PUSHDOWN AUTOMATA
PDA

$$A = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$

STATES (FINITE)

LANG. ALPHABET (FINITE)

STACK SYMBOLS (FINITE

START STATE

OPTIONAL BOTTOM OF STACK MARKER

OPTIONAL FINAL STATES

TRANSITION FUNCTION

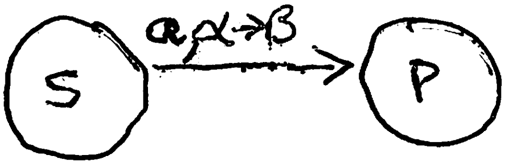
$$\delta : Q \times \Sigma_e \times \Gamma_e \to 2^{Q \times \Gamma^*}$$

CAN EXTEND TO Γ^*

CAN LIMIT TO Γ_e BY GROWING STATES

PDA DIAGRAMS

SOME TEXTS USE $a, X / Y$ WHICH IS FINE ALSO



$$\delta(s, a, \alpha) \supseteq \{(p, \beta)\}$$

CAN BE λ

PDA LANGUAGES
$\equiv$ CFLs

INSTANTANEOUS DESCRIPTIONS (ID)

$[q, w, \gamma]$

q - CURRENT STATE
w - REMAINING INPUT
γ - STACK CONTENTS
READ LEFT (TOP) TO RIGHT (BOTTOM)

SINGLE STEP

$[q, ax, Z\alpha] \vdash [p, x, \beta\alpha]$ IF $\delta(q, a, Z) \in (p, \beta)$

MULTISTEP $\vdash^*$ REFLEXIVE TRANSITIVE CLOSURE OF $\vdash$

GIVEN $A = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$
THERE CAN BE THREE NOTIONS OF ACCEPTANCE

FINAL STATE

$\mathcal{L}(A) = \{w \mid [q_0, w, Z_0] \vdash^* [f, \lambda, \beta]\}, f \in F$

EMPTY STACK

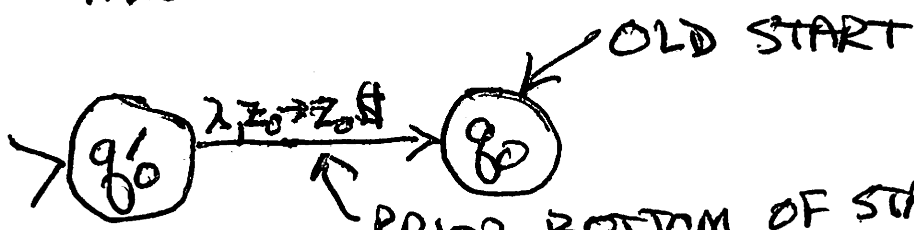
$N(A) = \{w \mid [q_0, w, Z_0] \vdash^* [q, \lambda, \lambda]\}, q \in Q$

EMPTY STACK AND FINAL STATE

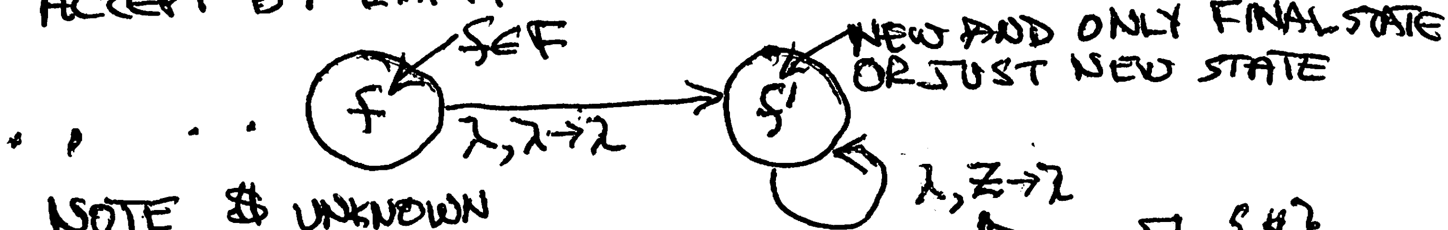
$E(A) = \{w \mid [q_0, w, Z_0] \vdash^* [f, \lambda, \lambda]\}, f \in F$

