -1$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else **B[i,w] = B[i-1,w]** // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4		
3	0					
4	0					

$i = 2$
$v_i = 4$
$w_i = 3$
w = 3
$w\text{-}w_i = 0$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else $B[i,w] = B[i\text{-}1,w]$ // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	
3	0					
4	0					

$i = 2$
$v_i = 4$
$w_i = 3$
w = 4
$w\text{-}w_i = 1$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else $B[i,w] = B[i\text{-}1,w]$ // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0					
4	0					

$i = 2$
$v_i = 4$
$w_i = 3$
w = 5
$w\text{-}w_i = 2$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else $B[i,w] = B[i\text{-}1,w]$ // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4		
4	0					

$i = 3$
$v_i = 5$
$w_i = 4$
w = 1..3
$w\text{-}w_i = -3..-1$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else **B[i,w] = B[i-1,w]** // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	
4	0					

$i = 3$
$v_i = 5$
$w_i = 4$
w = 4
$w\text{-}w_i = 0$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else $B[i,w] = B[i\text{-}1,w]$ // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0					

$i = 3$
$v_i = 5$
$w_i = 4$
w = 5
$w\text{-}w_i = 1$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        B[i,w] = B[i-1,w]
else $B[i,w] = B[i\text{-}1,w]$ // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	

$i = 4$
$v_i = 6$
$w_i = 5$
w = 1..4
$w\text{-}w_i = -4..-1$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        $B[i,w] = B[i\text{-}1,w]$
else **B[i,w] = B[i-1,w]** // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	7

$i = 4$
$v_i = 6$
$w_i = 5$
w = 5
$w\text{-}w_i = 0$

if $w_i <= w$ //item i can be in the solution
    if $v_i + B[i\text{-}1,w\text{-}w_i] > B[i\text{-}1,w]$
        $B[i,w] = v_i + B[i\text{-}1,w\text{-}w_i]$
    else
        B[i,w] = B[i-1,w]
else B[i,w] = B[i-1,w] // $w_i > w$

Knapsack 0-1 Example

Items:
1: (2,3)
2: (3,4)
3: (4,5)
4: (5,6)

i / w	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	3	3	3	3
2	0	0	3	4	4	7
3	0	0	3	4	5	7
4	0	0	3	4	5	**7**

We're DONE!!

The max possible value that can be carried in this knapsack is ***$7***

Knapsack 0-1 Problem – Run Time

```
for w = 0 to W
    B[0,w] = 0
```
O(*W*)

```
for i = 1 to n
    B[i,0] = 0
```
O(*n*)

```
for i = 1 to n
    for w = 0 to W
        < the rest of the code >
```
Repeat *n* times

O(*W*)

What is the running time of this algorithm?

O(*n*W*) – *of course, W can be mighty big*

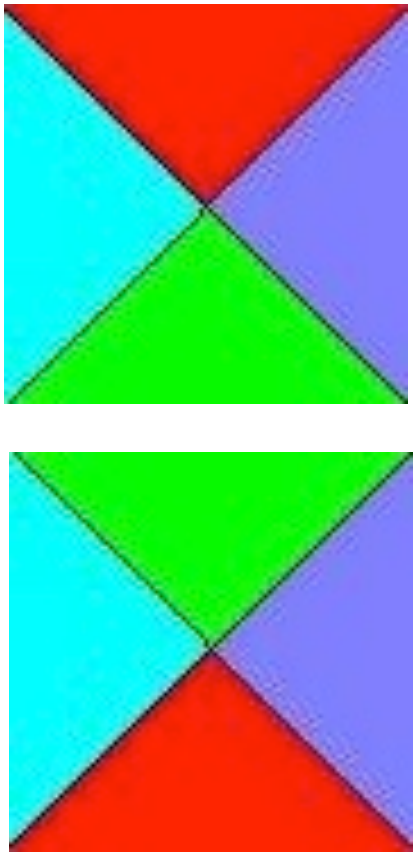
What is an analogy in world of sorting?

Remember that the brute-force algorithm takes: **O(2^n)**

Tiling

Undecidable and NP-Complete Variants

Basic Idea of Tiling

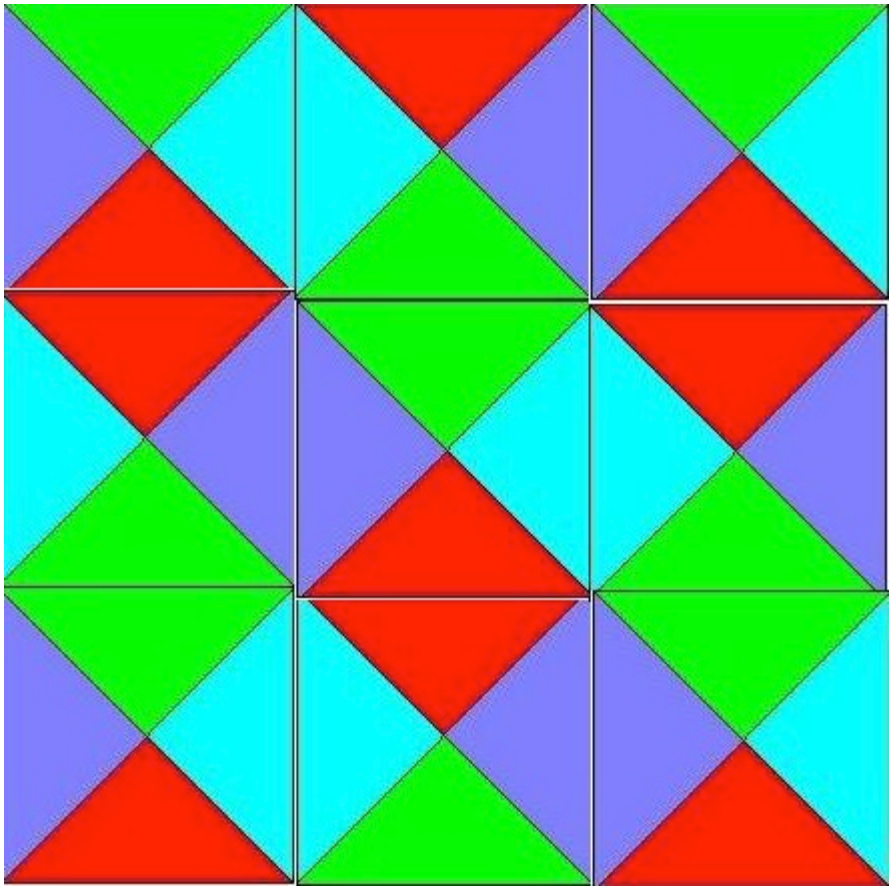


A single tile has colors on all four sides. Tiles are often called dominoes as assembling them follows the rules of placing dominoes. That is, the color (or number) of a side must match that of its adjacent tile, e.g., tile, t2, to right of a tile, t1, must have same color on Its left as is on the right side of t1. This constraint applies to top and bottom as well as sides. Boundary tiles do not have constraints on their sides that touch the boundaries.

Instance of Tiling Problem

- A finite set of tile types (a type is determined by the colors of its edges)
- Some 2d area (finite or infinite) on which the tiles are to be laid out
- An optional starting set of tiles in fixed positions
- The goal is to tile the plane following the adjacency constraints and whatever constraints are indicated by the starting configuration.

A Valid 3 by 3 Tiling of Tile Types from Previous Slide



Some Variations

- Infinite 2d plane (impossible, co-re-non-rec) in general
  - Our two tile types can easily tile the 2d plane
- Finite 2d plane (hard in general)
  - Our two tile types can easily tile any finite 2d plane
  - This is called the Bounded Tiling Problem

- One dimensional space
  - This is actually related to circuits in a directed graph (Why so? If tiles are vertices, what are edges?)

Tiling the Plane

- We will start with a Post Machine, $M = (Q, \Sigma, \delta, q_0)$, with tape alphabet $\Sigma = \{B,1\}$ where B is blank and δ maps pairs from $Q \times \Sigma$ to $Q \times (\Sigma \cup \{R,L\})$. M starts in state q_0
  - (Turing Machine with each action being L, R or Print)
- We will consider the case of M starting with a blank tape
- We will constrain our machine to never go to the left of its starting position (semi unbounded tape)
- We will mimic the computation steps of M
- Termination occurs if in state q reading b and $\delta(q,b)$ is not defined
- We will use the fact that halting when starting at the left end of a semi unbounded tape in its initial state with a blank tape is undecidable; we will actually look at complement of this

The Tiling Decision Problem

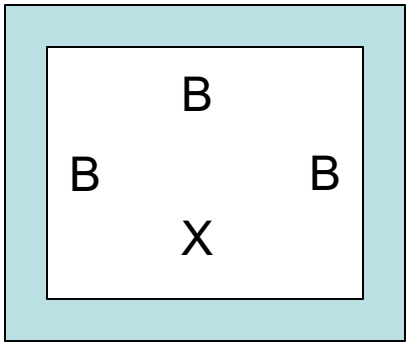
- Given a finite set of tile types and a starting tile in lower left corner of 2d plane, can we tile all places in the plane?
- A place is defined by its coordinates (x,y), $x \geq 0$, $y \geq 0$
- The fixed starting tile is at (0,0)

Colors

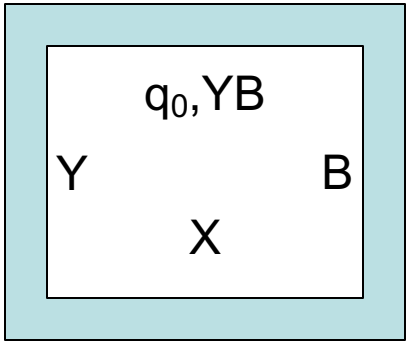
- Given M, define our tile colors as
- $\{X, Y, *, B, 1, YB, Y1\} \cup Q \times \{B,1\} \cup Q \times \{YB,Y1\} \cup Q \times \{R,L\}$
- X appears only on bottom of any and all tiles that are resting on the X-axis
- Y appears only on left of any and all tiles that are adjacent to the Y-axis
- Y is part of the label on top of any tile with its left side adjacent to the Y-axis

Simple Tiles

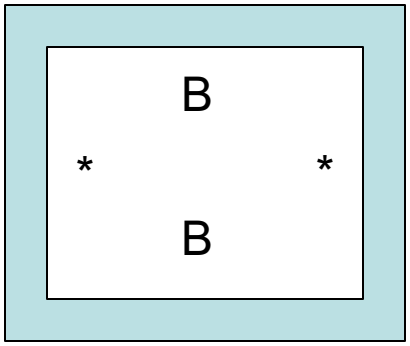
- Simplest tile (represents Blank on X axis)

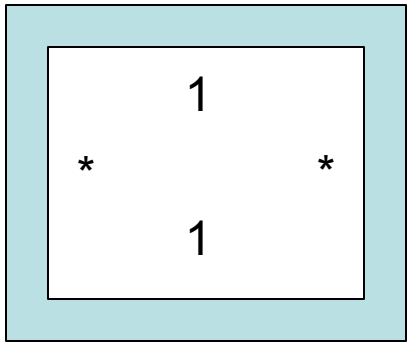


- Start tile (state q0; scanned symbol blank)

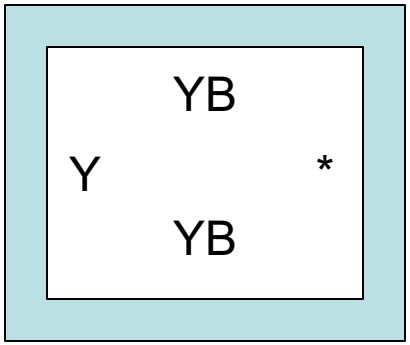


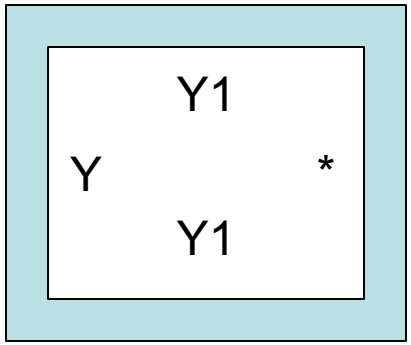
Tiles for Copying Tape Cell





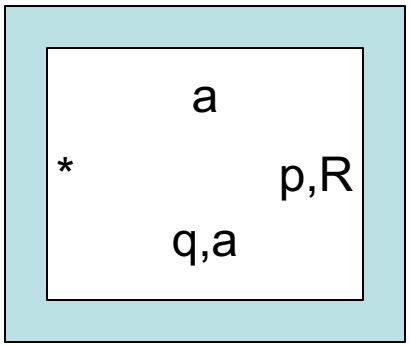
Copy cells not on left boundary and not scanned

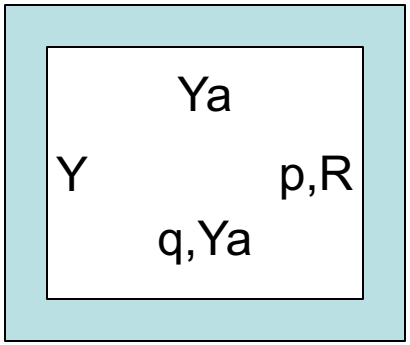


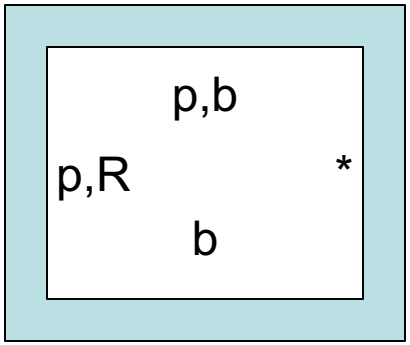


Copy cells on left boundary and not scanned

Right Move δ(q,a) = (p,R)

