Assignment#3 Key

1. Show prfs are closed under three-way mutual induction

Three-way mutual induction means that each induction step after calculating the base is computed using the previous value of the other function.

The formal hypothesis is:
Assume **g1, g2, g3, h1, h2,** and **h3** are already known to be prf, then so are **f1, f2,** and **f3,** where
f1(x,0) = g1(x); f1(x,y+1) = h1(f2(x,y),f3(x,y));
f2(x,0) = g2(x); f2(x,y+1) = h2(f3(x,y),f1(x,y))
f3(x,0) = g3(x); f3(x,y+1) = h3(f1(x,y),f2(x,y))

Proof is by construction

Three-Way Mutual Induction (Co-Recursion)

F will do all three computations in "parallel"

$\mathbf{F(x,0) = <g1(x), g2(x), g3(x)>}$ // bases for all three

$\mathbf{F(x, y+1) = < h1(<F(x,y)>_2,<F(x,y)>_3), h2(<F(x,y)>_3,<F(x,y)>_1), h3(<F(x,y)>_1,<F(x,y)>_2) >}$

F produces triples containing the values of **f1**, **f2**, and **f3**, in its first, second and third, component, respectively. The above shows **F** is a prf.

f1, **f2**, and **f3** are then defined from **F** as follows:

$\mathbf{f1(x,y) = <F(x,y)>_1}$

$\mathbf{f2(x,y) = <F(x,y)>_2}$

$\mathbf{f3(x,y) = <F(x,y)>_3}$

This shows that **f1**, **f2**, and **f3** are also prf's, as was desired.

2. Show every non-empty, re set is the range of a prf

Let **S** be an arbitrary non-empty, re set. Furthermore, let **S** be the range of some partial recursive function $\mathbf{f_s}$. Show that **S** is the range of some primitive recursive function, call it $\mathbf{h_s}$. First, since S is non-empty, it contains some element, call this element $\boldsymbol{a}$.

$$\mathbf{h_s(<x,t>) = STP(f_s, x, t) * VALUE(f_s, x, t) + (1\text{-}STP(f_s, x, t)) *} \ \boldsymbol{a}$$

Note that $\mathbf{h_s}$ is a prf since we defined it from just **STP**, **VALUE**, **multiply**, **addition**, and **limited subtraction**, all of which we previously showed were prf's. Also, this is very close to the prf we already showed on Page 155 of Computability Notes where we started with a semi-decision procedure rather than an enumerating partial recursive function.

Discussion of #2 (the actual proof it works)

All and only the elements in **S** will be enumerated by fs, which mean that, if **y ∈ S**, then **∃x** such that $\mathbf{f_S(x) = y}$. But that **y ∈ S** iff exists a **x** and **t** such that $\mathbf{STP(f_s, x, t\ )}$ and $\mathbf{VALUE(f_s, x, t\ ) = y}$. $\mathbf{h_s}$ enumerates all such values **y** in the function below. To make it an algorithm, we handle the case where $\mathbf{\sim STP(f_s, x, t\ )}$ by having $\mathbf{h_s}$ enumerate ***a***, a value whose existence we know by the assumption that **S≠Ø**.

$$\mathbf{h_s(\langle x,t\rangle) = STP(f_s, x, t\ ) * VALUE(f_s, x, t\ ) + (1\text{-}STP(f_s, x, t\ )) *} \boldsymbol{a}$$

This analysis shows that $\mathbf{h_s}$ enumerates all and only those values in **S**.