EQUIVALENCY OF LANGUAGE CLASSES, $\mathcal{L}(A)$, $N(A)$, $E(A)$, WHERE A RANGES OVER ALL PDAs

- CONVERTING ONE FORM TO ANOTHER
  FOR EACH CASE, ASSUME q_0' IS A NEW STATE (NEW START) AND \$ IS A NEW STACK SYMBOL PLACED ON BOTTOM OF STACK. FOR ALL CASES, START WITH

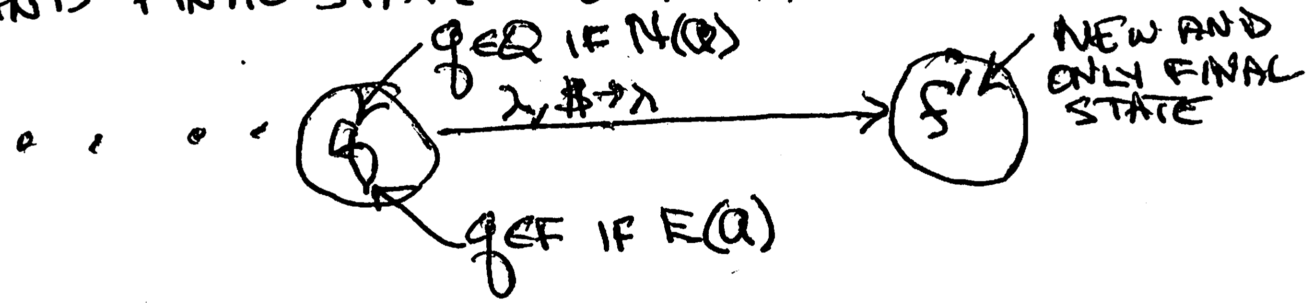


- CHANGE ACCEPT BY FINAL STATE TO ACCEPT BY EMPTY STACK OR EMPTY AND FINAL STATE



NOTE \$ UNKNOWN IN OLD MACHINE

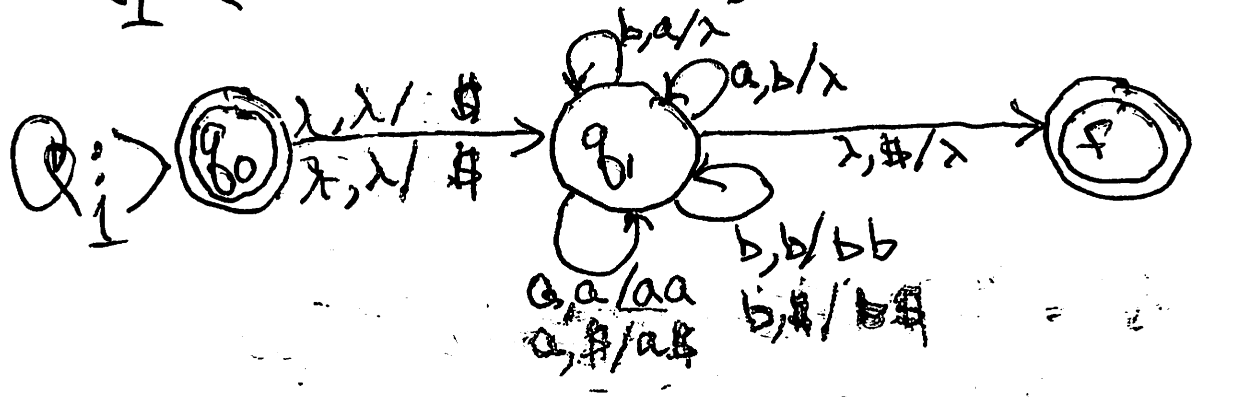
- CHANGE ACCEPT BY EMPTY STACK OR EMPTY AND FINAL STATE TO ACCEPT BY FINAL STATE