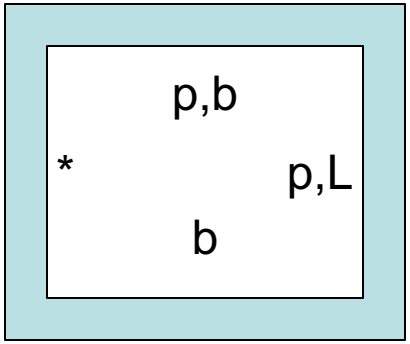


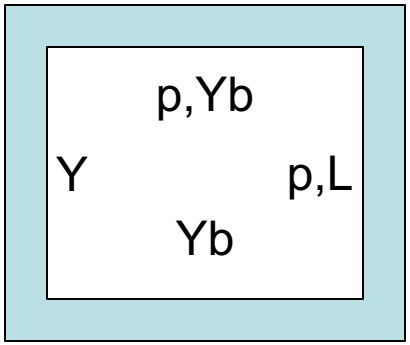


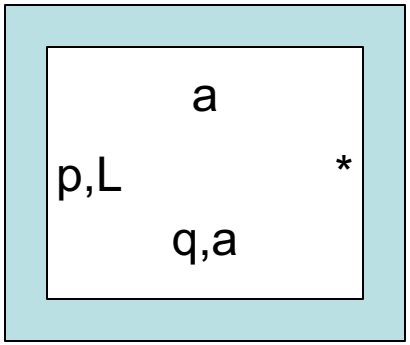


where $b \in \Sigma = \{B,1\}$

Left Move δ(q,a) = (p,L)

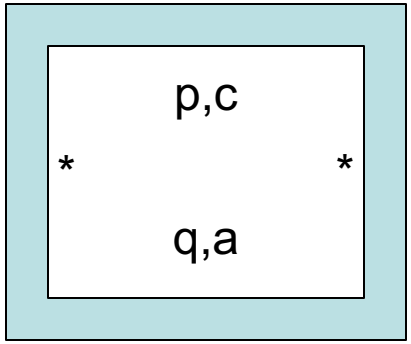


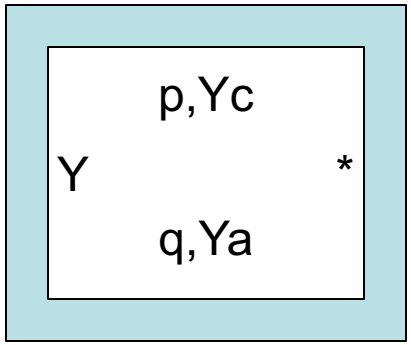




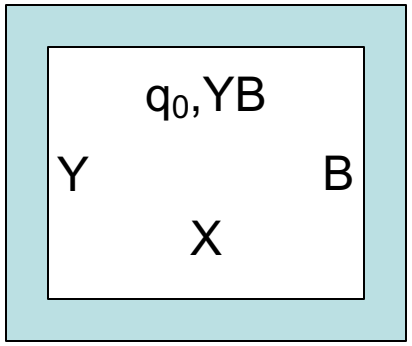
where $b \in \Sigma = \{B,1\}$

Print δ(q,a) = (p,c)

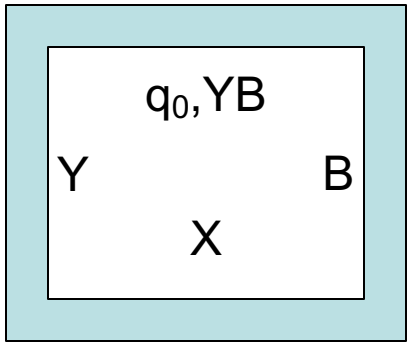


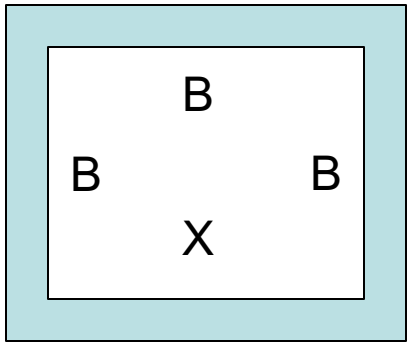


Corner Tile and Bottom Row

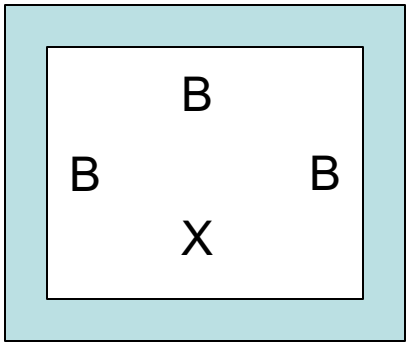


Zero-ed Row is forced to be



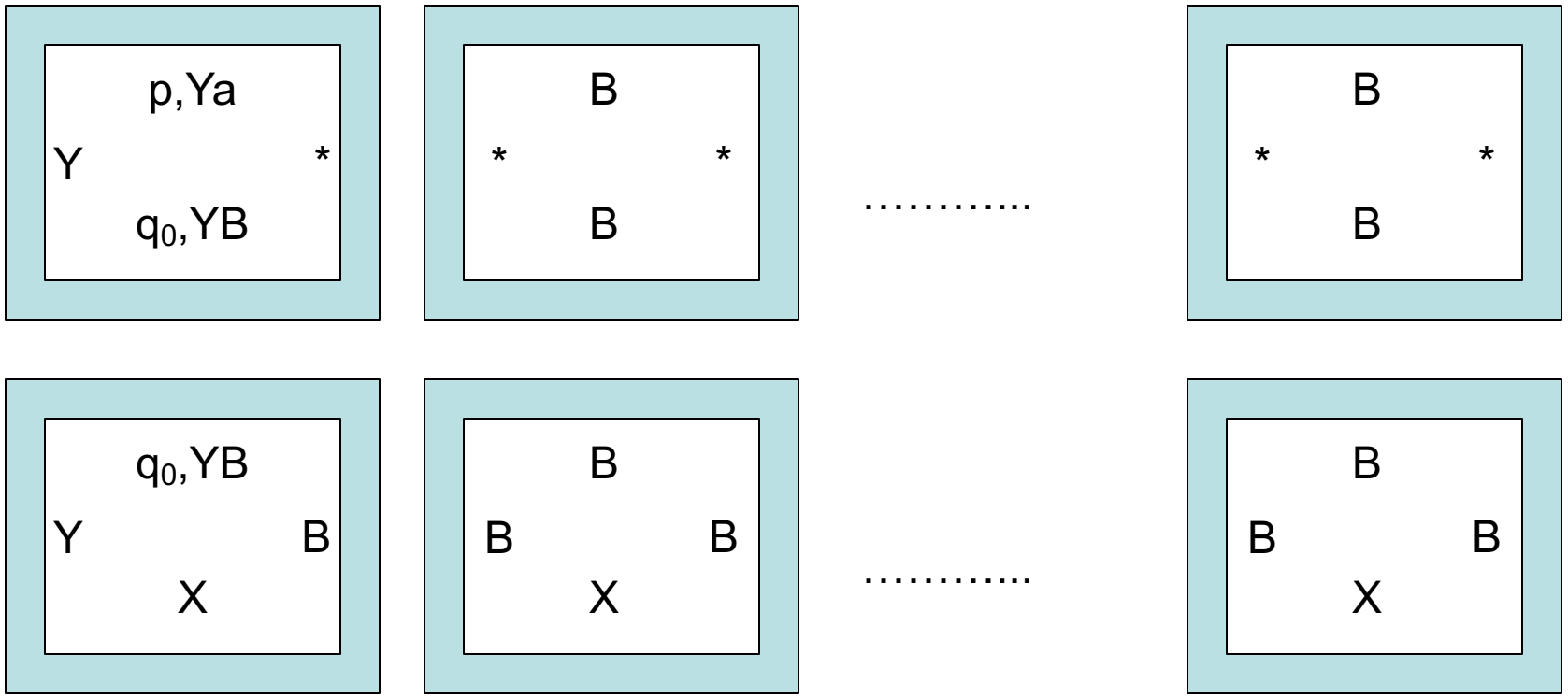


.....



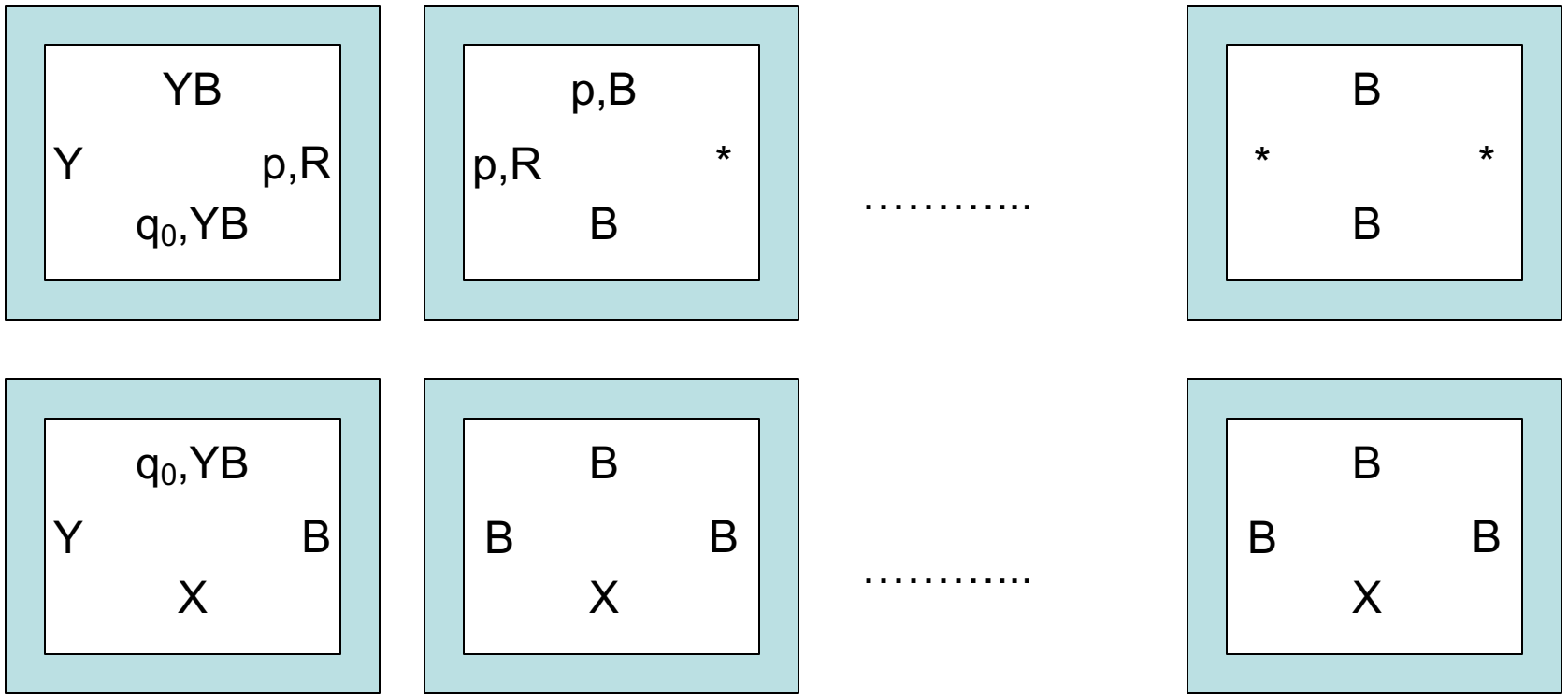
First Action Print

As we cannot move left of leftmost character first action is either right or print. Assume for now that $\delta(q_0,B) = (p,a)$



First Action Right Move

As we cannot move left of leftmost character first action is either right or print. Assume for now that $\delta(q_0,B) = (p,R)$



The Rest of the Story Part 1

- Inductively we can show that, if the i-th row represents an infinite transcription of the Turing configuration after step i then the (i+1)-st represents such a transcription after step i+1. Since we have shown the base case, we have a successful simulation.

The Rest of the Story Part 2

- Consider the case where M eventually halts when started on a blank tape in state q_0. In this case we will reach a point where no actions fill the slots above the one representing the current state. That means that we cannot tile the plane.
- If M never halts, then we can tile the plane (in the limit).

The Rest of the Story Part 3

- The consequences of Parts 1 and 2 are that Tiling the plane is as hard as the complement of the Halting problem ($\forall t [\sim STP(M, x, t)]$) or of accepts 0 ($\forall t [\sim STP(M, 0, t)]$) which are both co-RE Complete.
- This is not surprising as this problem involve a universal quantification over all coordinates (x,y) in the plane.

