1. QUOTIENT OF REGULAR

$L/R = \{x \mid \exists y \in R, xy \in L\}$

$\mathcal{Q}_L = (Q_L, \Sigma, \delta_L, q_{1L}, F_L)$ $L = \mathcal{L}(\mathcal{Q}_L)$

$\mathcal{Q}_R = (Q_R, \Sigma, \delta_R, q_{1R}, F_R)$ $R = \mathcal{L}(\mathcal{Q}_R)$

$\mathcal{Q}_{L/R} = (Q_L \cup Q_L \times Q_R, \Sigma, \delta_{L/R}, q_{1L/R}, F_{L/R})$

$F_{L/R} = F_L \times F_R$

$\delta_{L/R}(q, a) = \{\delta_L(q, a)\}$ $q \in Q_L, a \in \Sigma$

$\delta_{L/R}(q, \lambda) = \{\langle q, q_{1R}\rangle\}$ $q \in Q_L$

$\delta_{L/R}(\langle q, p\rangle, \lambda) = \{\langle \delta_L(q, a), \delta_R(p, a)\rangle\}$

$q \in Q_L, p \in Q_R, a \in \Sigma$

$L/R = \mathcal{L}(\mathcal{Q}_{L/R})$

WHY???

# 2. CLOSURE METATECHNIQUE

ASSUME CLOSURE UNDER

a) SUBSTITUTION / HOMOMORPHISM

b) INTERSECTION WITH REGULAR LANGUAGES

DEFINE

$f(a) = \{a, a'\}$ ; $g(a) = a'$

$h(a) = a$ ; $h(a') = \lambda$

GENERAL FORM OF CLOSURE

$OP(L) = h(f(L) \cap R)$

REGULAR PATTERN TO USE AS FILTER

EXAMPLE 1

$\text{PREFIX}(L) = h(f(L) \cap \Sigma^* g(\Sigma^*))$

$= h(\{xy' \mid xy \in L\})$

$= \{x \mid \exists y \in \Sigma^*, xy \in L\}$

EXAMPLE 2

$L/R = h(f(L) \cap \Sigma^* g(R))$

$= h(\{xy' \mid xy \in L, y \in R\})$

$= \{x \mid \exists y \in R, xy \in L\}$

3. MORE EXAMPLES:

$$\text{SUFFIX}(L) = h(f(L) \cap g(\Sigma^*)\Sigma^*)$$

$$\text{SUBSTRING}(L) = h(f(L) \cap g(\Sigma^*)\Sigma^* g(\Sigma^*))$$

4. OTHER WAYS TO GET SUFFIX, SUBSTRING ASSUMING PREFIX AND REVERSE

$$\text{SUFFIX}(L) = (\text{PREFIX}(L^R))^R$$

$$\begin{aligned}\text{SUBSTRING}(L) &= \text{SUFFIX}(\text{PREFIX}(L)) \\ &= \text{PREFIX}(\text{SUFFIX}(L))\end{aligned}$$

QUOTIENT

5. PROOF OF PREFIX(L) USING DFAs

$L = \mathcal{L}(Q)$ $Q = (Q, \Sigma, \delta, q_0, F)$

$PREFIX(L) = \mathcal{L}(Q')$ WHERE

$Q' = (Q, \Sigma, \delta, q_0, F')$

$F' = \{q \mid \exists w \; \delta^*(q, w) \in F, w \in \Sigma^*\}$

6. PROOF OF SUFFIX(L) USING DFAs

$L = \mathcal{L}(Q)$ $Q = (Q, \Sigma, \delta, q_0, F)$

$SUFFIX(L) = \mathcal{L}(Q'')$ WHERE

$Q'' = (Q, \Sigma, \delta, S, F)$

$S = \{q \mid \exists w \; \delta^*(q_0, w) = q, w \in \Sigma^*\}$

REALLY WE ARE LIMITED TO ONE START STATE IN FORMAL MODEL SO CREATE NEW START STATE $S_0$

$Q'' = (Q \cup \{S_0\}, \Sigma, \delta'', S_0, F)$

$\delta''(q, a) = \{\delta(q, a)\}$ $q \in Q, a \in \Sigma$

$\delta''(S_0, \lambda) = \{q \mid \exists w \; \delta^*(q_0, w) = q, w \in \Sigma^*\}$

~~THIS~~ IS AN NFA

# 7. PUMPING LEMMA FOR REGULAR LANGUAGES

LET L BE REGULAR THEN
$\exists N > 0$ SUCH THAT, IF $w \in L$ AND
$|w| \geq N$ THEN $w$ CAN BE WRITTEN
IF FORM $xyz$ $(w = xyz)$
WHERE $|xy| \leq N$, $|y| > 0$ AND
$\forall i \geq 0$ $xy^i z \in L$

USES "PIGEON HOLE PRINCIPLE"

IF I HAVE N CONTAINERS
AND I HAVE M>N ITEMS TO
PUT IN CONTAINERS THEN AT
LEAST ONE CONTAINER MUST
ACCOMMODATE MORE THAN ONE
ITEM.