EXAMPLE PDA

$L_1 = \{ w \mid |w|_a = |w|_b \}$. ASSUME λ ON STACK

Q_1:



I USED / WHERE BOOK USES $\rightarrow$

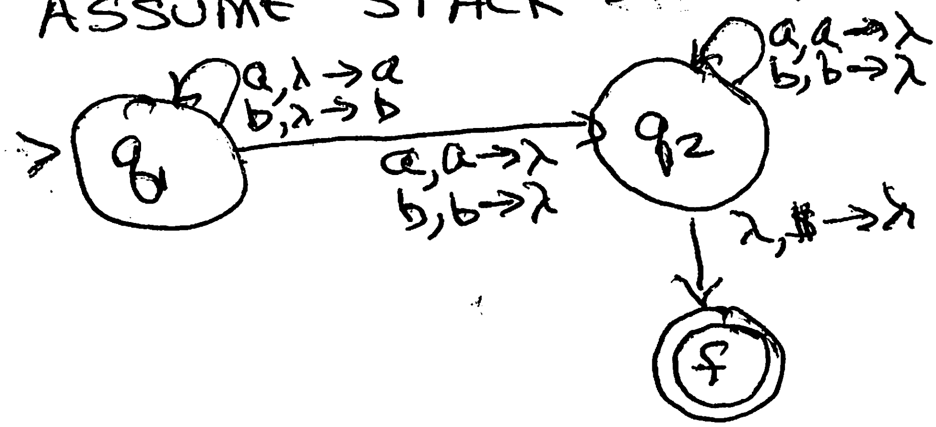
THIS GETS $L_1 = E(Q_1)$

THIS IS NON-DETERMINISTIC

$L_2 = \{ w w^R \mid w \in \{a, b\}^+ \}$

ASSUME STACK STARTS WITH \$

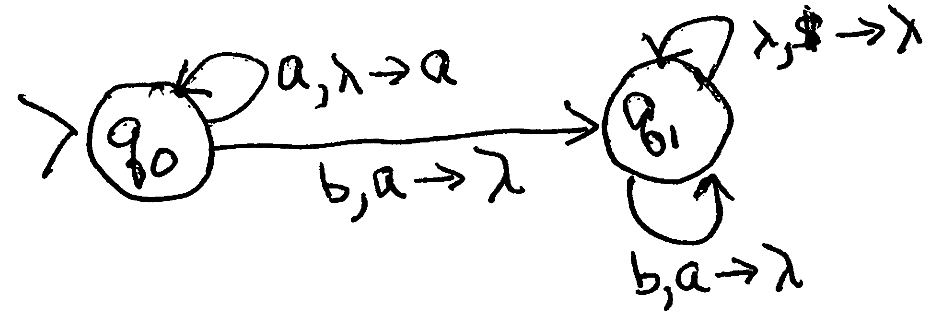
Q_2:



THIS GETS $L_2 = \mathcal{L}(Q_2) = E(Q_2)$

THIS IS ALSO NON-DETERMINISTIC

$L_3 = \{a^n b^n \mid n > 0\}$. ASSUME \$ ON STACK

Q_3: 

THIS GETS $L_3 = N(Q_3)$

THIS IS DETERMINSTIC

DETERMINISM FOR PDAs

1) FOR EACH $q \in Q$ & $z \in \Gamma$ & $a \in \Sigma$
   IF $|\delta(q, \lambda, z)| > 0$ THEN $|\delta(q, a, z)| = 0$

2) FOR NO $q \in Q$, $z \in \Gamma$ & $a \in \Sigma_\epsilon$
   IS $|\delta(q, a, z)| > 1$