Constraints on M

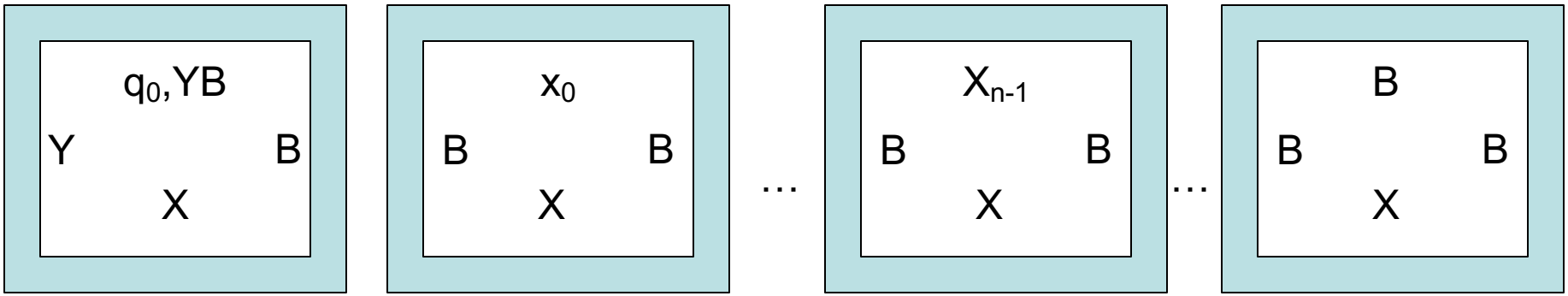
- The starting blank tape is not a real constraint as we can create M so its first actions are to write arguments on its tape.
- The semi unbounded tape is not new. If you look back at Standard Turing Computing (STC), we assumed there that we never moved left of the blank preceding our first argument.
- If you prefer to consider all computation based on the STC model then we can add to M the simple prologue $(R1)^{x1}R(R1)^{x2}R\ldots(R1)^{xk}R$ so the actual computation starts with a vector of x1 … xk on the tape and with the scanned square as the blank to right of this vector. The rest of the tape is blank.
- Think about how, in the preceding pages, you could actually start the tiling in this configuration.

Bounded Tiling Problem #1

- Consider a slight change to our machine M. First, it is non-deterministic, so our transition function maps to sets.
- Second, we add two auxiliary states $\{q_a, q_r\}$, where q_a is our only accept state and q_r is our only reject state.
- We make it so the reject state has no successor states, but the accept state always transitions back to itself rewriting the scanned square unchanged.
- We also assume our machine accepts or rejects in at most n^k steps, where n is the length of its starting input which is written immediately to the right of the initial scanned square.

Bounded Tiling Problem #2

- We limit our rows and column to be of size n^k+1. We change our initial condition of the tape to start with the input to M. Thus, it looks like



- Note that there are n^k – n of these blank representations at the end. We really only need the first as the tiling constraint will force all the others to be of the same form.

Bounded Tiling Problem #3

- The finitely bounded Tiling Problem we just described mimics the operation of any given polynomially-bound non-deterministic Turing machine.
- This machine can tile the finite plane of size $(n^k+1) * (n^k+1)$ just in case the initial string is accepted in n^k or fewer steps on some path (really a trace of at most n^k).
- If the string is not accepted, then we will hit a reject state on all paths and never complete tiling.
- This shows that the bounded tiling problem is NP-Hard
- Is it in NP? Yes. How? Well, we can be shown a tiling (posed solution takes space polynomial in n) and check it for completeness and consistency (this takes linear time in terms of proposed solution). Thus, we can verify the solution in time polynomial in n.

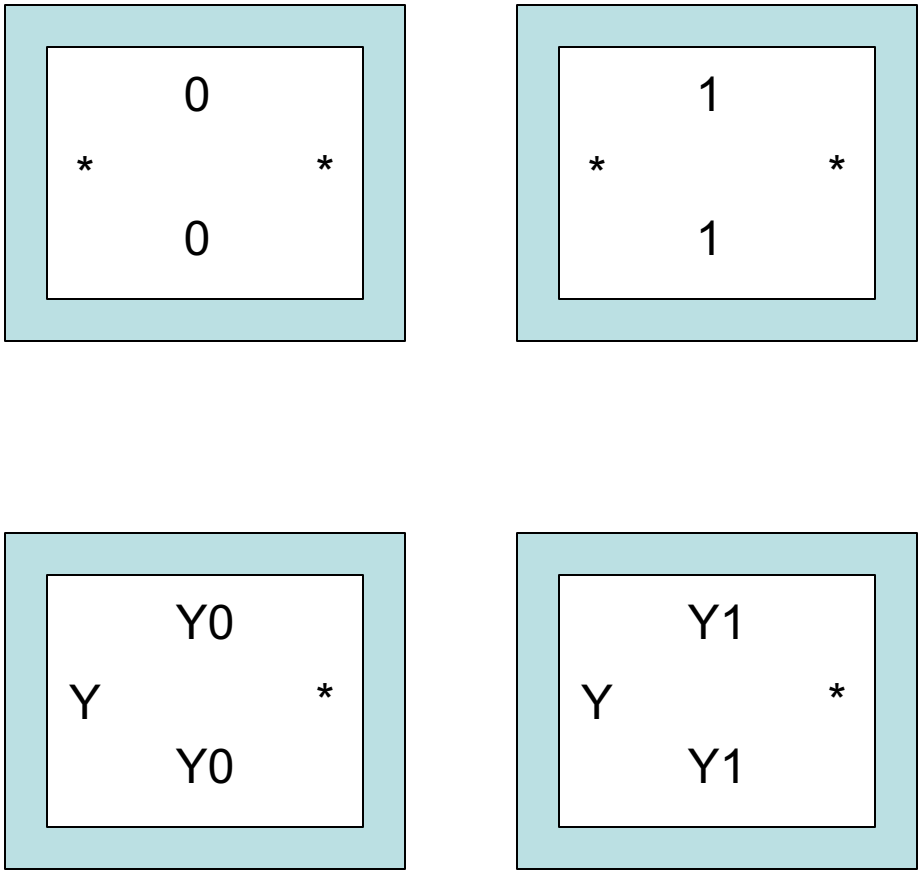
A Final Comment on Tiling

- If you look back at the unbounded version, you can see that we could have simulated a non-deterministic Turing machine there, but it would have had the problem that the plane would be tiled if any of the non-deterministic choices diverged and that is not what we desired.
- However, we need to use a non-deterministic machine for the finite case as we made this so it tiled iff some path led to acceptance. If all lead to rejection, we get stalled out on all paths as the reject state can go nowhere.

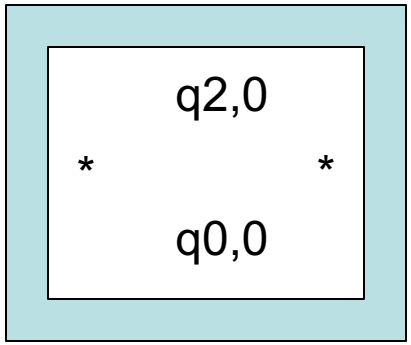
Tiling Example

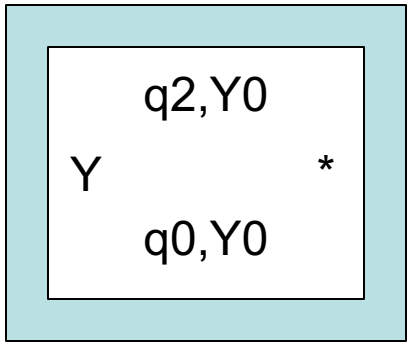
- Turing Machine Recognizes strings of at least two 1's in succession.
- q0 0 0 q2
- q0 1 R q1
- q1 0 L q2
- q1 1 1 q3
- q2 0 0 q2
- q2 1 1 q2
- No q3 rules so entering here stops tiling