# 8. PUMPING LEMMA IN DETAIL

LET $L = \mathcal{L}(Q)$ $Q = (Q, \Sigma, \delta, q_0, F)$

LET $N = |Q|$

LET $w \in \Sigma^+$ WHERE $|w| \geq N$

$w = v_1 v_2 \ldots v_m$ $m \geq N$ $v_i \in \Sigma$

AS $Q$ STARTS IN $q_0$, IT VISITS ONE STATE PRIOR TO READING $v_1$

AT OR BEFORE READING $v_N$, $Q$ MUST VISIT AT LEAST ONE STATE MORE THAN ONCE

LET $v_1 \ldots v_j$ BE SHORTEST STRING TO VISIT SOME REPEATED STATE, $q_r$, AND $v_{j+1} \ldots v_k$ BE SHORTEST NON-EMPTY STRING TO REVISIT $q_r$

SO $\delta^*(q_0, v_1 \ldots v_j) = q_r$

$\delta^*(q_0, v_{j+1} \ldots v_k) = q_r$

$\delta^*(q_r, v_{k+1} \ldots v_m) \in F$

LET $x = v_1 \ldots v_j$ $y = v_{j+1} \ldots v_k$

$z = v_{k+1} \ldots v_m$

THEN $\delta^*(q_0, x y^i) = q_r$ $\forall i \geq 0$

AND $\delta^*(q_0, x y^i z) \in F$ $\forall i \geq 0$

9. POMPING LEMME APPLICATION USING ADVERSARIAL PROCESS

ME: ASSUME $L = \{a^n b^n \mid n > 0\}$ REGULAR

P.L.: PROVIDES $N > 0$. I CAN ASSUME NOTHING ABOUT $N$ EXCEPT $N > 0$

ME: CHOOSE SOME STRING WITH LENGTH $\geq N$. I'LL CHOOSE $a^N b^N$

P.L.: $a^N b^N = xyz$, $|xy| \leq N$, $|y| > 0$ AND $\forall i \geq 0$, $xy^i z \in L$

I CANNOT CHOOSE SPLIT BUT P.L. IS CONSTRAINED BY $N$

ME: I CHOOSE $i = 0$

I CAN CHOOSE ANY $i \geq 0$

P.L.: FROM ABOVE MUST ASSERT

$a^{N-|y|} b^N \in L$.

ME: GOTCHA AS WITH $|y| > 0$

$N - |y| \neq N$ SO

$a^{N-|y|} b^N \notin L$

CONTRADICTION SO L IS <u>NOT</u> REGULAR

# 10. MYHILL-NERODE THEOREM

BASED ON NOTION OF
RIGHT-INVARIANT EQUIV. RELATIONS
OVER STRINGS

$$xRy \Rightarrow \forall z \; xzRyz \quad x,y,z \in \Sigma^*$$

CONSIDER $Q = (Q, \Sigma, \delta, q_0, F)$, $Q$ DFA

DEFINE $R_Q$ BY

$$x R_Q y \Leftrightarrow \delta^*(q_0, x) = \delta^*(q_0, y)$$

$R_Q$ IS AN EQUIV. RELATION
AND IT IS RIGHT-INV. AS

$$\delta^*(q_0, x) = \delta^*(q_0, y)$$
$$\Rightarrow \forall z \; \delta^*(q_0, xz) = \delta^*(q_0, yz)$$
$$\Rightarrow \forall z \; xz R_Q yz$$

MOREOVER, THE INDEX OF $R_Q$
(# OF PARTITIONS INDUCED BY $R_Q$)
IS FINITE ($|Q|$)

11. MYHILL-NERODE THEOREM STATEMENT

(a), (b), & (c) ARE EQUIVALENT

(a) L IS REGULAR

(b) L IS THE UNION OF SOME OF THE EQUIV. CLASSES OF SOME R.I.E.R OF FINITE INDEX

(c) THE SPECIFIC R.I.E.R. $R_L$ DEFINED BY

$$x R_L y \Leftrightarrow \forall z\, [xz \in L \Leftrightarrow yz \in L]$$

HAS FINITE INDEX

13. MYHILL-NERODE PROOF (LAST STAGE)

(c) $\Rightarrow$ (a)

DEFINE $Q_{R_L} = (Q, \Sigma, \delta, [\lambda], F)$

$Q = \{[x]_{R_L} \mid x \in \Sigma^*\}$ FINITE

$F = \{[x]_{R_L} \mid x \in L\}$ SUBSET OF $Q$

NOTE: $[x]_{R_L}$ IS EQUIV. CLASS CONTAINING $x$

$\delta([x], a) = [xa]$

14. CONSEQUENCES

A. MIN DFA IS UNIQUE AS ALL DFAs FOR $L$ ARE REFINEMENTS OF ONE DERIVED FROM $R_L$

B. A LANGUAGE $L$ IS <u>NOT</u> REGULAR IF $R_L$ HAS INFINITE INDEX

## APPLICATION OF MYHILL-NERODE

15. $L = \{a^n b^n \mid n > 0\}$ IS NOT REG.

CONSIDER $[a^i]_{R_L}$ $i > 0$

$a^i b^i \in L$ BUT
$a^i b^j \notin L$ WHEN $j \neq i$
THUS, $[a^i]_{R_L}$ AND $[a^j]_{R_L}$ ARE DISTINCT
WHEN $i \neq j$
SO $R_L$ HAS INFINITE INDEX

---

$L = \{a^{n^2} \mid n \geq 0\}$ IS NOT REG.
CONSIDER $[a^{i^2}]_{R_L}$ $i \geq 0$
$a^{i^2} a^{2i+1} \in L$
BUT
$a^{j^2} a^{2i+1} \notin L$ WHEN $j > i$
THUS, $[a^{i^2}]_{R_L}$ AND $[a^{j^2}]_{R_L}$
ARE DISTINCT WHEN $i \neq j$
$(i > j$ OR $i < j)$
SO $R_L$ HAS INFINITE INDEX