TOP DOWN PARSER

GIVEN $G = (N, \Sigma, R, S)$

DEFINE $A_G = (\{q\}, \Sigma, \Sigma \cup V, q, S, \emptyset)$

$$\delta(q, a, a) = \{(q, \lambda)\} \quad \forall a \in \Sigma$$

$$\delta(q, \lambda, A) = \{(q, \alpha) \mid A \to \alpha \in R\}$$

PREDICTIVE

$$N(A) = \mathcal{L}(G)$$

NOTE! THIS HAS ONE STATE SO ACTUAL STATEFULNESS IS STACK CONTENT

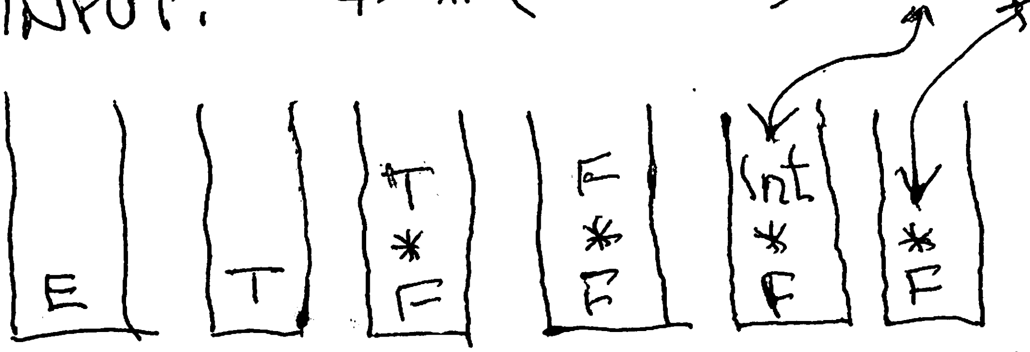
TOP-DOWN PARSER EXAMPLE

$E \to E+T \mid T$

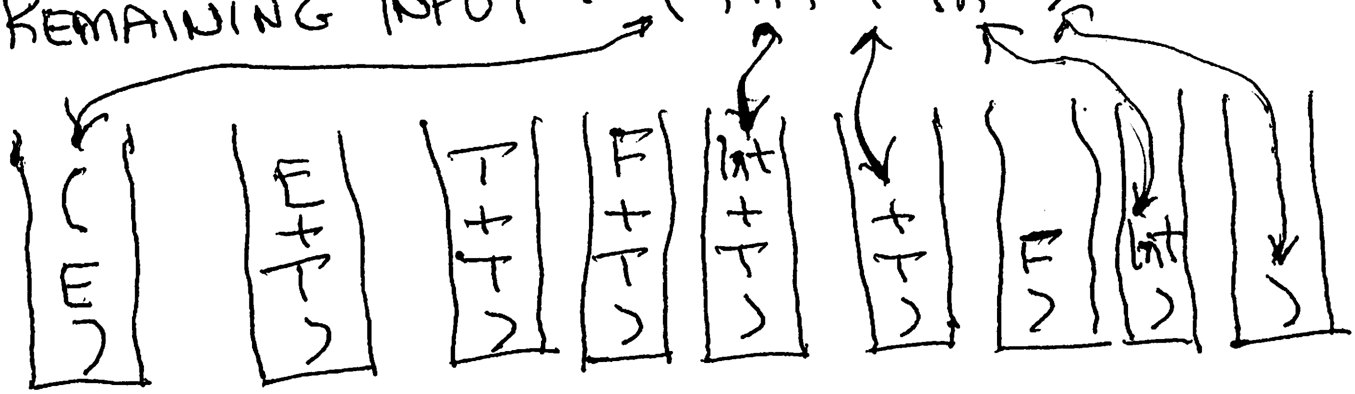
$T \to T*F \mid F$

$F \to (E) \mid int$

INPUT: $7 * (3+21) \Rightarrow int * (int+int)$



REMAINING INPUT: (Int + Int)