Tile Replication

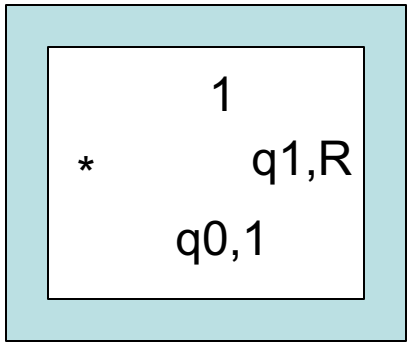


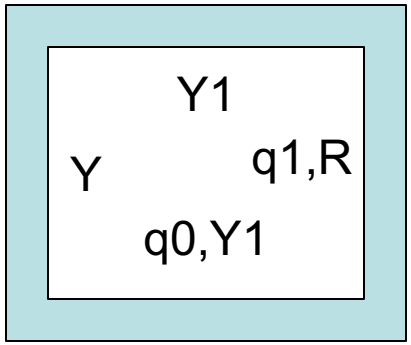
q0 0 0 q2

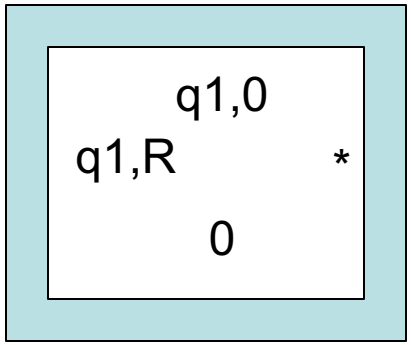


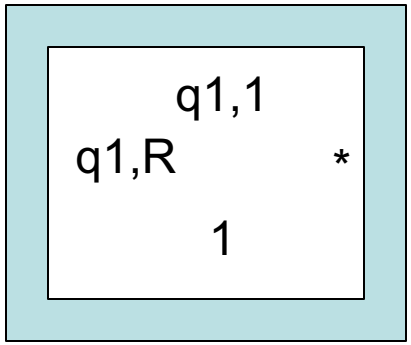


q0 1 R q1



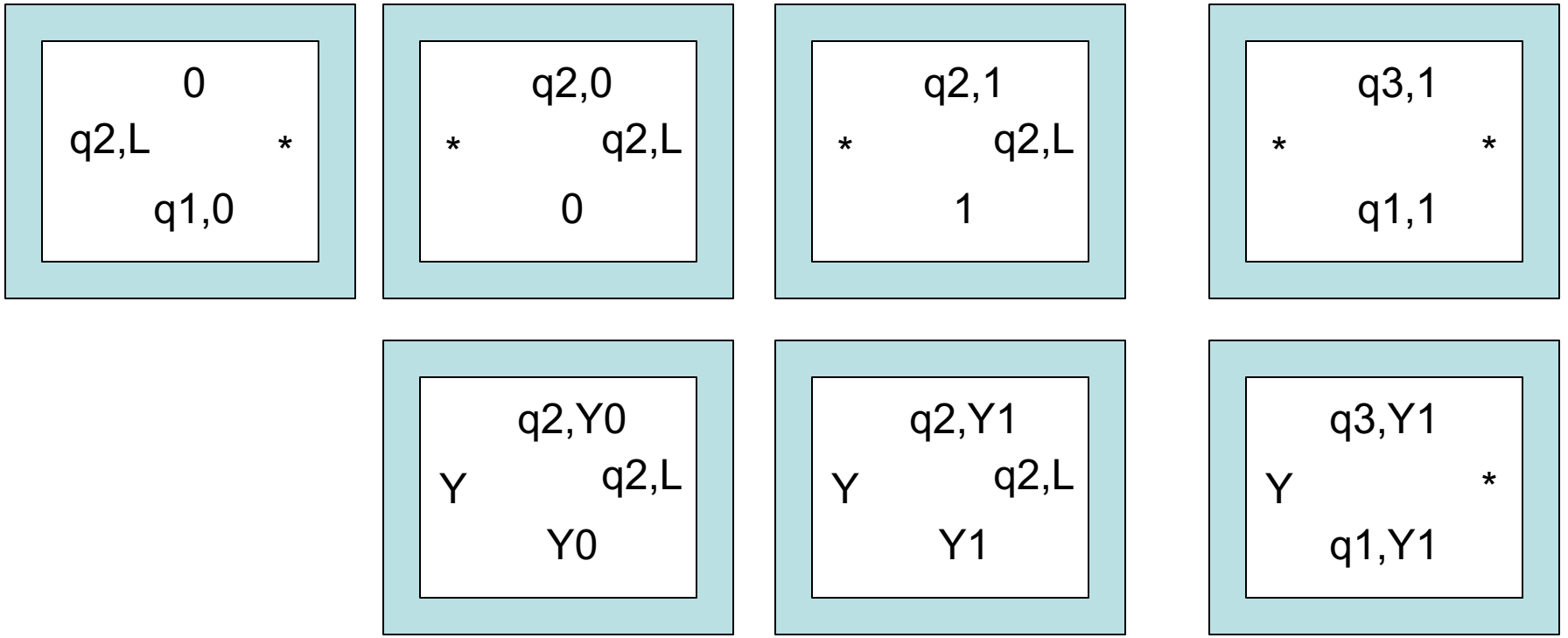




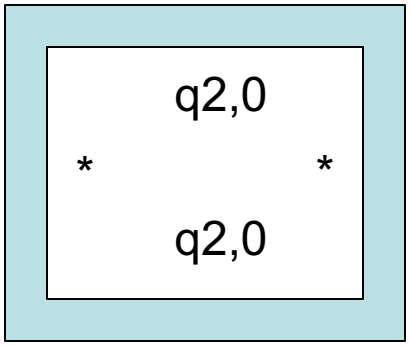


q1 0 L q2

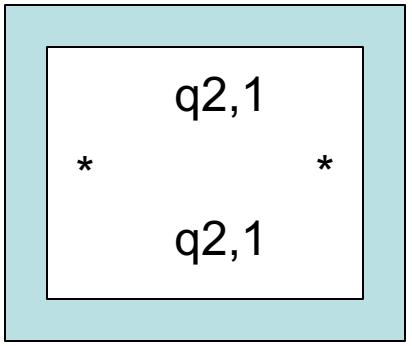
q1 1 1 q3

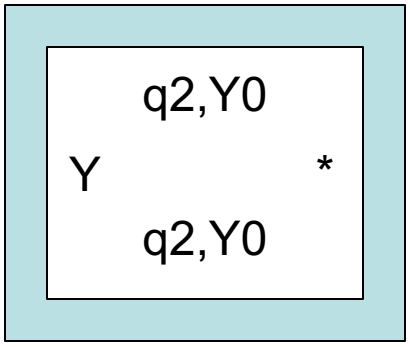


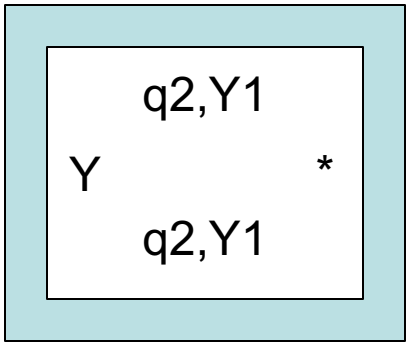
q2 0 0 q2



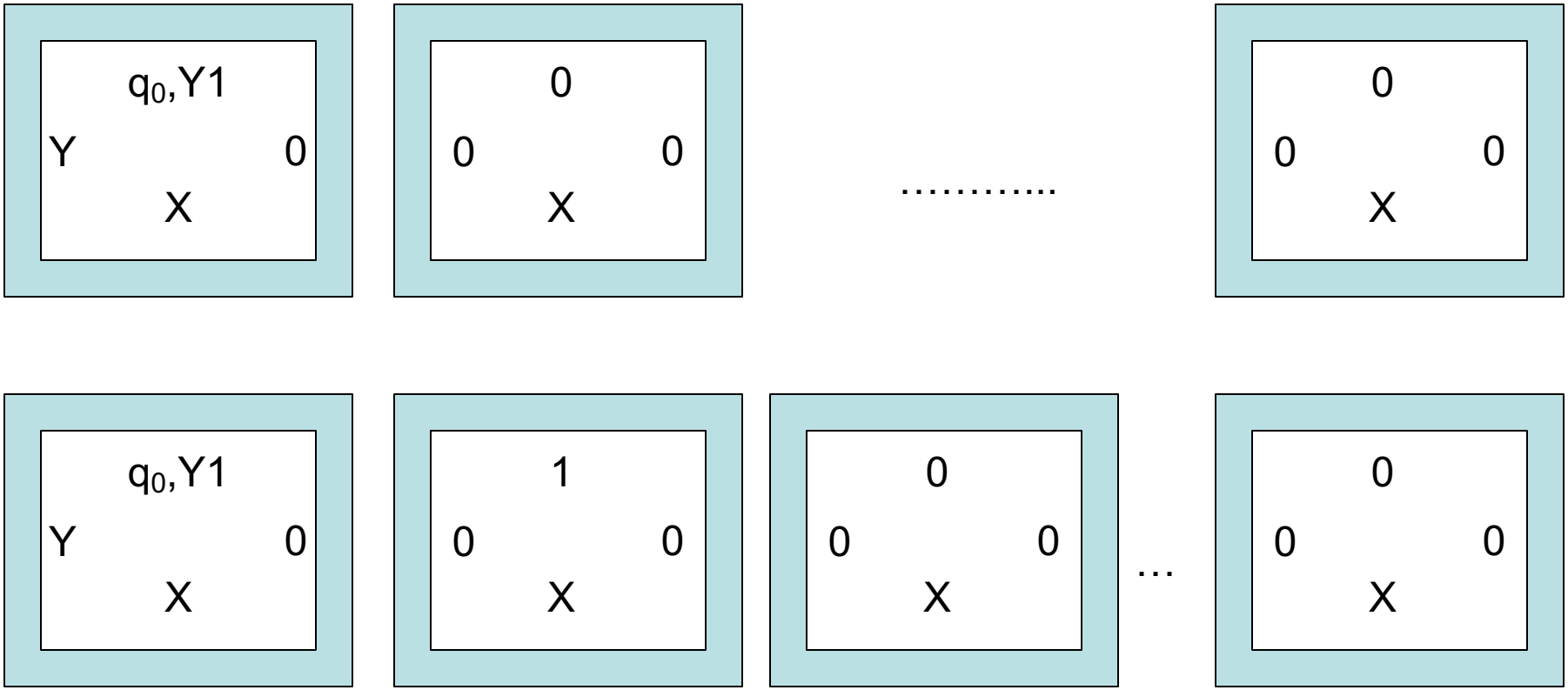
q2 1 1 q2



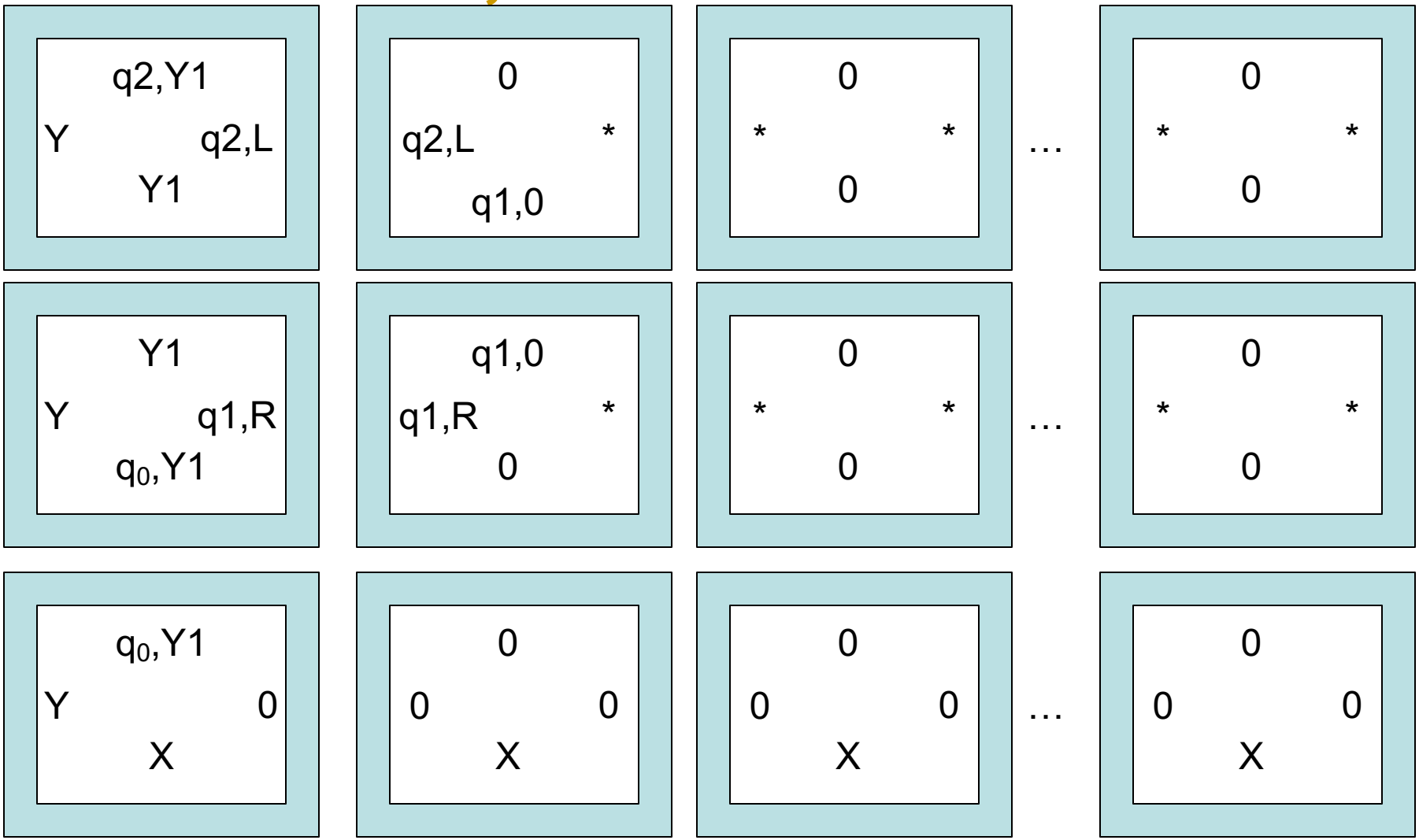




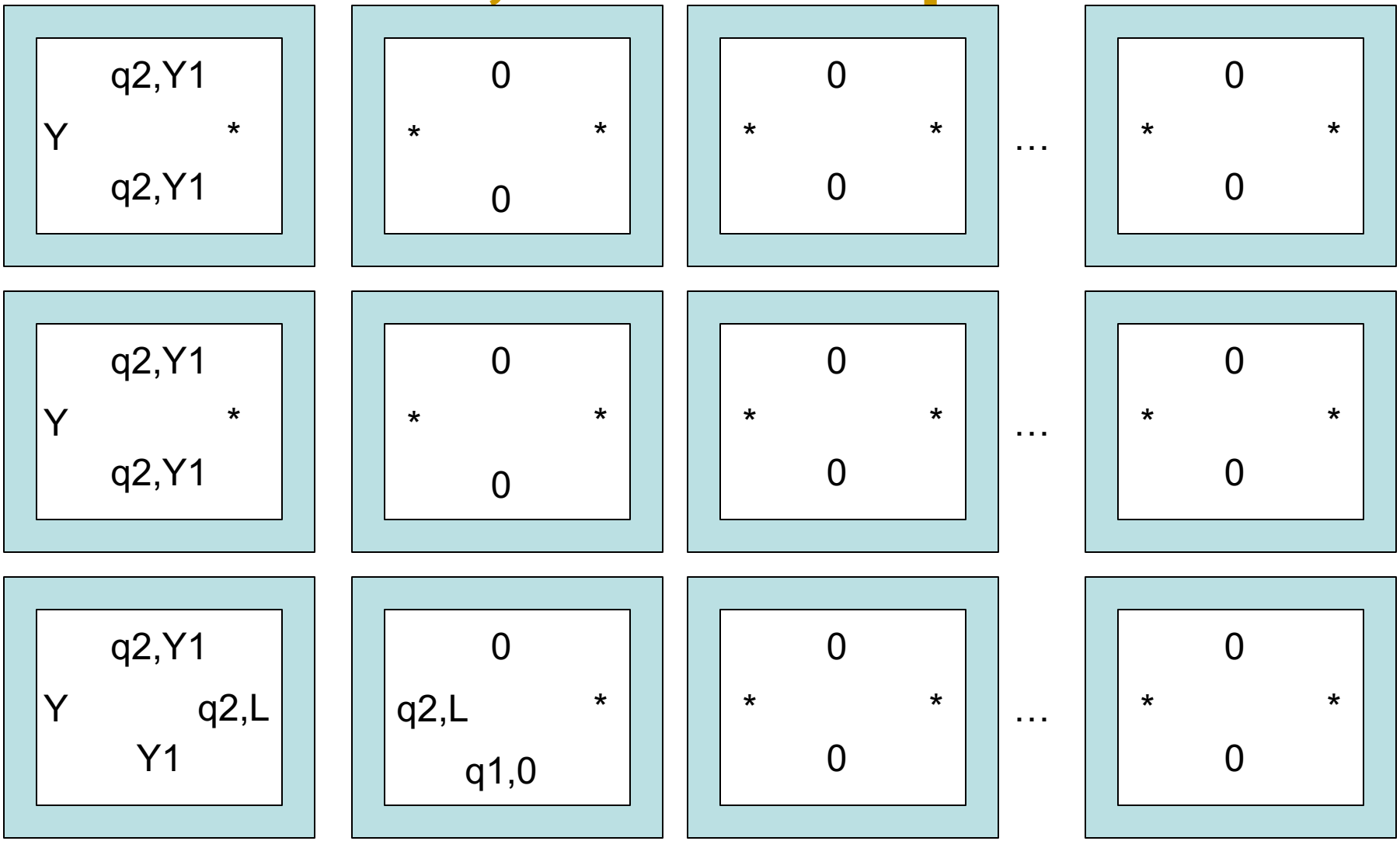
Sample Starting Rows



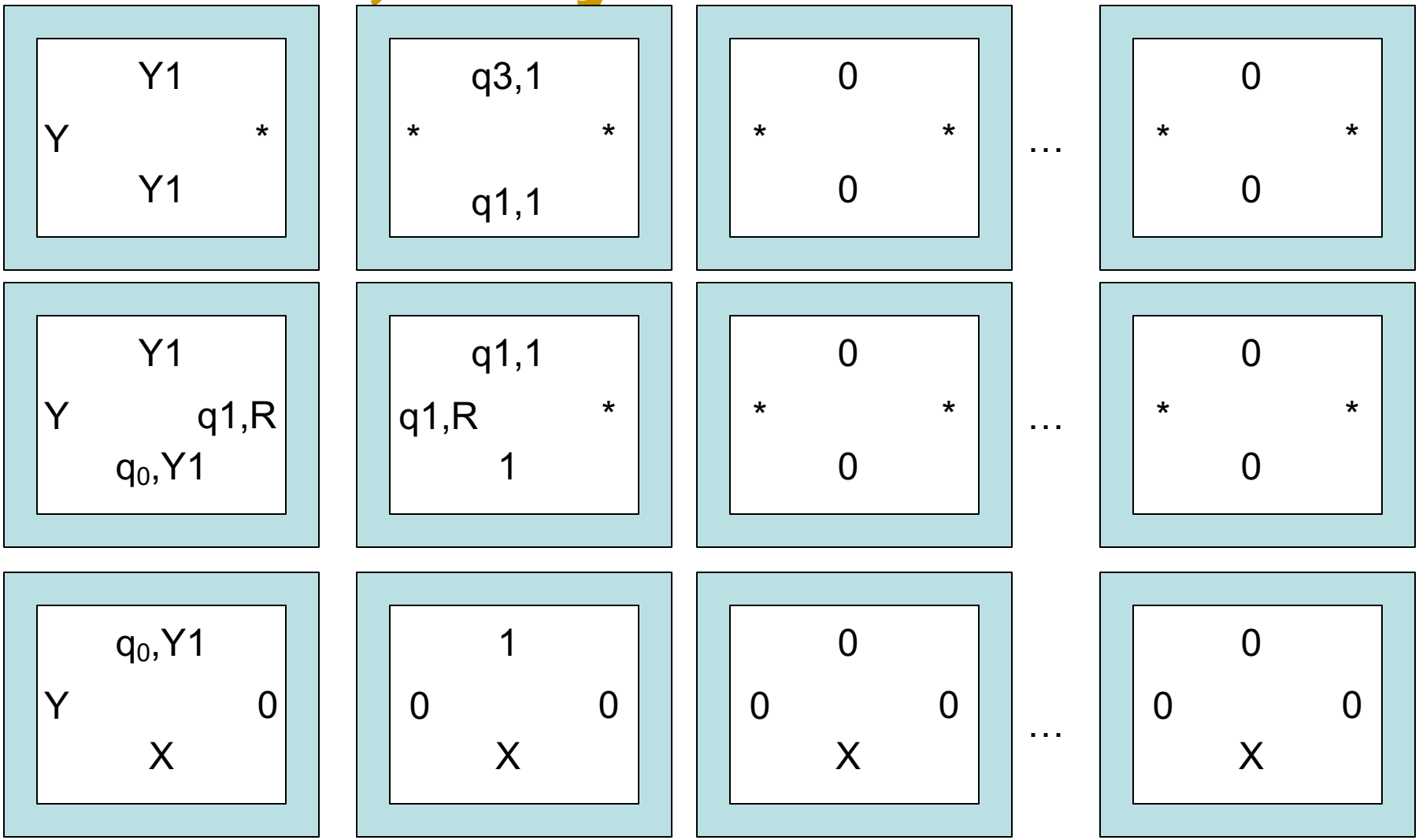
Case 1; Two More Rows



Case 1; Row 3 repeated



Case 2; Only Two More Rows



More on Variations

- One dimensional space (I asked you to think about that on an earlier slide)

- Infinite 3d space (really impossible in general)
  - This become there an exists, for all, problem – Does there exist an initial tape for which M never halts
  - In fact, one can mimic acceptance on no inputs here, meaning M is not an algorithm iff we can tile the 3d space

PCP Revisited

Bounded Post Correspondence

Bounded Variation

- Limit correspondence to a length that is polynomial in n, where n is length of initial input string.
- Outline of proof we can get for almost free
  - Convert halting problem for a Non-deterministic Turing machine to word problem for a Semi-Thue System
    Note: we originally did for deterministic machines, but the construction works for non-determinism and maps nicely to Semi-Thue systems which are non-deterministic by definition.
  - Recast as an instance of PCP
  - Limit the length of word to $(n+2)^k$, where original TM accepts or rejects in n^k steps.

Another Approach

- There is a tighter bound on Bounded PCP.

- Given sequences (x1, x2, …, xn) and (y1, y2, …, yn), and a positive integer
  $K \leq p(\max(|x1|+\ldots+|xn|, |y1|+\ldots+|yn|)$,
  where p is some polynomial, is there a solution to this instance involving indices i1, …,ik, $k \leq K$ (not necessarily distinct), of integers between 1 and n, such that the corresponding x and y strings are identical.
- Follows from Constable, Hunt and Sahni (1974). "On the Computational Complexity of Program Scheme Equivalence," *Siam Journal of Computing* 9(2), 396-416.

Co-NP

Fourth Significant Class of Problems

Co–NP

For any decision problem A in NP, there is a ‘complement’ problem Co–A defined on the same instances as A, but with a question whose answer is the negation of the answer in A. That is, an instance is a "yes" instance for A if and only if it is a "no" instance in Co–A.

Notice that the complement of the complement of a problem is the original problem.

Co–NP

Co–NP is the set of all decision problems whose complements are members of NP.

For example: consider Graph Color GC

- **Given: A graph G and an integer k.**
- **Question: Can G be properly colored with k colors?**

Co–NP

The complement problem of GC

Co–GC

Given: A graph G and an integer k.

Question: Do all proper colorings of G require more than k colors?

Co–NP

Notice that Co–GC is a problem that does not appear to be in the set NP. That is, we know of no way to check in polynomial time the answer to a "Yes" instance of Co–GC.

What is the "answer" to a Yes instance that can be verified in polynomial time?

Co–NP

Not all problems in NP behave this way. For example, if X is a problem in class P, then both "yes" and "no" instances can be solved in polynomial time.

That is, both "yes" and "no" instances can be verified in polynomial time and hence, X and Co–X are both in NP, in fact, both are in P.

This implies P = Co–P and, further,

$$\mathbf{P = Co\text{–}P \subseteq NP \cap Co\text{–}NP}.$$

Co–NP

This gives rise to a second fundamental question:

NP = Co–NP?

If P = NP, then NP = Co–NP.

This is not "if and only if."

It is possible that NP = Co–NP and, yet, P ≠ NP.

Co–NP

If $A \leq_P B$ and both are in NP, then the same polynomial transformation will reduce Co-A to Co–B. That is,
Co–A $\leq_P$ Co–B. Therefore, Co–SAT is 'complete' in Co–NP.

In fact, corresponding to NP–Complete is the complement set Co–NP–Complete, the set of hardest problems in
Co–NP.

Turing Reductions

Now, return to Turing Reductions.

Recall that Turing reductions include polynomial transformations as a special case. So, we should expect they will be more powerful.

Turing Reductions

(1) Problems A and B can, but need not, be decision problems.

(2) No restriction placed upon the number of instances of B that are constructed.

(3) Nor, how the result, $Answer_A$, is computed.

In effect, we use an Oracle for B.

Turing Reductions

Technically, Turing Reductions include Polynomial Transformations, but it is useful to distinguish them.

Polynomial transformations are often the easiest to apply.

NP–Hard

Fifth Significant Class of Problems

NP–Hard

To date, we have concerned ourselves with decision problems. We are now in a position to include additional problems. In particular, optimization problems.

We require one additional tool – the second type of transformation discussed above – Turing reductions.

NP–Hard

Definition: Problem B is NP–Hard if there is a polynomial time Turing reduction A $\leq_{PT}$ B for some problem A in NP–Complete.

This implies NP–Hard problems are at least as hard as NP–Complete problems. Therefore, they cannot be solved in polynomial time unless P = NP (and maybe not then).

This use of an oracle, allows us to reduce co-NP-Complete problems to NP-Complete ones and vice versa.

QSAT

- **QSAT is the problem to determine if an arbitrary fully quantified Boolean expression is true. Note: SAT only uses existential.**
- **QSAT is NP-Hard but may not be in NP.**
- **QSAT can be solved in polynomial space (PSPACE).**

NP–Hard

Polynomial transformations are Turing reductions.

Thus, NP–Complete is a subset of NP–Hard.

Co–NP–Complete also is a subset of NP–Hard.

NP–Hard contains many other interesting problems.

NP-Easy

- **NP-Easy** is the set of function problems that are solvable in polynomial time by a deterministic Turing machine with an oracle for some decision problem in **NP**.
- That is, given an Oracle for some **NP** problem **Y**, if **X** is Turing reducible to **Y** in polynomial time then **X** is **NP-Easy**.

NP–Easy

Problem X need not be, but often is, NP–Complete.

In fact, X can be any problem in NP or Co–NP.

More to the point, an NP-Easy problem does not even need to be a decision problem – it can be an optimization problem or some other problem seeking a numerical rather than binary (yes/no answer).

NP–Equivalent

Problem B in NP–Hard is *NP–Equivalent* when B reduces to some problem X in NP, That is, $B \leq_{PT} X$. This is, when B is also NP-Easy.

Since B is in NP–Hard, we already know there is a problem A in NP–Complete that reduces to B. That is, $A \leq_{PT} B$.

Since X is in NP, $X \leq_{PT} A$. Therefore, $X \leq_{PT} A \leq_{PT} B \leq_{PT} X$.

Thus, X, A, and B are all polynomially equivalent, and we can say

<u>Theorem.</u> Problems in NP–Equivalent are polynomial if and only if P = NP.

Example: Optimization version of Subset-Sum is NP-Equivalent.

NP-Easy and Equivalent

- **NP-Easy -- these are problems that are polynomial when using an NP oracle (≤pt)**
- **NP-Equivalent is the class of NP-Easy and NP-Hard problems (assuming Turing rather than many-one reductions)**
  - **In essence this is the functional equivalent of NP-Complete but also of Co-NP-Complete since we can negate answers**

SubsetSum Optimization

NP-Equivalence

SubsetSum Optimization (SSO)

$S = \{s_1, s_2, \ldots, s_n\}$

set of positive integers

and an integer B.

Optimization: Find a subset of S whose values sum to the largest attainable value ≤B?

Strategy: Use Oracle for SubsetSum Decision Problem but only use it a polynomial number of times – Great care must be taken here as B takes only lg B bits to represent

SSO is NP-Hard

- We can show **SS** $\leq_{PT}$ **SSO**
- Let **[(s1, s2, …, sn), B]** be an instance of **SubsetSum** (we'll call it **SS**)
- We can ask the oracle for **SSO** for the largest value **G ≤ B** such that some subsequence of **(s1, s2, …, sn)** equals **G**. If its answer is **B** we say "YES"; else we say "NO"

SSO is NP-Easy

- We can show **SSO $\leq_{PT}$ SubsetSum**
- Let **[(s1, s2, …, sn), B]** be an instance of **SSO**
- Again, our goal is to find the largest value **G ≤ B** such that some subsequence of **(s1, s2, …, sn)** equals **G**
- The challenge is to do this in a number of steps that is polynomial in the size of the question. As any integer **k** can be represented in **$\log_2 k$** bits, we need to make sure we don't ask more than **$\log_2 S$** questions of our oracle, where **S** is the length of the representation of **[(s1, s2, …, sn), B]**.
- Read the next slide very carefully

A Subtle Failure

- Let **[(s1, s2, …, sn), B]** be an instance of **SSO**. Below **sequence A** is **(s1, s2, …, sn)**

```
SUBSET-SUM-OPTIMIZATION(sequence A,  int B) {
    for i=B downto 1
          if ( SubsetSum(A, i) ) then return i;
    return 0;
}
```

- This calls the oracle **SS** up to **B** times
- As **B** is $2^{\log_2(B)}$, we might ask an exponential number of questions relative to the representation of our input parameter **B**
- As **B** can be as large as the sum of the sequence **(s1, s2, …, sn)**, the value **B** can be exponential in the size of the representation of our input and so our reduction is not polynomially bounded.

Using SubsetSum Oracle

```
SUBSET-SUM-OPTIMIZATION(sequence A, int B) {
   int best = B;
   for i = floor(log2B) downto 0 do
      A = A + { 2^i };    // add to set; will succeed now
   for i = floor(log2B) downto 0 do {
      A = A - { 2^i };           // remove from set
      if !SUBSET-SUM(A, best) then // 2^i was essential
          best = best - 2^i;  // reduce best
   }
   return best;
}
```

Example of SubsetSum Opt

- **Initial Values:**
- **A = {1, 4, 5, 7}, best = b = 15**
- **A = {1, 4, 5, 7, 8, 4, 2, 1}, best = 15**
- **A = {1, 4, 5, 7, 4, 2, 1}, best = 15**
- **A = {1, 4, 5, 7, 2, 1}, best = 15**
- **A = {1, 4, 5, 7, 1}, best = 15-2 = 13**
- **A = {1, 4, 5, 7}, best = 13**

Another Example

- **Initial Values:**
- **A = {1, 4, 5, 7}, best = b = 20**
- **A = {1, 4, 5, 7, 16, 8, 4, 2, 1}, best = 20**
- **A = {1, 4, 5, 7, 8, 4, 2, 1}, best = 20**
- **A = {1, 4, 5, 7, 4, 2, 1}, best = 20**
- **A = {1, 4, 5, 7, 2, 1}, best = 20**
- **A = {1, 4, 5, 7, 1}, best = 20-2 = 18**
- **A = {1, 4, 5, 7}, best = 18-1 = 17**

Analysis

- Each loop has $O(\log_2 b)$ iterations, which is linear with respect to the size of b.
- Note that if we tried all values less that b, we would have O(b) tries and that is exponential in $\log_2 b$, the size of b.
- The correct solution takes advantage of the NP-complete power of the oracle.