ACCEPTS BY EMPTY STACK

BOTTOM-UP PARSER

GIVEN $G = (V, \Sigma, R, S)$

DEFINE $A_G = (\{q, f\}, \Sigma, \Sigma \cup V \cup \{\$\}, \delta, \$, \{f\})$

$\delta(q, a, \lambda) = \{(q, a)\} \quad \forall a \in \Sigma$ SHIFT

$\delta(q, \lambda, \alpha^R) = \{(q, A) \mid A \to \alpha \in R\}$ REDUCE

NOTE: LOOKING AT HANDLE ON STACK
(MORE THAN ONE SYMBOL IS POSSIBLE)

$\delta(q, \lambda, S) \supseteq \{(f, \lambda)\}$

$\delta(f, \lambda, \$) = \{(f, \lambda)\}$

$E(A) = \mathcal{L}(G)$

COULD USE

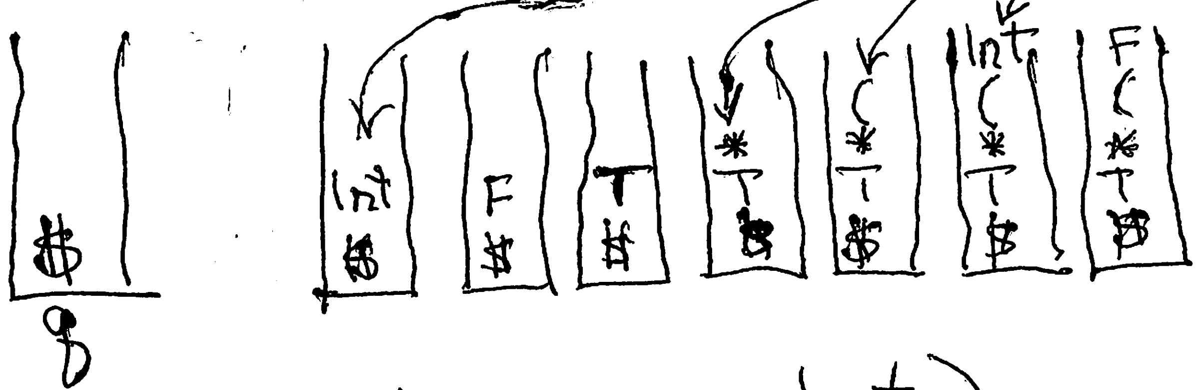
$\delta(q, \lambda, S\$) \supseteq \{(q, \lambda)\}$

$N(A) = \mathcal{L}(G)$

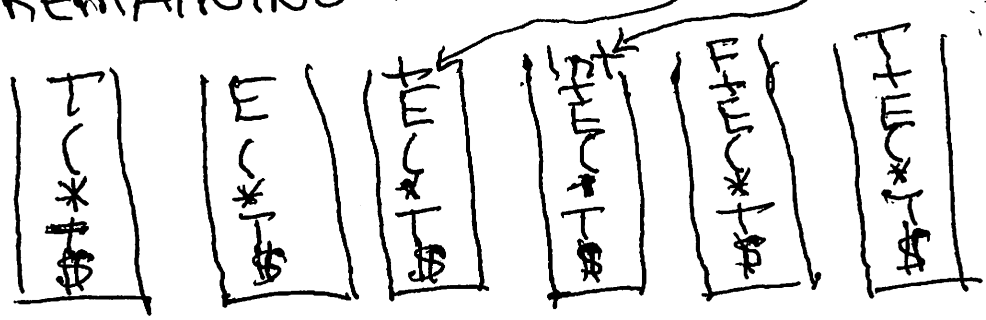
BOTTOM-UP PARSER EXAMPLE

E → E+T | T
T → T*F | F
F → (E) | Int

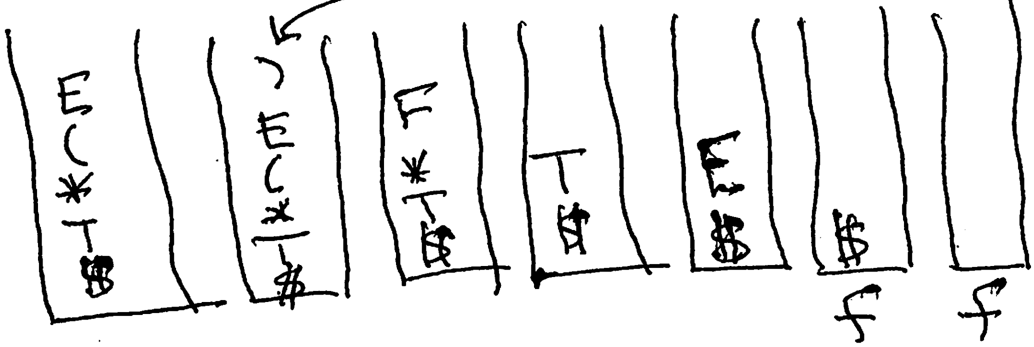
INPUT: 7*(3+21) ⇒ Int*(Int+Int)



REMAINING INPUT +Int)



REMAINING INPUT)



ACCEPTS BY FINAL STATE AND
EMPTY STATE
OR JUST EMPTY STACK (CAN ALTER FOR FINAL)

LOOK AT

PARSING.PPTX

LIMITING PDA TO PUSH/POP

PUSH: PUSH(α) IS EQUIVALENT TO

$$\delta(q, a, z) \supseteq \{(p, \alpha z)\}$$

WHERE WE JUST USE

$$\delta(q, a, z) \supseteq \{(p, \text{PUSH}(\alpha))\}$$

POP: POP IS EQUIVALENT TO

$$\delta(q, a, z) \supseteq \{(p, \lambda)\}$$

WHERE WE JUST USE

$$\delta(q, a, z) \supseteq \{(p, \text{POP})\}$$

IF WANT TO SIMULATE STANDARD OPERATION OF

$$\delta(q, a, z) \supseteq \{(p, \alpha)\}$$

CAN DO

$$\delta(q, a, z) \supseteq \{(p', \text{POP})\}$$

$$\delta(p', \lambda, X) \supseteq \{(p, \text{PUSH}(\alpha))\}$$

ANY ELEMENT OF Γ OR λ IF ALLOWED

$$[q_0, w, \$] \vdash^* [f, \lambda, \lambda]$$

PDA TO CFG

$Q = (Q, \Sigma, \Gamma, \delta, q_0, Z, F)$

{f}

USED IN TECHNIQUE 1 Z = \$; NOT USED IN BOOK'S APPROACH SO ∅

TECHNIQUE #1:

LIMITED TO PUSH & POP

NON-TERMINALS ARE OF FORM

$\langle p, z, q \rangle$

GOAL IS $\langle p, z, q \rangle \stackrel{*}{\Rightarrow} w \in \Sigma^*$

WHEN $[p, w, z] \vdash^* [q, \lambda, \lambda]$

START SYMBOL IS

$S \to \langle q_0, \$, f \rangle$

CAN ACTUALLY DO FOR EMPTY STACK ONLY BY HAVING

$S \to \langle q_0, \$, q \rangle \quad \forall q \in Q$

RULES, OTHER THAN START, ARE

$\langle q, X, p \rangle \to a \langle s, Y, t \rangle \langle t, X, p \rangle \quad \forall t \in Q$

WHENEVER $\delta(q, a, X) \supseteq \{(s, \text{PUSH}(Y))\}$

$\langle q, X, p \rangle \to a$

WHENEVER $\delta(q, a, X) \supseteq \{(p, \text{POP})\}$

GOAL: $\langle q_0, \$, f \rangle \stackrel{*}{\Rightarrow} w$, WHENEVER $w \in L(Q)$

CFL CLOSURE

EASY: UNION

$G_A = (V_A, \Sigma, R_A, S_A)$

$G_B = (V_B, \Sigma, R_B, S_B)$

$G = (\{S\} \cup V_A \cup V_B, \Sigma, \{S \to S_A | S_B\} \cup R_A \cup R_B, S)$

MODERATE: INTERSECTION WITH REGULAR

$Q_0 = (Q_0, \Sigma, \Gamma, \delta_0, q_0, \$, F_0)$ PDA, $L_0 = \mathcal{L}(Q_0)$