Minimum Colors for a Graph

- We know K-Color (**KC**) is NP Complete
- We can reduce **KC** to **MinColor** problem just by seeing if **MinColor** is **≤ K**. Thus, **MinColor** is NP-Hard
- How do we reduce **MinColor** to **KC** asking only a log number of questions of the oracle for **KC**?
- Consider, if **N** nodes, then can easily **N-Color**
- Can we **N/2-Color**?
  - If so, then try **N/4**
  - If not, then try **3N/4**
- This is a simple binary search for optimal value

PSPACE

- PSPACE is set of problems solvable in polynomial space with unlimited time
  PSPACE = $\cup$ SPACE(n^k)
- PSPACE = co-PSPACE = NPSPACE
- PSPACE is a strict superset of CSLs
- PSPACE-Complete Problem is, given a regular expression e over Σ, does e denote all strings in Σ^*?
- The above, while solvable, is potentially hard
- Another PSPACE-Complete problem is QSAT
- PSPACE is suspected to outside the P/NP hierarchy

EXPTIME and EXPSPACE

- EXPTIME is the set of problems solvable in $2^{p(n)}$ where is p is some polynomial.
- NEXPTIME is the set of problems solvable in $2^{p(n)}$ on a non-deterministic TM.
- EXPSPACE is set of problems solvable in $2^{p(n)}$ space and unbounded time

Elementary Functions

$$\mathsf{ELEMENTARY} = \bigcup_{k \in \mathbb{N}} \text{k-EXP}$$

$$= \mathsf{DTIME}(2^n) \cup \mathsf{DTIME}(2^{2^n}) \cup \mathsf{DTIME}(2^{2^{2^n}}) \cup \cdots$$

Alternating TM (ATM)

- ATM adds to NDTM notation the notion where, for each state q, q has one of the following properties: (accept, reject, $\vee$, $\wedge$)
  - $\vee$ means mean accept the string if any final state reached after q is accepting
  - $\wedge$ means mean accept the string if all final states reached after q are accepting
- AP = PSPACE where AP is class of problems solvable in polynomial time on an ATM

QSAT, Petri Net, Presburger

- QSAT is solvable by an alternating TM in polynomial time and polynomial space
- As noted, before, QSAT is PSPACE-Complete
- Petri net reachability is EXPSPACE-hard and requires 2-EXPTIME
- Presburger arithmetic is at least in 2-EXPTIME, at most in 3-EXPTIME, and can be solved by an ATM with n alternating quantifiers in doubly exponential time

Complexity Hierarchy

- $P \subseteq NP \subseteq PSPACE = NPSPACE \subseteq EXPTIME \subseteq NEXPTIME \subseteq EXPSPACE \not\subseteq 2\text{-}EXPTIME \not\subseteq 3\text{-}EXPTIME \not\subseteq \ldots \not\subseteq ELEMENTARY \not\subseteq PRF \not\subseteq REC$
- $P \neq EXPTIME$; At least one of these is true
  - $P \not\subseteq NP$
  - $NP \not\subseteq PSPACE$
  - $PSPACE \not\subseteq EXPTIME$
- $NP \neq NEXPTIME$
  - Note that EXPTIME = NEXPTIME iff P=NP
  - Note that k-EXPTIME $\not\subseteq$ (k+1)-EXPTIME, k>0
- $PSPACE \neq EXPSPACE$; At least one of these is true
  - $PSPACE \not\subseteq EXPTIME$
  - $EXPTIME \not\subseteq EXPSPACE$

FP and FNP

- **FP is functional equivalent to P**
  R(x,y) in FP if can provide value y for input x via a deterministic polynomial time algorithm
- **FNP is functional equivalent to NP;**
  R(x,y) in FNP if can verify any pair (x,y) via a deterministic polynomial time algorithm

TFNP

- **TFNP is the subset of FNP where a solution always exists, i.e., there is a y for each x such that R(x,y).**
  - **Task of a TFNP algorithm is to find a y, given x, such that R(x,y)**
  - **Unlike FNP, the search for a y is always successful**
- **FNP properly contains TFNP contains FP (we don't know if proper)**

Prime Factoring

- **Prime factoring is defined as, given n and k, does n have a prime factor < k?**
- **Factoring is in NP and co-NP**
  - **Given candidate factor can check its primality in poly time and then see if it divides n**
  - **Given candidate set of factors can check their primalities, and see if product equals n; if so, and no candidate < k, then answer is no**

Prime Factoring and TFNP

- **Prime Factoring as a functional problem is in TFNP, but is it in FP?**
- **If TFNP in FP then TFNP = FP since FP contained in TFNP**
- **If that is so, then carrying out Prime Factoring is in FP and its decision problem is in P**
  - **If this is so, we must fear for encryption, most of which depends on difficulty of finding factors of a large number**

More TFNP

- There is no known recursive enumeration of TFNP but there is of FNP
  - This is similar to total versus partially recursive functions (analogies are everywhere)
- It appears that TFNP does not have any complete problems!!!
  - But there are subclasses of TFNP that do have complete problems!!

Another Possible Analogy

- **Is P = (NP intersect Co-NP)?**
- **Recall that REC = (RE intersect co-RE)**
- **The analogous result may not hold here**

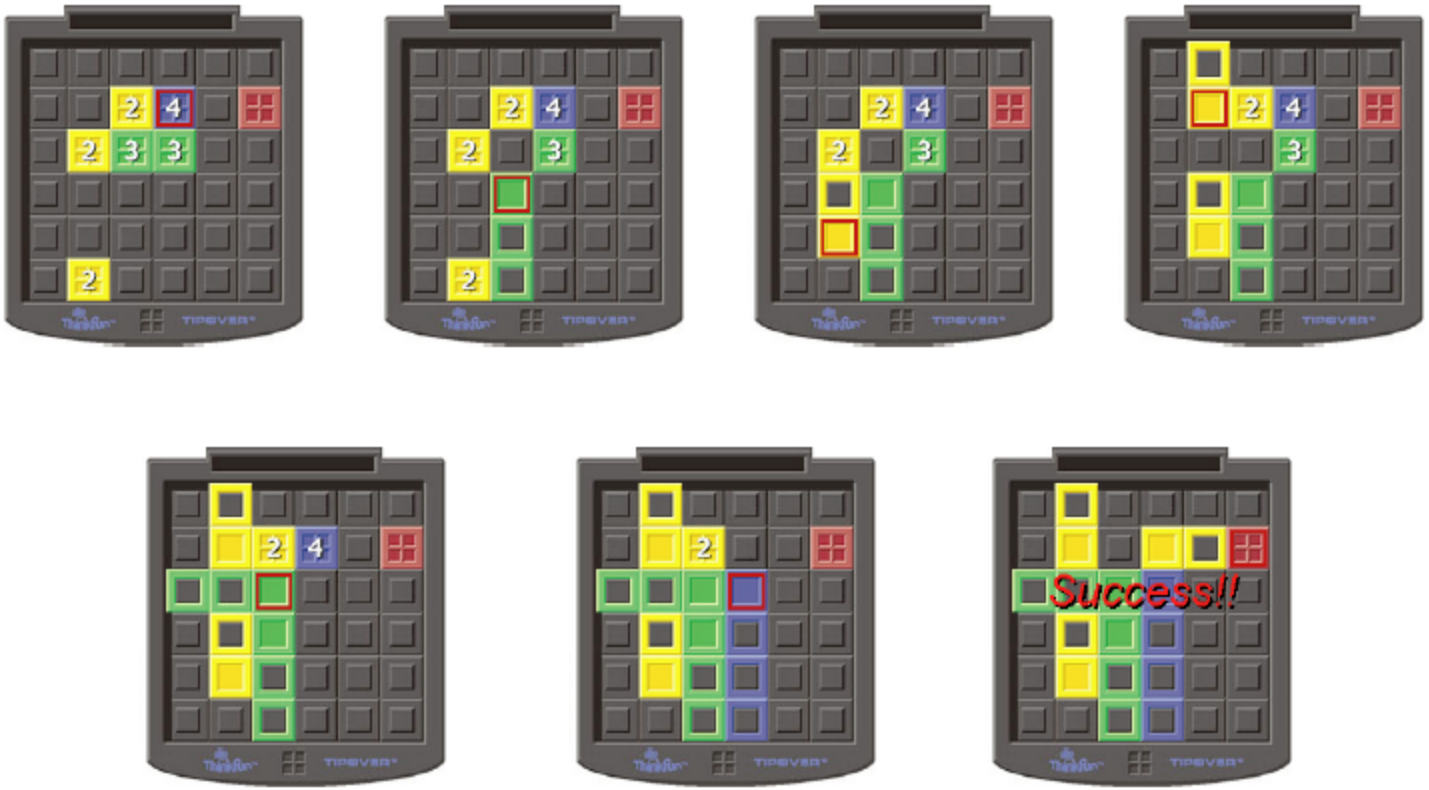
Turing vs m-1 Reductions

- **In effect, our normal polynomial reduction (≤p) is a many-one polynomial time reduction as it just asks and then accepts its oracle's answer**
- **In contrast, NP-Easy and NP-Equivalent employ a Turing machine polynomial time reduction (≤pt) that uses rather than mimics answers from its oracle**

More Examples of NP Complete Problems

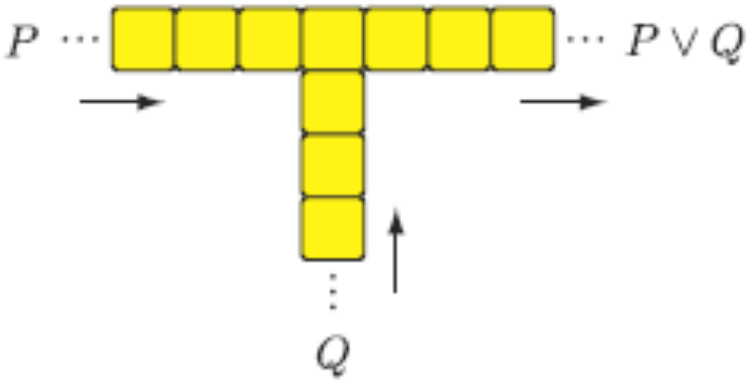
TipOver

Rules of Game



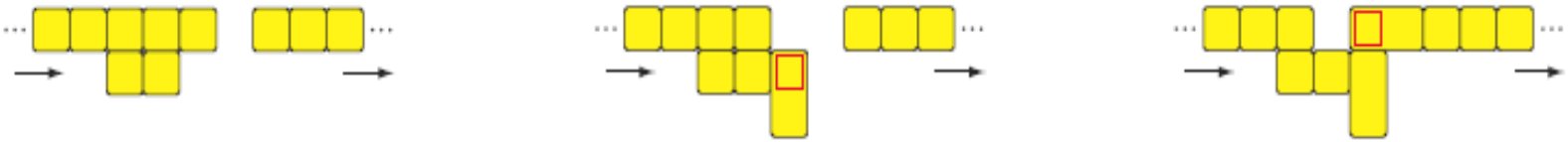
Numbers are height of crate stack;
If could get 4 high out of way we can attain goal