$Q_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ DFA $L_1 = \mathcal{L}(Q_1)$

DEFINE $Q_2 = (Q_0 \times Q_1, \Sigma, \Gamma, \delta_2, \langle q_0, q_1 \rangle, \$, F_0 \times F_1)$

$\delta_2(\langle q, s \rangle, a, X) \supseteq \{(\langle q', s' \rangle, \alpha)\}, X \in \Gamma, a \in \Sigma_e$

IFF $\delta_0(q, a, X) \supseteq \{(q', \alpha)\}$

AND $\delta_1(s, a) = s'$ IF $a \in \Sigma$

$\delta_1(s, a) = s$ IF $a = \lambda$

ELSE $\delta_1(s, a) = s$

NOW INDUCTIVELY CAN SHOW

$[\langle q_0, q_1 \rangle, w, \$] \vdash^* [\langle t, s \rangle, \lambda, B]$ IFF

$[q_0, w, \$] \vdash^* [t, \lambda, B]$ IN Q_0

AND $[q_1, w] \vdash^* [s, \lambda]$ IN Q_1

SO $w \in F(Q_2)$ IFF $t \in F_0$ & $s \in F_1$ IFF $w \in F(Q_0) \cap F(Q_1)$

CFL CLOSURE

MODERATE: SUBSTITUTION

$G = (V, \Sigma, R, S) \quad L = \mathcal{L}(G)$

SUBSTITUTION

$f(a) = L_a \quad a \in \Sigma \quad L_a$ A CFL

$G_a = (V_a, \Sigma, R_a, S_a) \quad L_a = \mathcal{L}(G_a)$

IN R, CHANGE ALL INSTANCES OF $a \in \Sigma$ IN RHS's TO S_a

THUS, IF ORIGINALLY,

$S \Rightarrow a_1, \ldots, a_k$

THEN, IN NEW

$S \overset{*}{\Rightarrow} S_{a_1} \cdots S_{a_k}$

AND THEN

$S \overset{*}{\Rightarrow} w_{a_1} \cdots w_{a_k}$

WHERE $w_{a_i} \in f(a_i)$

$G' = (V \cup V_A \cup V_B, \Sigma, R', S)$

$R' = R^{\text{CHANGED}} \cup R_a \quad a \in \Sigma$

WHERE R^{CHANGED} IS AS ABOVE

CFL NON-CLOSURE

INTERSECTION

BY EXAMPLE OF CONTRADICTORY CASE

$L_1 = \{a^n b^n c^m \mid n, m > 0\}$

$L_2 = \{a^m b^n c^n \mid n, m > 0\}$

$S_1 \to S_1 c \mid T_1 c$ $\quad S_2 \to a S_2 \mid a T_2$

$T_1 \to a T_1 b \mid a b$ $\quad T_2 \to b T_2 c \mid b c$

BOTH ARE CFLs

HOWEVER,

$L_1 \cap L_2 = \{a^n b^n c^n \mid n > 0\}$

WHICH IS NOT A CFL

COMPLEMENT

BY FACT THAT CLOSURE UNDER UNION AN COMPLEMENT IMPLIES CLOSURE UNDER INTERSECTION

$\sim(\sim A \cup \sim B) = A \cap B$

CSG

$\alpha \to \beta \qquad |\alpha| \le |\beta|,$

$\alpha \in (V \cup \Sigma)^* V (V \cup \Sigma)^*$

$\beta \in (V \cup \Sigma)^+$

ONE EXCEPTION IS

$S \to \lambda$ IF $\lambda \in L$

AND THEN S CANNOT BE ON RHS

$L = \{a^n b^n c^n \mid n > 0\}$

$G = (\{A\ \}, \{a, b, c\}, R, A)$

$A \to aBbc \mid abc$

$B \to aBbC \mid abC$

$Cb \to bC$

$Cc \to cc$

$1\langle 0\rangle$

$0\langle 1\rangle$

$010 \cdots \ 1\langle 0\rangle$

$L = \{ww \mid w \in \{0,1\}^+\}$

$S \to 00 \mid 11 \mid 0A\langle 0\rangle \mid 1A\langle 1\rangle$

$A \to 0AZ \mid 1AX \mid 0Z \mid 1X$

$\left.\begin{array}{l} Z0 \to 0Z \\ Z1 \to 1Z \\ X0 \to 0X \\ X1 \to 1X \end{array}\right\}$ SHUTTLE

$Z\langle 0\rangle \to 0\langle 0\rangle$

$Z\langle 1\rangle \to 1\langle 0\rangle$

$X\langle 0\rangle \to 0\langle 1\rangle$

$X\langle 1\rangle \to 1\langle 1\rangle$

$\langle 0\rangle \to 0$

$\langle 1\rangle \to 1$

LBA

SIMPLE VIEW

R/W TAPE

q_0 ¢ | w | \$

ACCEPT BY FINAL STATE

ACTIONS ARE

READ WRITE MOVE (LEFT/RIGHT/STAY)

OFTEN EASIEST TO VIEW
OPERATIONS AS BEING ABLE
TO LOOK LEFT OR RIGHT (BASICALLY
MOVES EITHER WAY)

CAN ALSO VIEW AS MULTITRACK
(FINITE # OF TRACKS)
FOR EXAMPLE $(\{¢,\$\} \cup \Sigma) \times (\{¢,\$\} \cup \Gamma)$
AS TAPE ALPHABET WITH CHANNEL
1 HAVING INPUT.

QUICK EXAMPLE OF LBA

$L = \{a^n b^n c^n \mid n > 0\}$

$q_0 ¢ \rightarrow ¢ q_1$

$q_1 a \rightarrow x q_2$

$q_2 b \rightarrow y q_3$

$q_3 c \rightarrow q_4 z$

$q_2 a \rightarrow a q_2$

$q_3 b \rightarrow b q_3$

$z q_4 \rightarrow q_4 z$

$q_2 y \rightarrow y q_2$

$q_3 z \rightarrow z q_3$

$b q_4 \rightarrow q_4 b$

$y q_4 \rightarrow q_4 y$

$a q_4 \rightarrow q_4 a$

$x q_4 \rightarrow x q_1$

$q_f z \rightarrow z q_f$

$q_1 y \rightarrow y q_f$

$q_f y \rightarrow y q_f$

$q_f \$ \rightarrow \$ q_f$

$q_0 ¢ a^n b^n c^n \$ \vdash^* ¢ x^k q_1 a^{n-k} y^k b^{n-k} z^k c^{n-k} \$ \vdash^* ¢ x^n y^n z^n \$ q_f$

$\vdash^* ¢ x^n q_1 y^n z^n \$$

TAPE ALPHABET IS

$\{¢, \$, x, y, z, a, b, c\}$

ALL RE SETS ARE HOMOMORPHIC IMAGES OF CSLs

LET $G = (V, \Sigma, R, S)$; CONSTRUCT $G' = (V \cup \{D\}, \Sigma \cup \{*\}, R', S')$

FOR EACH RULE

$$\alpha \to \beta \in R$$

WHERE $|\alpha| \leq |\beta|$,

$$\alpha \to \beta \in R'$$

FOR EACH RULE

$$\alpha \to \beta \in R$$

WHERE $|\alpha| > |\beta|$, LET $d = |\alpha| - |\beta|$

$$\alpha \to \beta D^d \in R'$$

ALSO R' CONTAINS

$$DX \to XD \quad \forall X \in \Sigma \cup V$$

$$D* \to **$$

$$S' \to S*$$

$\mathcal{L}(G') = \{ w *^k \mid w \in \mathcal{L}(G)$ AND $k > 0$ BUT UNKNOWN $\}$

$h(\mathcal{L}(G')) = \mathcal{L}(G)$; $h(a) = a$, $a \in \Sigma$

$h(*) = \lambda$