Problematic OR Gadget



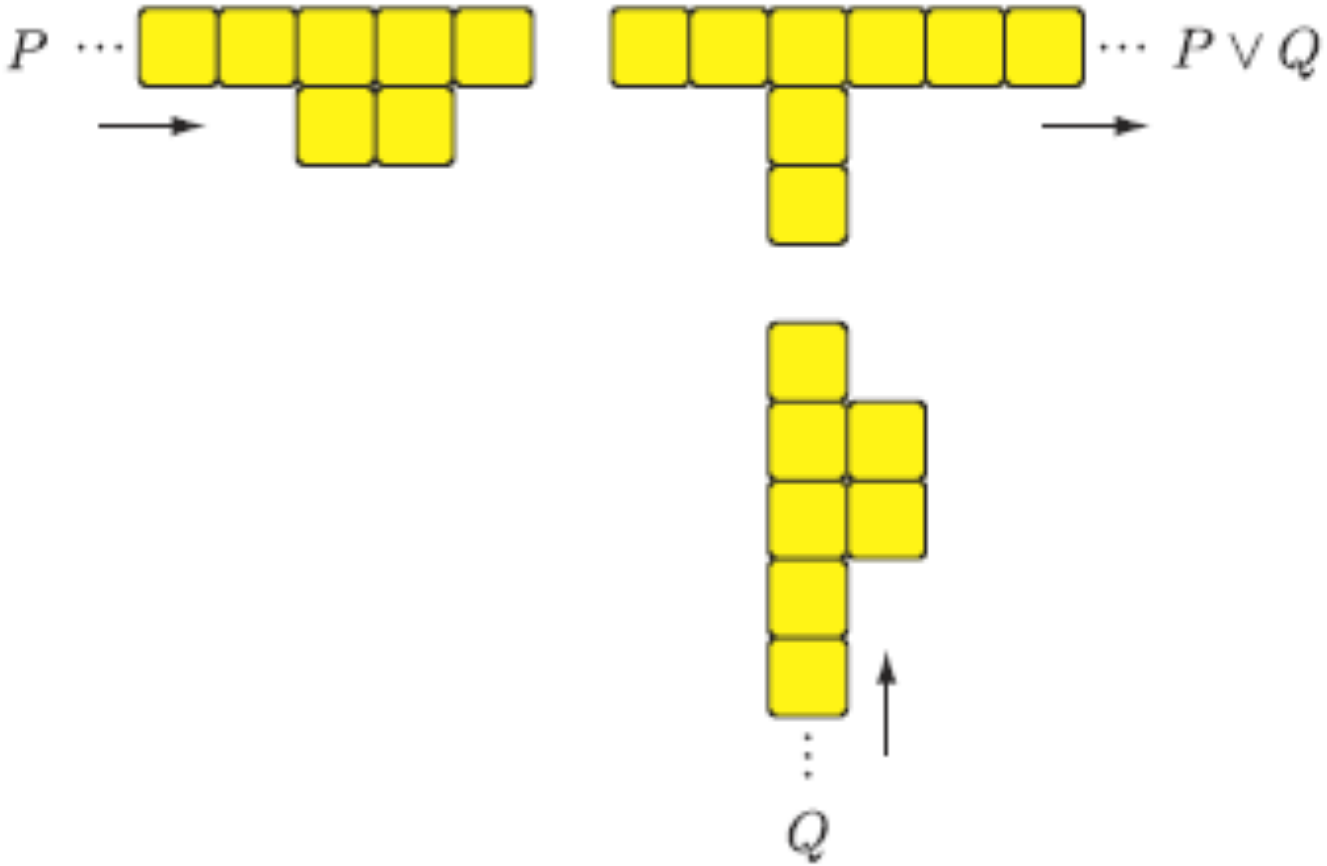
Can go out where did not enter

Directional gadget

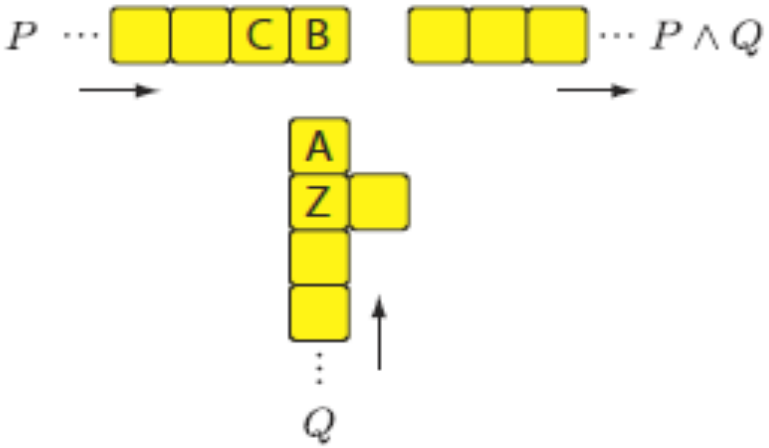


Single stack is two high;
tipped over stack is one high, two long;
red square is location of person travelling the towers

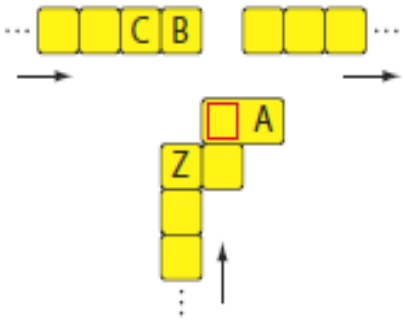
One directional Or gadget

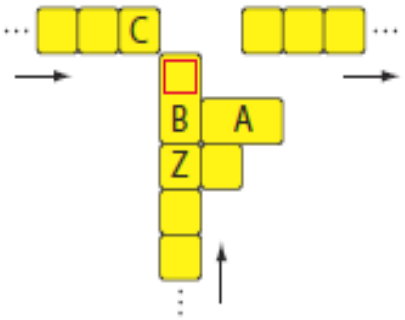


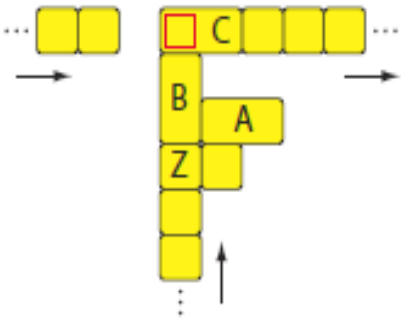
AND Gadget



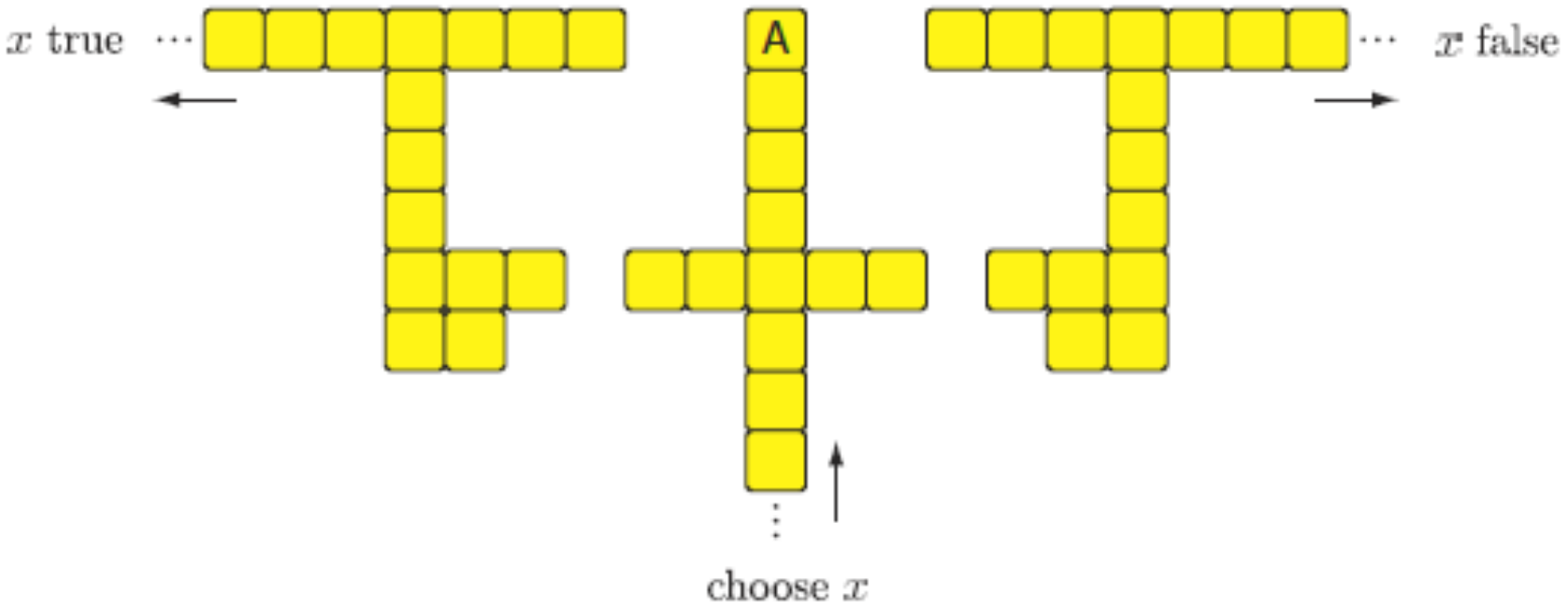
How AND Works





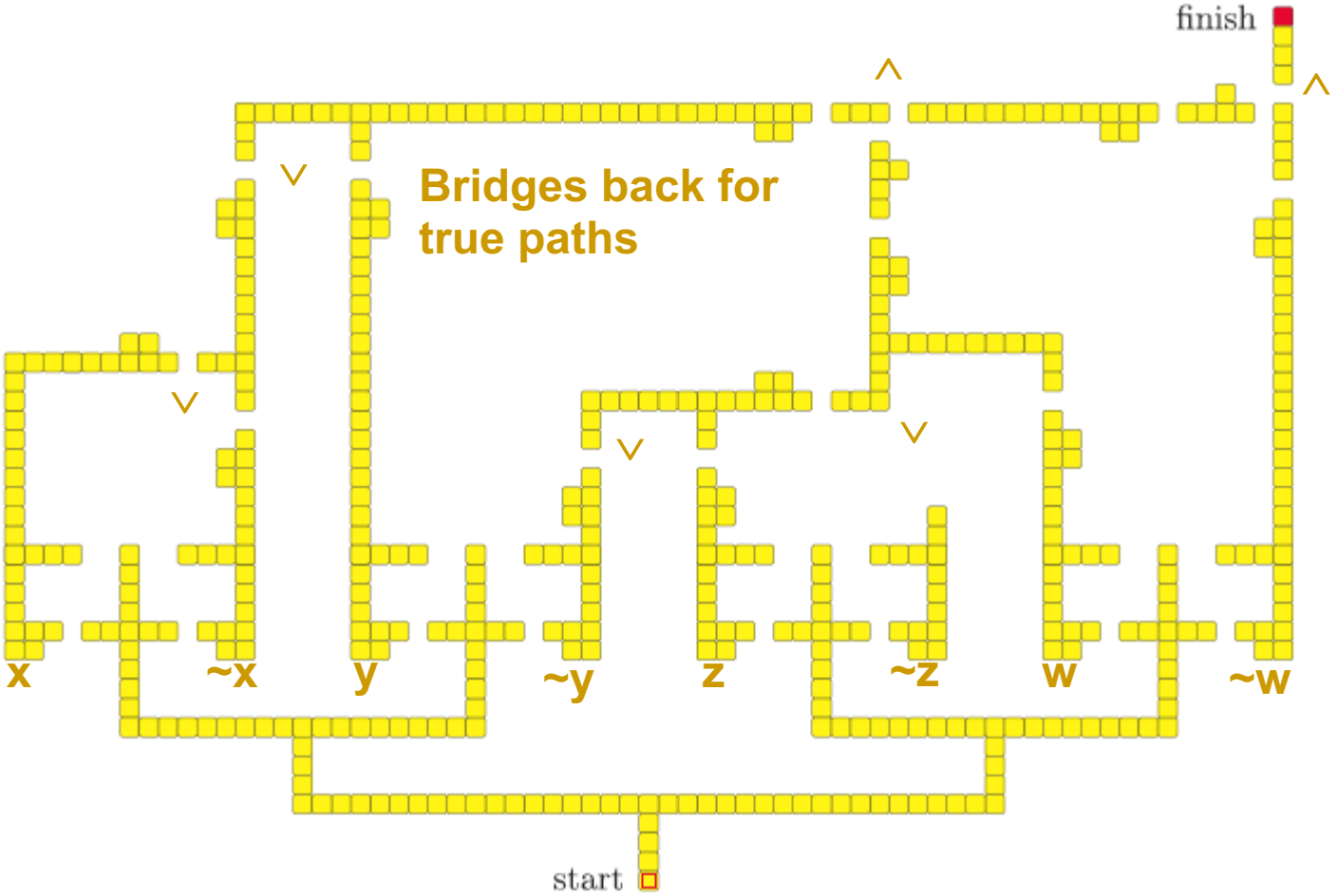


Variable Select Gadget



Tip A left to set x true; right to set x false
Can build bridge to go back but never to change choice

((x∨~x∨y)∧(~y∨z∨w)∧~w)



Win Strategy is NP-Complete

- TipOver win strategy is NP-Complete
- Minesweeper consistency is NP-Complete
- Phutball single move win is NP-Complete
  - Do not know complexity of winning strategy
- Checkers is really interesting
  - Single move to King is in P
  - Winning strategy is PSpace-Complete

Finding Triangle Strips

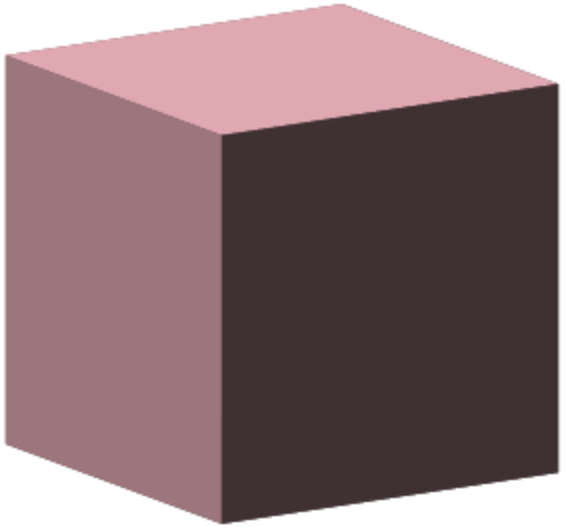
Adapted from presentation by
Ajit Hakke Patil
Spring 2010

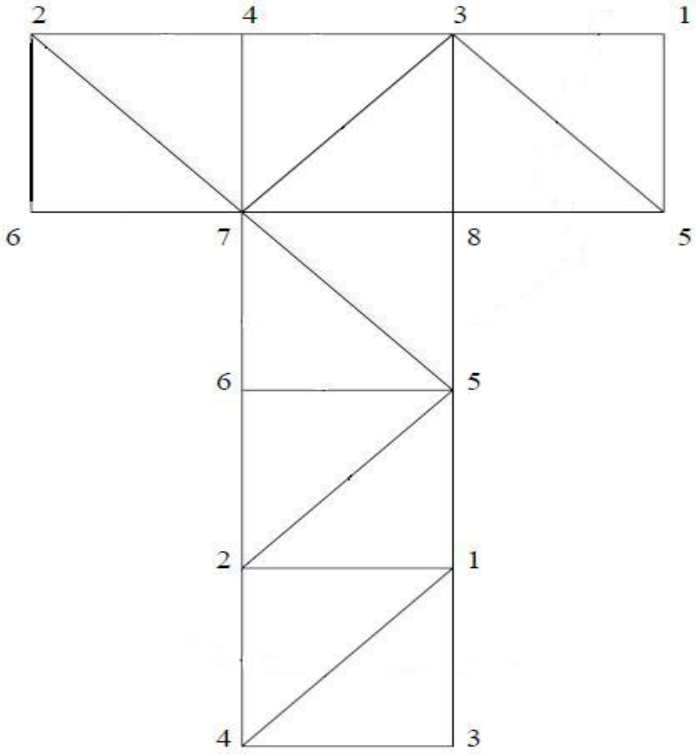
Graphics Subsystem

- **The graphics subsystem (GS) receives graphics commands from the application running on CPU/GPU over a bus, builds the image specified by the commands, and outputs the resulting image to display hardware**
- **Graphics Libraries:**
  - **OpenGL, DirectX.**

Surface Visualization

- As Triangle Mesh
- Generated by triangulating the geometry



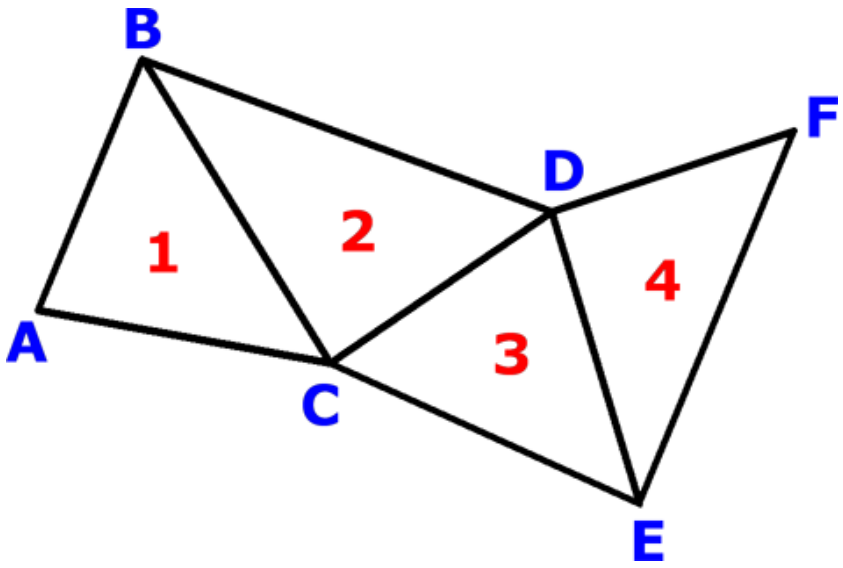


Triangle List vs Triangle Strip

- Triangle List: *Arbitrary ordering of triangles.*
- Triangle Strip: *A triangle strip is a sequential ordering of triangles*. i.e consecutive triangles share an edge
- In case of triangle lists we draw each triangle separately.
- So for drawing N triangles you need to call/send 3N vertex drawing commands/data.
- However, using a Triangle Strip reduces this requirement from 3N to N + 2, provided a single strip is sufficient.

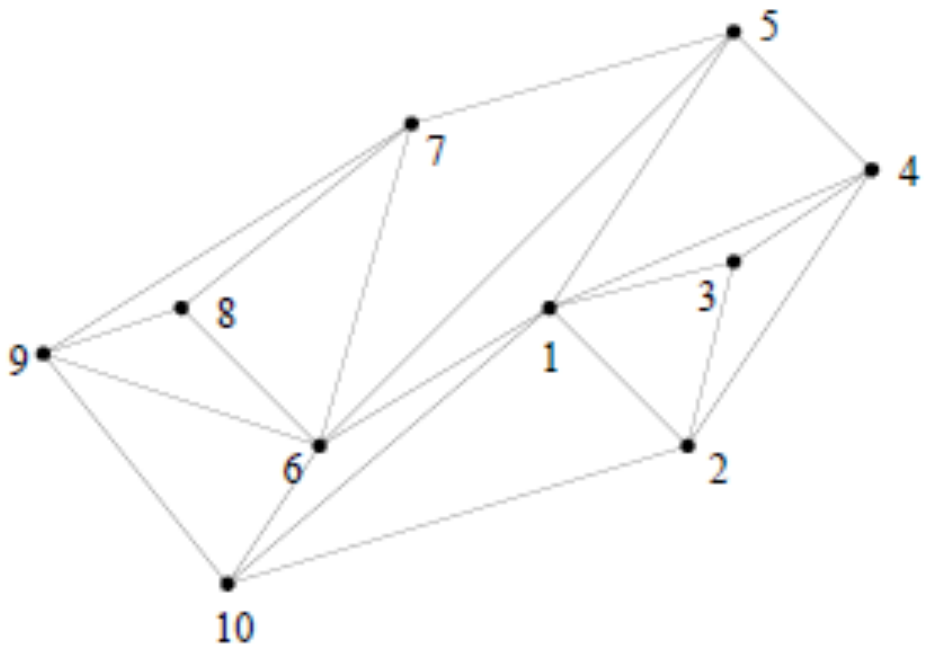
Triangle List vs Triangle Strip

- four separate triangles: ABC, CBD, CDE, and EDF
- But if we know that it is a triangle strip or if we rearrange the triangles such that it becomes a triangle strip, then we can store it as a sequence of vertices ABCDEF
- This sequence would be decoded as a set of triangles ABC, BCD, CDE and DEF
- Storage requirement:
  - 3N => N + 2



Tri-strips example

- Single tri-strip that describes triangles is: 1,2,3,4,1,5,6,7,8,9,6,10,1,2



K-Stripability

- Given some positive integer K (less than the number of triangles).
- Can we create K tri-strips for some given triangulation – no repeated triangles.

Triangle List vs Triangle Strip

```
// Draw Triangle Strip
glBegin(GL_TRIANGLE_STRIP);
 For each Vertex
{
    glVertex3f(x,y,z); //vertex
}
glEnd();
```

```
// Draw Triangle List
glBegin(GL_TRIANGLES);
For each Triangle
{
      glVertex3f(x1,y1,z1);// vertex 1
      glVertex3f(x2,y2,z2);// vertex 2
      glVertex3f(x3,y3,z3);// vertex 3
}
glEnd();
```

Problem Definition

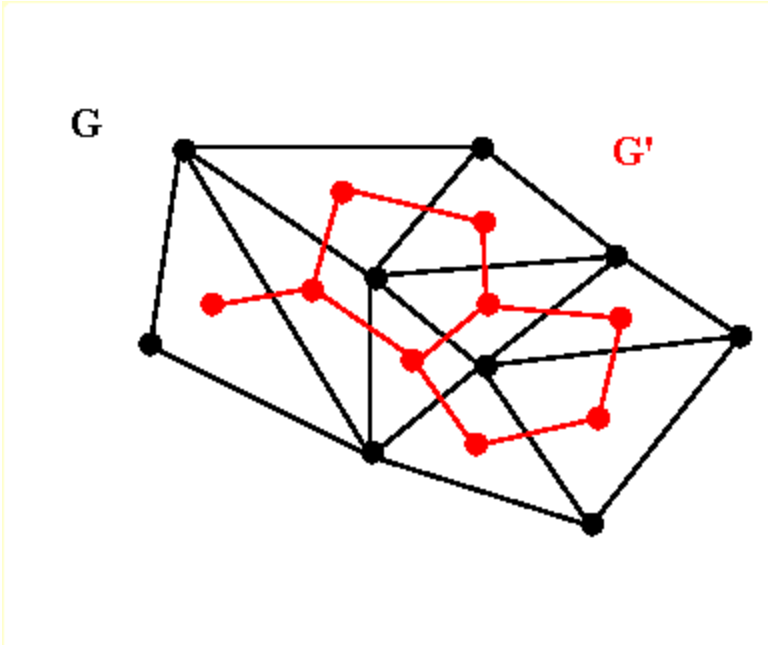
- Given a triangulation T = {t_1, t_2, t_3 ,... t_n}. Find the triangle strip (sequential ordering) for it?
- Converting this to a decision problem.
- Formal Definition:
  Given a triangulation T = {t_1, t_2, t_3 ,... t_N}. Does there exists a triangle strip?

NP Proof

- Provided a witness of a 'Yes' instance of the problem. we can verify it in polynomial time by checking if the sequential triangles are connected.
- Cost of checking if the consecutive triangles are connected
  - For i to N -1
    - Check of i_{th} and $i+1_{th}$ triangle are adjacent (have a common edge)
    - Three edge comparisions or six vertex comparisions
  - ~ 6N
- Hence it is in NP.

Dual Graph

- The *dual graph* of a triangulation is obtained by defining a vertex for each triangle and drawing an edge between two vertices if their corresponding triangles share an edge
- This gives the triangulations *edge-adjacency* in terms of a graph
- Cost of building a Dual Graph
  - $O(N^2)$
- e.g G' is a dual graph of G.



NP-Completeness

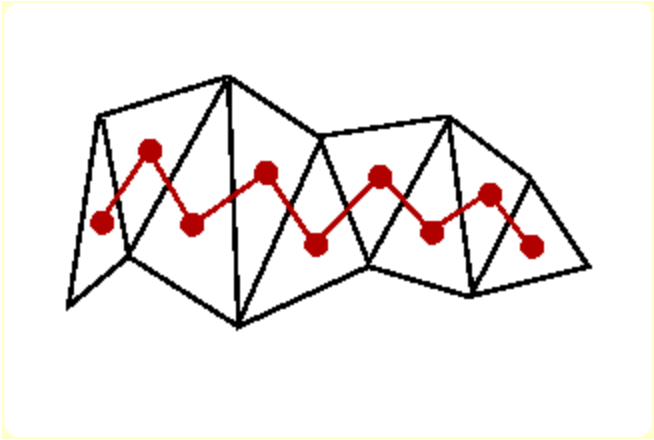
- To prove it's NP-Complete we reduce a known NP-Complete problem to this one; the Hamiltonian Path Problem.
- Hamiltonian Path Problem:
  - Given: A Graph G = (V, E). Does G contains a path that visits every vertex exactly once?

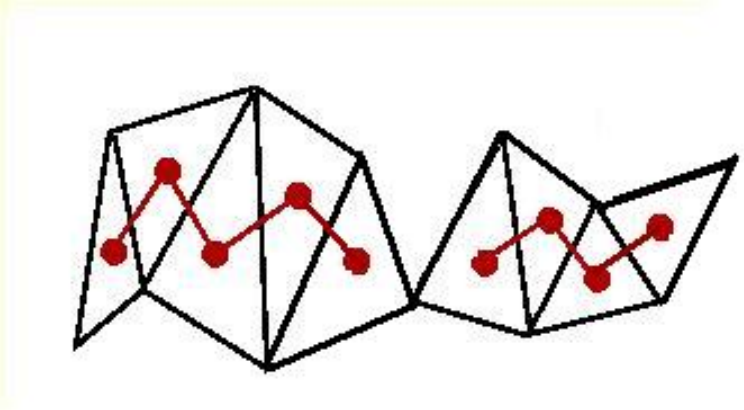
NP-Completeness proof by restriction

- Accept an Instance of Hamiltonian Path, G = (V, E), we restrict this graph to have max. degree = 3.The problem is still NP-Complete.
- Construct an Instance of HasTriangleStrip
  - G' = G
    - V' = V
    - E' = E
  - Let this be the dual graph G' = (V', E') of the triangulation T = {t1, t2, t3 ,.. tN}.
    - V' ~ Vertex v_i represents triangle t_i, i = 1 to N
    - E' ~ An edge represents that two triangles are *edge-adjacent* (share an edge)
- Return HasTriangleStrip(T)

NP-Completeness

- G will have a Hamiltonian Path iff G' has one (they are the same).
- G' has a Hamiltonian Path iff T has a triangle strip of length N – 1.
- T will have a triangle strip of length N – 1 iff G (G') has a Hamiltonian Path.
- 'Yes' instance maps to 'Yes' instance. 'No' maps to 'No.'





HP $\leq_P$ HasTriangleStrip

- The ‘Yes/No’ instance maps to ‘Yes/No’ instance respectively and the transformation runs in polynomial time.
- Polynomial Transformation
- Hence finding Triangle Strip in a given triangulation is a NP-Complete Problem